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Abstract

We provide an axiomatic foundation for efficiency measurement in the full ⟨input, output⟩ space, referred to as "graph efficiency" measurement by Färe, Grosskopf, and Lovell [1985]. We posit four types of axioms: indication, monotonicity, independence of units of measurement, and continuity. We analyze six well-known inefficiency indexes from the operations-research and economics literature and discuss several other related indexes.

We present two impossibility results demonstrating that no index can satisfy all of the axioms on a general class of (well-behaved) technologies. Specifically, no inefficiency index can satisfy both indication and continuity (in either quantities or technologies), and no inefficiency index can satisfy both monotonicity and unit independence. We present a full evaluation of the trade-offs involved in selecting among the indexes.

JEL classification: C43; C61; D24.

Keywords: Technical efficiency indexes; technical efficiency axioms.

1. Introduction.

Analysis of the axiomatic foundations of efficiency measurement began with Färe and Lovell [1978], who proposed three axioms that an input-based efficiency index should satisfy: *indication* (the index is equal to one if and only if the input vector is efficient in the sense of Koopmans [1951]), *monotonicity* (an increase in any input, holding other inputs as well as all outputs constant, reduces the value of the index), and *homogeneity* (*e.g.*, doubling all inputs, holding outputs constant, cuts the value of the index in half). Subsequently, Russell [1985, 1987, 1990] clarified the initial Färe and Lovell axioms and introduced two additional axioms for input-based efficiency measurement: *invariance with respect to units of measurement* (also known as *commensurability*) and *continuity* (in technologies as well as input and output quantities).

In recent years, empirical research on efficiency measurement has focused much more on measurement in the full space of inputs and outputs, which we refer to as $\langle$input, output$\rangle$ space. To our knowledge, however, no work in the Färe-Lovell tradition has been carried out in this space. While an extensive operations-research (DEA) literature has assessed the ability of several indexes to satisfy certain properties in this space, this literature differs in two important respects from the literature in the Färe-Lovell tradition. First, the DEA literature treats efficiency indexes as functions of production data, which determine the technology. Second, the DEA-constructed technologies are convex polyhedrons—usually convex polyhedral cones. We treat the efficiency indexes as functions of the $\langle$input, output$\rangle$ production vector and an exogenous technology. The class of permissible technologies is general and in particular is not restricted to the DEA class. Our analysis thus encompasses technologies constructed by a variety of techniques, including stochastic frontier methods.

In this paper, we provide an axiomatic foundation for efficiency measurement in the full $\langle$input, output$\rangle$ space, referred to as "graph efficiency" measurement by Färe, Grosskopf, and Lovell [1985]. We posit four types of axioms: indication, monotonicity, independence of units of measurement, and continuity.[1] We analyze six well-known inefficiency indexes from the operations-research as well as the economics literature: the hyperbolic index (Färe, Grosskopf, and Lovell [1985]), the directional distance index (Luenberger [1992] and Chung, Färe, and Grosskopf [1997]), the Briec index (Briec [1997]), the Färe-Grosskopf-Lovell (FGL) index (Färe, Grosskopf, and Lovell [1985]), the additive (slacks-based) index (Charnes, Cooper, Golany, Seiford, and Stutz [1985]), and the weighted additive index (typically attributed to Cooper and Pastor [1995]). Several other indexes are discussed and related to these six.

Although our axioms seem to be natural requirements for an inefficiency index, we present two impossibility results demonstrating that no index can satisfy all of the axioms.

[1] Homogeneity is not considered, since it is not obvious how this property should be extended to $\langle$input, output$\rangle$ space. We have formulated some possible extensions, but none is satisfied by any of the indexes we consider.

Specifically, no inefficiency index can satisfy both indication and continuity (in either quantities or technologies), and no inefficiency index can satisfy both monotonicity and unit independence. Trade-offs are confronted in the selection of an index.

We present a full evaluation of each of the indexes in terms of our axioms. Two of the indexes—the directional distance index and the additive index—are dominated by alternatives. The hyperbolic index and the Briec index satisfy weak monotonicity, unit independence, and continuity in both production vectors and technologies while failing to satisfy the remaining axioms. Since these two indexes have the same properties, they can be treated as equivalent in terms of our axiomatic structure. The FGL index satisfies indication, weak monotonicity, and unit independence, while the weighted additive index satisfies indication, monotonicity, and unit scalability (a weakening of unit independence). Thus, there are three distinct groups of axioms that are satisfied by an appropriate choice of the inefficiency index.

Section 2 introduces the notation and describes the general assumptions on technologies. Section 3 defines an inefficiency index and discusses six of the most prominent indexes in the literature. Section 4 describes the axioms we use to evaluate the indexes, while Section 5 presents two theorems stating the incompatibility of two pairs of axioms. Section 6 presents our main result, Theorem 3, cataloging the performance of each of the indexes in terms of our axiomatic structure. The proof of Theorem 3 is presented in the Appendix. Section 7 discusses the implications of these results for some additional inefficiency indexes. Section 8 concludes.

2. Preliminaries.

A firm (or other production unit) uses n inputs to produce m outputs with production vectors, denoted $\langle x, y\rangle$, contained in the $\langle$input, output$\rangle$ space $\mathbf{R}_+^{n+m}$. Denote the origin of this space by $\left\langle 0^{[n]}, 0^{[m]}\right\rangle$.

The firm's technology set $T \subset \mathbf{R}_+^{n+m}$ contains feasible production vectors. A production vector $\langle x, y\rangle \in T$ is technologically efficient (in the sense of Koopmans [1951]) if $\langle x, -y\rangle > \langle \bar{x}, -\bar{y}\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \notin T$.[2] Denote the efficient subset of T by $\text{Eff}(T)$.

The theoretical literature on technical efficiency measurement has focused on a general class of technologies satisfying only very weak regularity conditions. We consider the collection of non-empty, closed technology sets that satisfy the following conditions:[3]

[2] Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i.

[3] All but free disposability of these conditions are necessary to guarantee that our efficiency indexes are well defined. Free disposability could be dispensed with (theoretically); the only change that would be needed in what follows would be to redefine the inefficiency indexes on the free-disposal hull of T rather than on T itself (as in Russell [1987] for input-based efficiency indexes). Note, finally, that the free disposability assumption implies connectedness of the technology, a property that we exploit.

(i) $\langle x, y\rangle \in T$, $\bar{y} \in \mathbf{R}^m_+$, and $\langle \bar{x}, -\bar{y}\rangle > \langle x, -y\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \in T$ (free disposability of inputs and outputs),

(ii) $y > 0^{[m]} \implies \langle 0^{[n]}, y\rangle \notin T$, and

(iii) the production possibility set, $P(x) = \{\, y \in \mathbf{R}^m_+ \mid \langle x, y\rangle \in T \,\}$, is non-empty and bounded for all $x \in \mathbf{R}^n_+$.

We denote by $\mathcal{T}$ the set of non-empty, closed technologies satisfying these conditions.

3. Inefficiency Indexes.

A technological inefficiency index measures the "distance"[4] from the production vector to the frontier of T or to its efficient subset $\mathrm{Eff}(T)$. Typically, the production point is compared to a particular point—the reference vector—on the boundary or the efficient subset of T. The issues addressed in formulating a specific inefficiency index are (i) the selection of the reference vector corresponding to any production vector and (ii) the specification of the distance between the production vector and the reference vector.

Many of the inefficiency indexes in the literature are not well defined—or have unacceptable properties—at the boundary of $\langle$input, output$\rangle$ space. To avoid a loss of focus on the basic structures of these indexes and the relationships among them, we restrict our attention here to strictly positive quantities.[5] Formally, we define an inefficiency index as a mapping, $I : \Xi \to R(I)$, with image $I(x, y, T)$, where

$$\Xi = \{\, \langle x, y, T\rangle \in \mathbf{R}^{n+m}_{++} \times \mathcal{T} \mid \langle x, y\rangle \in T \,\} \tag{3.1}$$

and $R(I) \subseteq [0, +\infty)$, the effective range of I, varies across specific indexes. Although the inefficiency indexes are restricted to strictly positive production vectors, a technology may contain feasible or even efficient points on the boundary of $\mathbf{R}^n_+$.

To extend this definition to inefficiency indexes with parameters, we introduce a parameter space $G \subseteq \mathbf{R}^p$ and a parameter vector $g \in G$. Define a parameterized inefficiency index as a mapping $I^g : \Xi \times G \to R(I)$, with image $I^g(x, y, T, g)$. This extension is required for the directional distance inefficiency index and the axiom of unit scalability, each of which is defined below.

Many of the indexes we consider were originally defined on the particular subset of $\mathcal{T}$ generated by mathematical programming methods of constructing technology sets on a finite set of data points. This method, commonly referred to as Data Envelopment Analysis (DEA), generates convex polyhedral reference technologies (*i.e.*, intersections of

[4] We surround "distance" with quotation marks to underscore the informal nature of this notion, which is not consistent with the formal mathematical concept of distance.

[5] We do not mean to imply that these boundary issues are unimportant; on the contrary, since many data sets contain zero values of input or output quantities, these boundary issues need to be dealt with. See Russell and Schworm [2009] for an analysis of boundary problems.

finite numbers of half spaces).[6] Many of these indexes, however, can be applied to the more general class of technologies $\mathcal{T}$.[7]

To our knowledge, the first formulation of an inefficiency index in the full ⟨input, output⟩ space, attributable to Färe, Grosskopf, and Lovell [1985, pp. 110–111],[8] is the *hyperbolic inefficiency index*, defined by

$$I_H(x, y, T) = \max\{\, \lambda \mid \langle x/\lambda, \lambda y\rangle \in T \,\}\,. \tag{3.2}$$

This index contracts inputs and expands outputs along a (particular) hyperbolic path to the frontier and maps into the [1,+∞) interval.

Figure 1 illustrates the paths followed for a technology with one input and one output. The technology set is the shaded area with the boundary given by the thick solid line. The thick dashed lines identify the production vectors that are compared with one of the three efficient vertexes. All production vectors in the darkly shaded region are compared to efficient points. Those in the lightly shaded regions have inefficient points as reference points. Three examples are displayed, with (x, y) compared to the inefficient boundary point $(\bar{x}, \bar{y})$, (x', y') compared to the efficient point $(\bar{x}', \bar{y}')$, and (x'', y'') compared to the inefficient boundary point $(\bar{x}'', \bar{y}'')$. Note that the paths are not parallel; nor are they simple translates of one another.

The *directional distance inefficiency index*, I_{DD}, adapted from the shortage function of Luenberger [1992] to the measurement of inefficiency by Chung, Färe, and Grosskopf [1997],[9] is defined by

$$I_{DD}(x, y, T, g) = \max\{\, \lambda \mid \langle x - \lambda g_x,\ y + \lambda g_y\rangle \in T \,\}\,, \tag{3.3}$$

where $g = \langle g_x, g_y\rangle \in \mathbf{R}^{n+m}_{++}$.[10] This index measures the feasible contraction/expansion in the direction g and maps into $\mathbf{R}_+$.

Figure 2 displays the connection between the production vectors and their reference points for a particular technology and a particular choice of the direction g. Production vectors in the darkly shaded region, such as (x', y'), are compared to efficient points, as are vectors on the three dashed arrows. Those in the lightly shaded areas, such as $\langle x, y\rangle$ and $\langle x'', y''\rangle$, have reference points that are inefficient.

6 See Charnes, Cooper, Lewin, and Seiford [1994].

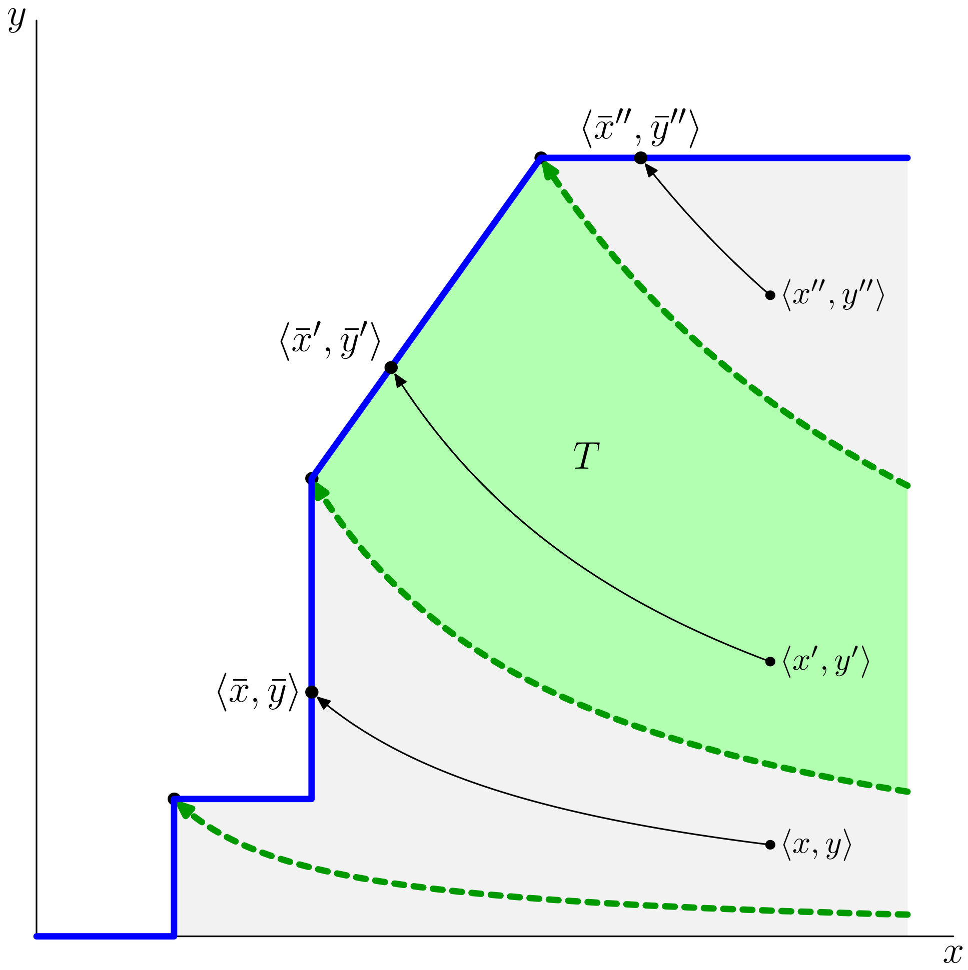
7 The Range Adjusted Measure of Inefficiency (RAM) attributable to Cooper, Park, and Pastor [1999] depends critically on the DEA construction of the technology and hence cannot be extended to the general class of technologies. This index is discussed in Section 7.

8 They refer to this index as the "Farrell Graph Measure of Technical Efficiency" and give it additional attention in Färe, Grosskopf, and Lovell [1994, Ch. 8].

9 See also Chambers, Chung, and Färe [1996], Färe and Grosskopf [2000], and Cherchye, Kuosmanen, and Post [2001].

10 Luenberger [1992] and others restrict the direction g only to the non-negative orthant, but we choose to restrict g to the positive orthant because this enhances the number of axioms—notably continuity axioms—that the directional distance index satisfies.

Figure 1: The Hyperbolic Inefficiency Index.



Briec [1997] proposes an alternative to the directional distance function by using the definition of I_{DD} with the direction $g = \langle x, y \rangle$. The *Briec inefficiency index*, defined by

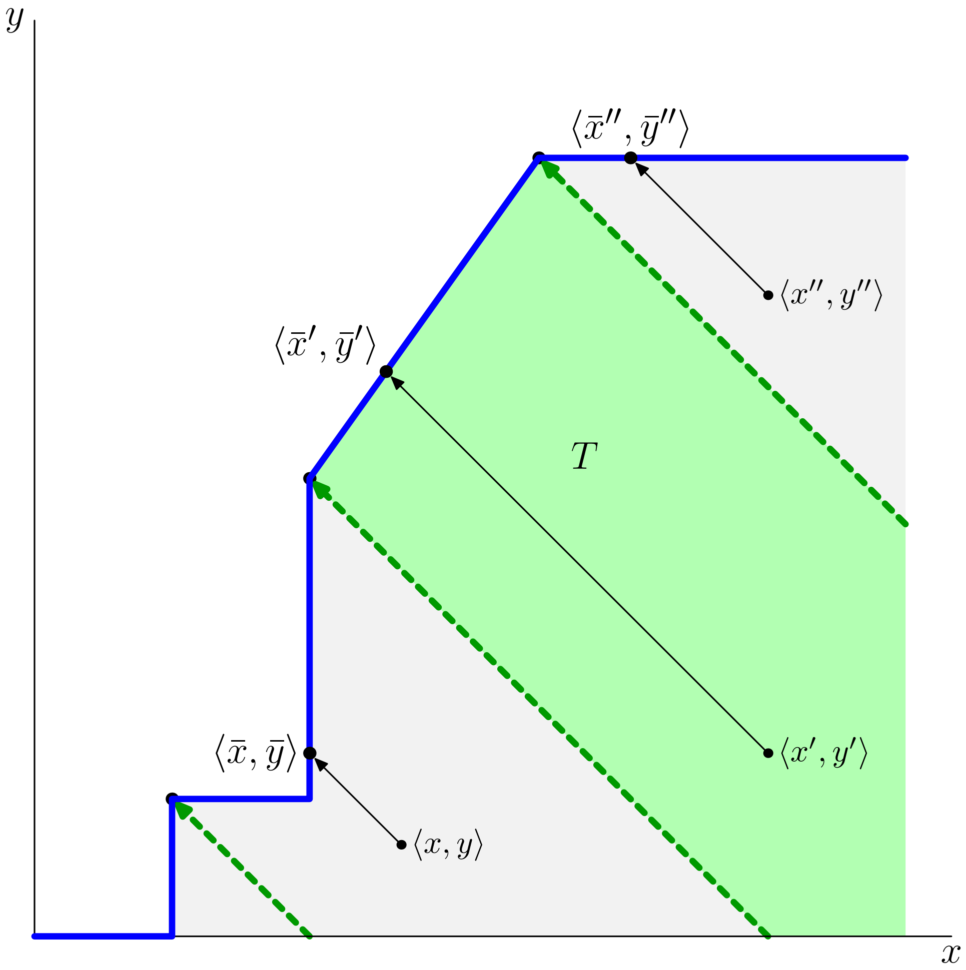
$$I_B(x, y, T) = \max \Big\{ \lambda \mid \big\langle (1 - \lambda)x,\ (1 + \lambda)y \big\rangle \in T \Big\}, \tag{3.4}$$

maps into the [0, 1] interval.[11] Figure 3 displays the reference points for the Briec index. The interpretation is the same as for Figures 1 and 2. Note that the paths to the frontier are not parallel.

Färe, Grosskopf, and Lovell [1985, pp. 153–154] have formulated an extension of the

[11] Briec and others argue that this inefficiency index is a special case of the directional distance function; this is not strictly correct since the formal definition of the directional distance function specifies a direction g that is independent of $\langle x, y \rangle$. In fact, if one were to specify a "generalized directional distance function" as in (3.3) but with g being a function of $\langle x, y \rangle$, then the standard directional distance inefficiency index (3.3) and the Briec inefficiency index (3.4) would each be special cases of the "generalized directional inefficiency index."

Figure 2: The Directional Distance Inefficiency Index.



input-based Färe-Lovell [1978] efficiency index to the full ⟨input, output⟩ space:[12,13]

$$E_{FGL}(x, y, T) = \min_{\alpha,\beta} \left\{ \frac{\sum_i \alpha_i + \sum_j \beta_j}{n+m} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x, y, T) \right\}, \tag{3.5}$$

where

$$\Omega(x, y, T) = \left\{ \langle \alpha, \beta \rangle \mid \langle\, \alpha \otimes x, y \oslash \beta \rangle \in T \;\wedge\; 0^{[n]} \leq \alpha \leq 1^{[n]} \;\wedge\; 0^{[m]} \ll \beta \leq 1^{[m]} \right\}, \tag{3.6}$$

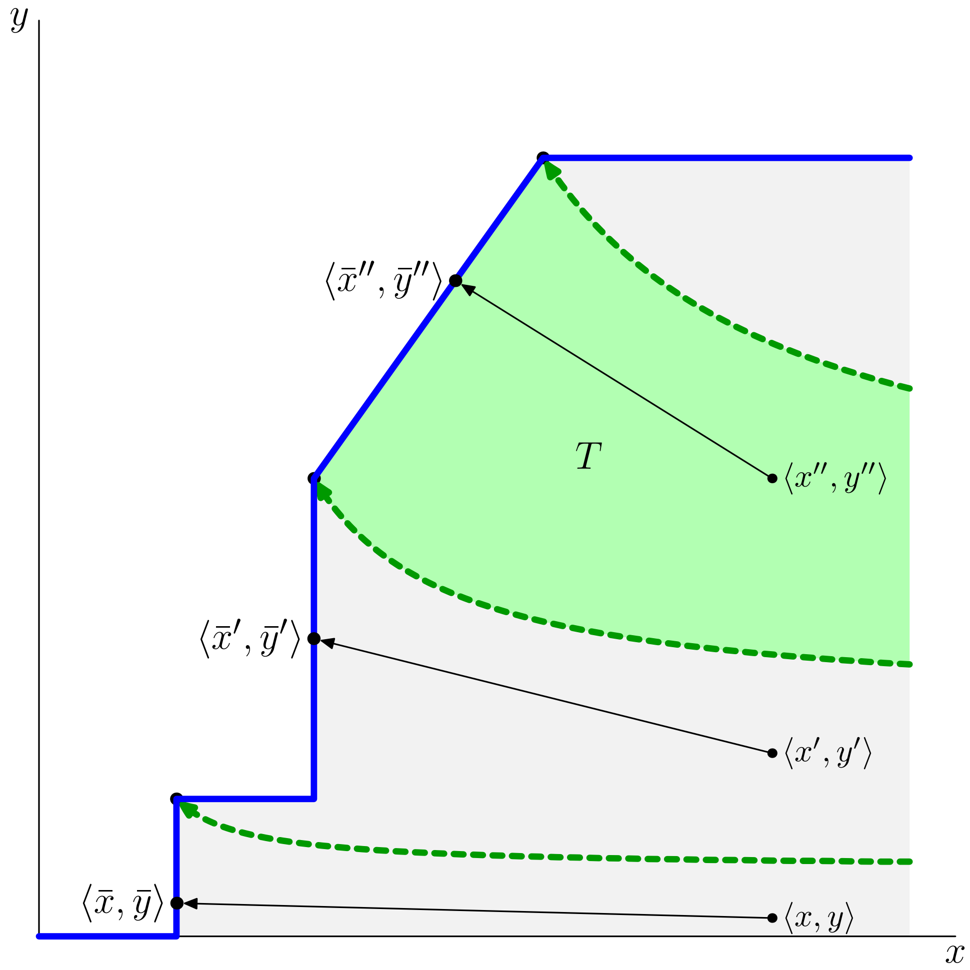
$\alpha \otimes x = \langle \alpha_1 x_1, \ldots, \alpha_n x_n \rangle$, and $y \oslash \beta = \langle y_1/\beta_1, \ldots, y_m/\beta_m \rangle$, for $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$ and $\beta = \langle \beta_1, \ldots, \beta_m \rangle$.[14] To make the index comparable to the other indexes in terms of the

[12] For obscure historical reasons, they refer to this index as the "Russell Graph Measure of Technical Efficiency."

[13] Russell and Schworm [2009] have recently modified this index to eliminate critical problems at the boundary of output space. Since we are considering production vectors that are strictly positive, these problems do not affect us here.

[14] Although $\Omega(x, y, T)$ is not a closed set, the min in (3.5) exists, owing to our restriction that $y \gg 0^{[m]}$ and our assumption that the production set $P(x)$ is bounded for all $x \in \mathbf{R}^n_+$.

Figure 3: The Briec Inefficiency Index.



axioms that follow, we define the *FGL inefficiency index* by

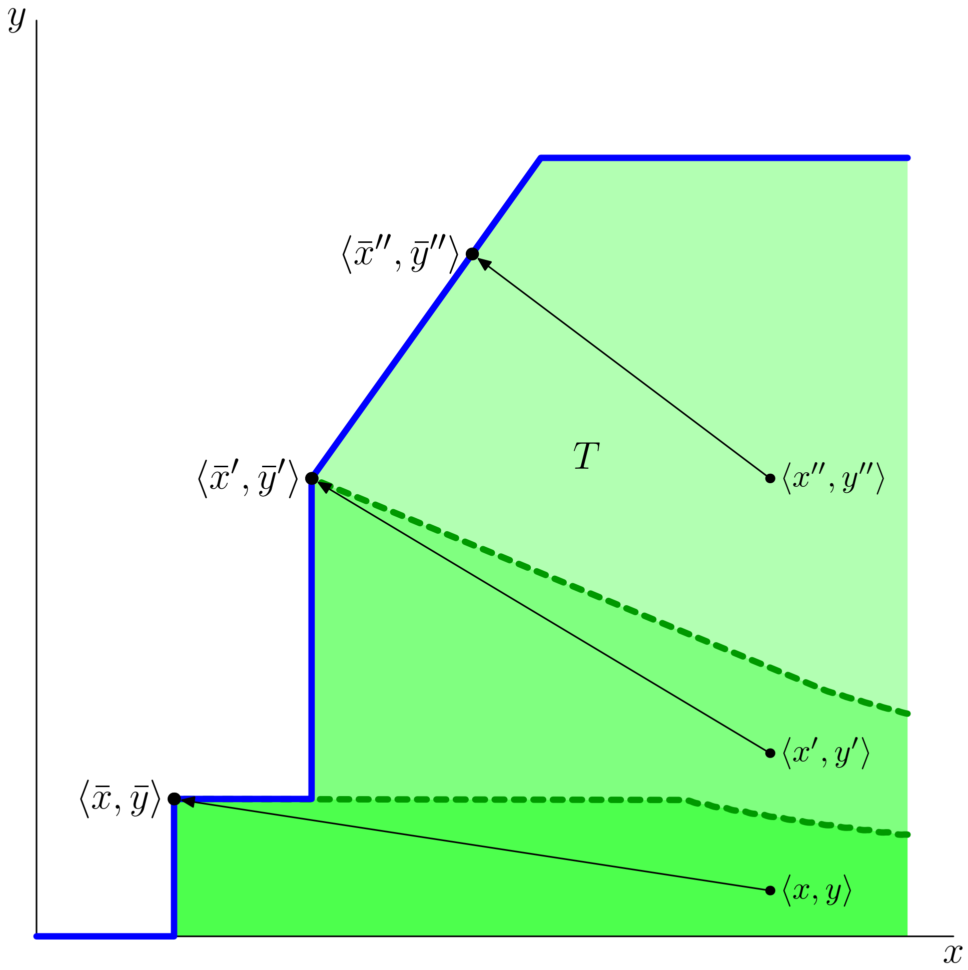
$$I_{FGL}(x, y, T) = 1 - E_{FGL}(x, y, T), \tag{3.7}$$

with range $[0, 1]$.[15] The FGL index is an average of the coordinate-wise maximal contractions of inputs and expansions of outputs.

The FGL index contracts all inputs and expands all outputs until an efficient production vector is achieved. Therefore, as illustrated in Figure 4, all feasible ⟨input, output⟩ vectors are compared to efficient points. The different shading in Figure 4 shows the points that are attracted to various efficient points. The top region with the lightest shading are the feasible points compared to an efficient point on the line segment. The middle region with the middle shading are the feasible points compared to the efficient vertex on the lower end of the line segment indicated by $\langle \bar{x}', \bar{y}' \rangle$. The bottom region with the darkest shading shows the points compared to $\langle \bar{x}, \bar{y} \rangle$.

[15] An alternative that satisfies the same properties is $I_{FGL}(x, y, T) = [E_{FGL}(x, y, T)]^{-1}$, with range $[1, +\infty)$.

Figure 4: The I_{FGL} Inefficiency Index.



The *additive index* of Charnes, Cooper, Golany, Seiford, and Stutz (CCGSS) [1985], defined on DEA technologies, can be extended to the class of technologies $\mathcal{T}$ as follows:

$$I_A(x, y, T) = \max_{s,t} \left\{ \sum_i s_i + \sum_j t_j \mid \langle s, t \rangle \in \Gamma(x, y, T) \right\} \tag{3.8}$$

where

$$\Gamma(x, y, T) = \left\{ \langle s, t \rangle \mid \langle x - s, y + t \rangle \in T \ \wedge \ s \geq 0^{[n]} \ \wedge \ t \geq 0^{[m]} \right\}. \tag{3.9}$$

This index maximizes the sum of the slacks and maps into the $[0, +\infty)$ interval. Since the additive index compares all feasible points to efficient points, a figure showing the reference points would be similar to Figure 4.

An acknowledged problem with the additive index I_A is its dependence on units of measurement. This problem is discussed in the original paper by CCGSS, who propose a unit-invariant modification of the index. We discuss this index briefly in Section 7. An alternative generalization, the *weighted additive index*, introduces weights for the slack variables as follows:

$$I_{WA}(x,y,T) = \max_{s,t} \left\{ \sum_i u_i s_i + \sum_j v_j t_j \;\middle|\; \langle s,t \rangle \in \Gamma(x,y,T) \right\}, \tag{3.10}$$

where $u \in \mathbf{R}^n_+$ and $v \in \mathbf{R}^m_+$ are pre-specified weights and Γ is defined in (3.9).[16]

Many other (in)efficiency indexes have been formulated in the literature, but we have chosen not to include them in our basic analysis because (i) they are defined only for special classes of technologies such as the DEA class and cannot be straightforwardly extended to the general class $\mathcal{T}$, (ii) they are dominated, in the context of our axiomatic structure, by indexes that we do consider,[17] or (iii) they are similar to, or even equivalent to, an index that we do analyze and hence have identical axiomatic properties. Several of these indexes are discussed briefly in Section 7.

4. Axioms.

We propose four types of axioms as suitable for inefficiency measures defined on the full space of inputs and outputs. Two of the axioms—indication and monotonicity—are obvious extensions of axioms proposed by Färe and Lovell [1978] for input-based measures of efficiency. As none of the inefficiency indexes satisfies monotonicity, we also consider a weaker version of monotonicity. We consider two nested axioms for invariance with respect to changes of units of measurement. Finally, we posit a continuity axiom in production vectors and technologies, as proposed by Russell [1990] for input efficiency indexes.

The most basic axiom requires that an inefficiency index distinguish between inefficient and efficient production vectors:[18]

Indication of Efficiency (I): For all $\langle x,y,T \rangle \in \Xi$, there exists a θ in the range of the index such that $I(x,y,T) = \theta$ if and only if $\langle x,y \rangle \in \mathrm{Eff}(T)$.

The two monotonicity axioms are as follows:

Monotonicity (M): For all pairs $\langle x,y,T \rangle \in \Xi$ and $\langle x',y',T \rangle \in \Xi$ satisfying $\langle x,-y \rangle < \langle x',-y' \rangle$, $I(x,y,T) < I(x',y',T)$.

[16] Subsequent literature indicates that this formulation was suggested in an unpublished paper by Cooper and Pastor [1995], but we have not been able to verify this citation.

[17] As shown in Theorem 3 below, the additive and the directional distance indexes are also dominated by other indexes, but these two indexes are included in our basic analysis because they are central to the development of efficiency measurement (in early, formative years in the case of the former and in recent years in the case of the latter).

[18] In the operations research literature (*e.g.*, Cooper, Park, and Pastor [1999]), an index satisfying (I) is said to be "comprehensive."

Weak Monotonicity (WM): For all pairs $\langle x, y, T\rangle \in \Xi$ and $\langle x', y', T\rangle \in \Xi$ satisfying $\langle x, -y\rangle < \langle x', -y'\rangle$, $I(x, y, T) \leq I(x', y', T)$.

To formulate changes in units of measurement, let $\mathcal{W}_x$ be the set of positive $n \times n$ diagonal matrices and $\mathcal{W}_y$ be the set of positive $m \times m$ diagonal matrices. The extension to ⟨input, output⟩ space of the unit independence axiom proposed by Russell [1987][19] is as follows:

Unit Independence (UI): For all $\langle x, y, T\rangle \in \Xi$ and for all $W_x \in \mathcal{W}_x$ and $W_y \in \mathcal{W}_y$, if $\hat{x} = W_x x$, $\hat{y} = W_y y$, and

$$\widehat{T} = \Big\{(\hat{x}, \hat{y}) \mid \big(W_x^{-1}\hat{x}, W_y^{-1}\hat{y}\big) \in T\Big\},$$

then

$$I(x, y, T) = I(\hat{x}, \hat{y}, \widehat{T}).$$

This axiom requires that unit changes (and the initial choice of units) have no effect on the inefficiency index. A weaker requirement for parameterized inefficiency indexes is invariance with respect to unit changes with compensating changes in parameters:

Unit Scalability (US): For all $\langle x, y, T\rangle \in \Xi$ and for all $W_x \in \mathcal{W}_x$ and $W_y \in \mathcal{W}_y$ with $\hat{x} = W_x x$, $\hat{y} = W_y y$ and

$$\widehat{T} = \Big\{(\hat{x}, \hat{y}) \mid \big(W_x^{-1}\hat{x}, W_y^{-1}\hat{y}\big) \in T\Big\},$$

there exists a parameter $\hat{g} \in G$ such that

$$I(x, y, T, g) = I\left(\hat{x}, \hat{y}, \widehat{T}, \hat{g}\right).$$

We stress that, in our view, this axiom is substantially weaker than (UI), since it allows the index to depend on the initial choice of units.

We consider three continuity axioms. Russell [1990] argued (page 256) that continuity is a compelling property, "for it provides assurance that 'small' errors of measurement (of, *e.g.*, input or output quantities) result only in 'small' errors of efficiency measurement." If the technology is constructed from data on production vectors, the argument for continuity in the technology is perhaps even more compelling. We therefore believe the strongest of these continuity axioms—continuity in both input and output quantities and in technologies—is a desirable property for an inefficiency index. Since some standard inefficiency indexes do not satisfy this strong continuity property, we consider weaker versions as well.

Continuity in production vectors (C–$\langle x, y\rangle$): I is continuous in $\langle x, y\rangle$.

Continuity in technologies (C–T): I is continuous in T.[20]

Joint continuity (C–$\langle x, y, T\rangle$): I is continuous in $\langle x, y, T\rangle$.

[19] Following the nomenclature of Eichhorn and Voeller [1976] in their axiomatic analysis of price and quantity indexes, Russell [1987] referred to this property as "commensurability."

[20] As in Russell [1990], we adopt the topology of closed convergence on $\mathcal{T}$.

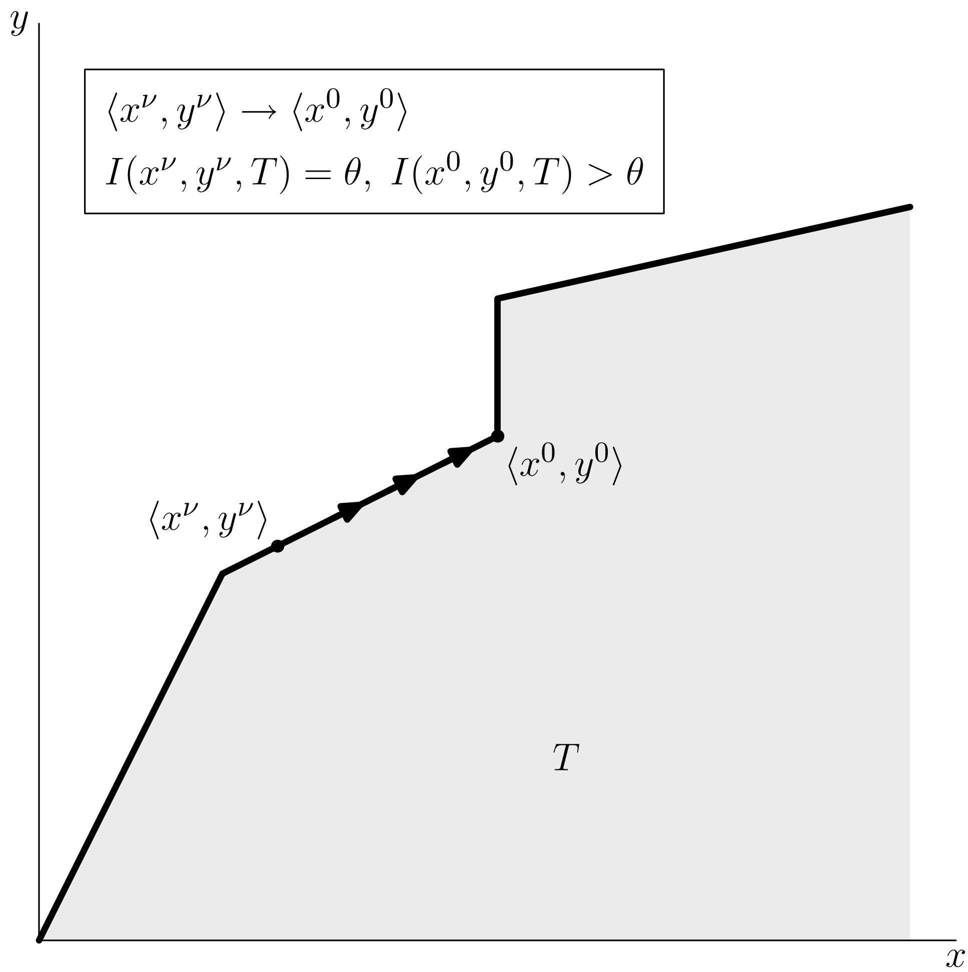
5. Impossibility Results.

A fundamental incompatibility among our axioms is encapsulated in the following result.

Theorem 1: *(a) There does not exist an inefficiency index satisfying (I) and (C–$\langle x, y\rangle$). (b) There does not exist an inefficiency index satisfying (I) and (C–T).*

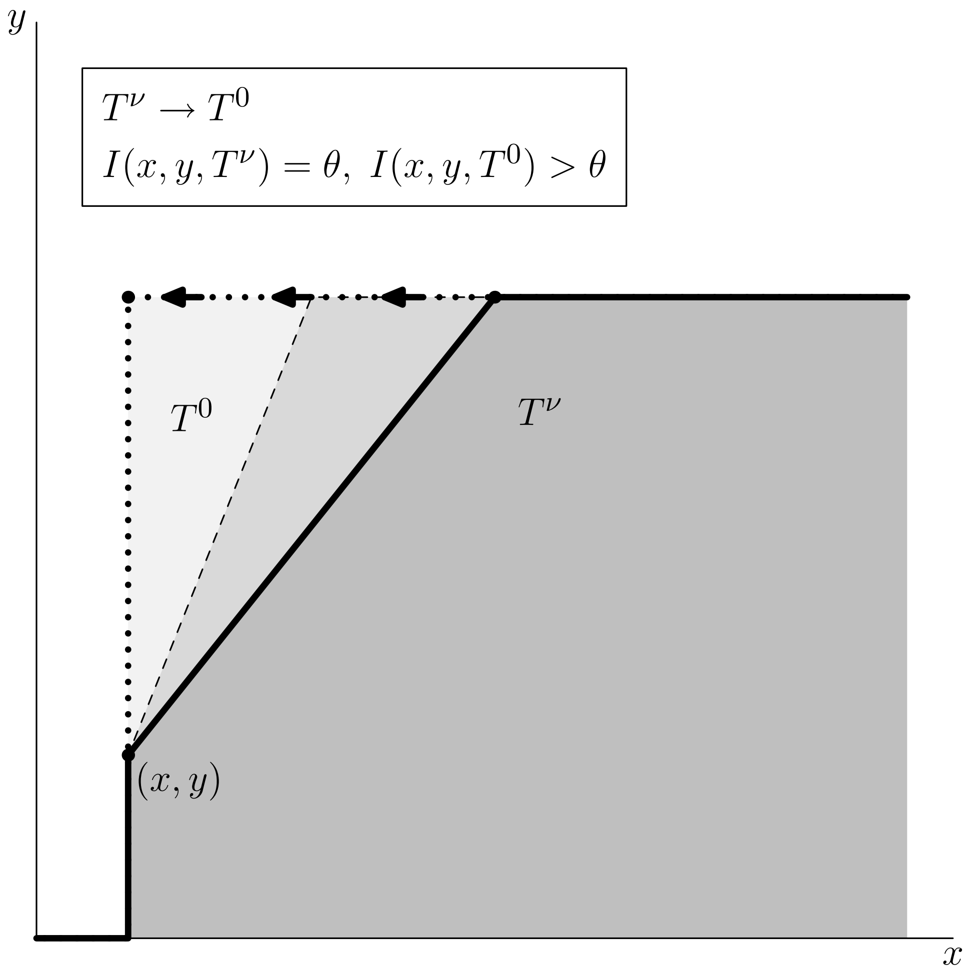
Proof: (a) In Figure 5, $\langle x^\nu, y^\nu\rangle \to \langle x^o, y^o\rangle$. As $\langle x^\nu, y^\nu\rangle$ is efficient for all ν and $\langle x^o, y^o\rangle$ is inefficient, (I) implies $I(x^\nu, y^\nu, T) = \theta$ for all ν and $I(x^o, y^o) > \theta$, violating (C–$\langle x, y\rangle$). (b) In Figure 6, $T^\nu \to T^o$. (I) implies $I(x, y, T^\nu) = \theta$ for all ν but also $I(x, y, T^o) > \theta$, violating (C–T). Each of these examples can be easily extended to higher dimensions. ■

Figure 5: Incompatibility of (I) and (C–$\langle x, y\rangle$).



Theorem 1 indicates that the set of all efficiency indexes defined on technologies $\mathcal{T}$ can be partitioned into two subsets: (1) those satisfying indication and hence violating continuity and (2) those satisfying continuity and hence violating indication. A further

Figure 6: Incompatibility of (I) and (C–T).



partitioning of the former is generated by the following result (essentially taken from Russell [1987]):

Theorem 2: *There does not exist an inefficiency index satisfying (M) and (UI).*

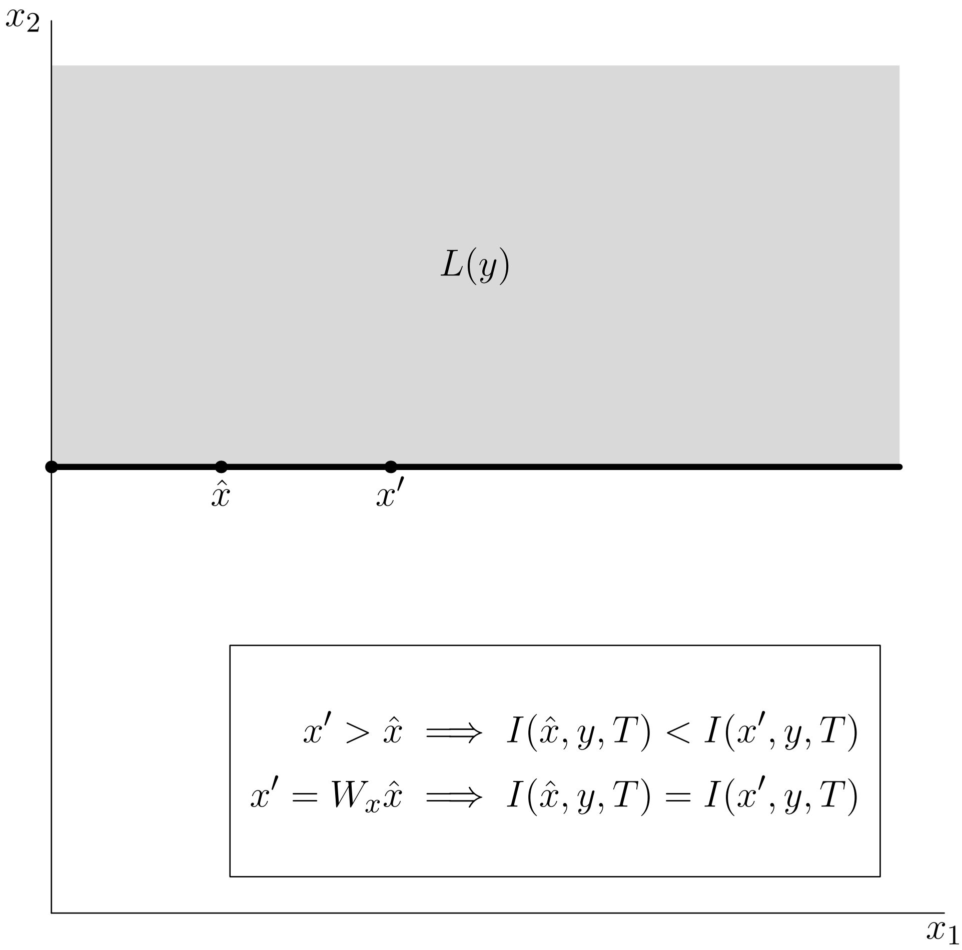
Proof: Assume efficiency in output space and consider the input requirement set in Figure 7 and input vectors $\hat{x}$ and x'. Clearly, (M) implies $I(x', y, T) > I(\hat{x}, y, T)$. Next note that $\hat{x} = W_x x$, where W_x is the two-by-two diagonal matrix with $\langle \kappa, 1 \rangle$ on the diagonal. Moreover, $\hat{y} = W_y y = y'$, where W_y is the m-dimensional identity matrix. Finally, $L(\hat{y}) = L(y)$, so that $\widehat{T} = T$ in the definition of unit independence, and (UI) implies $I(x', y, T) = I(\hat{x}, y, T)$, contradicting (M). ■

6. Properties of the Inefficiency Indexes.

The properties satisfied by the six inefficiency indexes are spelled out in the following theorem (proved in the Appendix):

Theorem 3:

Figure 7: Incompatibility of (M) and (UI).



- *I_H satisfies (WM), (UI), and (C–$\langle x, y, T\rangle$) but fails to satisfy (I) and (M).*
- *I_{DD} satisfies (WM), (US), and (C–$\langle x, y, T\rangle$) but fails to satisfy (I), (M), and (UI).*
- *I_B satisfies (WM), (UI), and (C–$\langle x, y, T\rangle$) but fails to satisfy (I) and (M).*
- *I_{FGL} satisfies (I), (WM), and (UI) but fails to satisfy (M), (C–$\langle x, y\rangle$), and (C–T).*
- *I_A satisfies (I) and (M) but fails to satisfy (US),[21] (C–$\langle x, y\rangle$), and (C–T).*
- *I_{WA} satisfies (I), (M), and (US) but fails to satisfy (UI), (C–$\langle x, y\rangle$), and (C–T).*

If we evaluate inefficiency indexes solely in terms of our axioms, then we see that two indexes should be excluded from further consideration since they are dominated by other indexes. The directional distance index I_{DD} is dominated by both the hyperbolic index I_H and the Briec index I_B, while the index additive I_A is dominated by the weighted additive index I_{WA}.

The axioms do not discriminate between the hyperbolic index and the Briec index, since both satisfy weak monotonicity, unit independence, and joint continuity in

[21] And, of course, (UI).

the production vector and the technology while failing to satisfy indication and monotonicity. The Färe-Grosskopf-Lovell index I_{FGL} and the weighted additive indexes are distinguished from the hyperbolic and Briec indexes by satisfying indication but failing to satisfy continuity in either the production vector or the technology. The Färe-Grosskopf-Lovell and weighted additive indexes are distinguished from each other by the Färe-Grosskopf-Lovell index satisfying unit independence and failing monotonicity and the weighted additive index satisfying monotonicity and unit scalability but failing to satisfy unit independence.

7. Discussion of Additional Indexes.

As noted in the closing paragraph of Section 3, a number of additional inefficiency indexes formulated in the literature are not included in the statement of Theorem 3. This theorem does, however, have implications, discussed briefly in this section, for these alternative indexes.[22]

To facilitate comparison of these indexes, we express them in terms of the sets, $\Gamma(x, y, T)$ and $\Omega(x, y, T)$, defined by (3.9) and (3.6). The set $\Gamma(x, y, T)$ is the set of feasible *additive* slacks while $\Omega(x, y, T)$ is the set of feasible *proportional* slacks. These are alternative but equivalent representations of the difference between an initial production vector and its reference vector. Their relation can be expressed as follows:

$$s_i = (1 - \alpha_i)x_i, \; i = 1, \ldots, n, \tag{7.1}$$

and

$$t_i = (1 - \beta_j)y_j/\beta_j, \; j = 1, \ldots, m. \tag{7.2}$$

Although the additive and weighted additive indexes were initially proposed for DEA technologies, they can be defined—as we have done above—for the general class of technologies $\mathcal{T}$ without modification. Some indexes, however, depend critically on the structure of the DEA framework. An example is the Range Adjusted Measure (RAM) of Inefficiency, a modification by Cooper, Park, and Pastor [1999] of the additive index to achieve unit independence. It depends explicitly on the quantity data for K "decision making units": $\{\, x_{ik} \,\}_{k=1}^{K}$, $i = 1, \ldots, n$, and $\{\, y_{jk} \,\}_{k=1}^{K}$, $j = 1, \ldots, m$. The RAM index is defined by

$$I_{RAM}(x, y, T) = \max_{s,t} \left\{ \sum_i \frac{s_i}{\Delta_i^x} + \sum_j \frac{t_j}{\Delta_j^y} \;\middle|\; \langle s, t \rangle \in \Gamma(x, y, T) \right\}, \tag{7.3}$$

where

$$\Delta_i^x = \max_k \{\, x_{ik} \,\}_{k=1}^{K} - \min_k \{\, x_{ik} \,\}_{k=1}^{K}, \quad i = 1, \ldots, n, \tag{7.4}$$

[22] We apologize for not acknowledging efficiency-index formulations that have escaped our attention.

and

$$\Delta_j^y = \max_k \{ y_{jk} \}_{k=1}^K - \min_k \{ y_{jk} \}_{k=1}^K, \quad j = 1, \ldots, m. \tag{7.5}$$

For this index to be well defined, the set of possible production vectors must be bounded so that Δ_i^x for $i = 1, \ldots, n$ and Δ_j^y for $j = 1, \ldots, m$ are defined. This cannot be achieved for the general class of technologies $\mathcal{T}$.

Some indexes have been excluded from Theorem 3 because they are dominated by other indexes in terms of our axiomatic structure. An interesting example is a generalized hyperbolic measure, introduced in in Section 5.7 of Färe, Grosskopf, and Lovell [1985]:

$$I_{GH}(x, y, T) = \max_{\langle \lambda_x, \lambda_y \rangle} \{ \lambda_x + \lambda_y \mid \langle x/\lambda_x, \lambda_y y \rangle \in T \}. \tag{7.6}$$

This index is based on two parameters, one to contract x and one to expand y. It combines some of the properties of (a) the hyperbolic index and the Briec index, each based on a single parameter, and (b) the FGL, additive, and weighted additive indexes, each based on $n + m$ parameters to achieve coordinate-wise contractions of x and coordinate-wise expansions of y. This index fails to satisfy either indication or continuity in either the production vector or the technology and fails to satisfy monotonicity (but is weakly monotonic and unit independent); it, therefore, is dominated by all of the unit-independent indexes considered above.[23]

Other proposed indexes can be shown to be equivalent to those we have analyzed above or are equivalent in terms of our axiomatic system. Several indexes have been introduced as modifications of the additive index with the objective of achieving unit independence. In the original paper introducing the additive index (CCGSS [1985]), the authors propose a unit-invariant modification of the additive index:

$$I_{AUI}(x, y, T) = \max_{s,t} \left\{ \sum_i \frac{s_i}{x_i} + \sum_j \frac{t_j}{y_j} \;\middle|\; \langle s, t \rangle \in \Gamma(x, y, T) \right\}. \tag{7.7}$$

Substituting for $s_i,\ i = 1, \ldots, n$, from (7.1) and for $t_j,\ j = 1, \ldots, m$, from (7.2) we arrive at

$$\begin{aligned} I_{AUI}(x, y, T) &= \max \left\{ n - \sum_i \alpha_i + \sum_j \frac{1}{\beta_j} - m \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x, y, T) \right\} \\ &= n - m - \min_{\alpha,\beta} \left\{ \sum_i \alpha_i - \sum_j \frac{1}{\beta_j} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x, y, T) \right\}. \end{aligned} \tag{7.8}$$

[23] Proofs of violation of identification and monotonicity are easily adapted from the corresponding proofs of these properties by the hyperbolic index; proof of violation of continuity are easily adapted from the proof of the same for the FGL index.

This reformulated AUI index, featuring a vector of coordinate-wise input contractions and output expansions, is similar to the FGL index and has the same axiomatic properties.[24]

Pastor, Ruiz, and Sirvent [1999] define an index for DEA technologies and treat it as a modification of the FGL index. Their formulation is[25]

$$I_{PRS}(x,y,T) = \min_{\alpha,\beta} \left\{ \frac{\frac{1}{m}\sum_j \frac{1}{\beta_j}}{\frac{1}{n}\sum_i \alpha_i} \;\middle|\; \langle \alpha, \beta \rangle \in \Omega(x,y,T) \right\}. \tag{7.9}$$

This index is also similar to the FGL index in that it begins with the vector of coordinate-wise input contractions and output expansions, but the PRS index is a ratio of the average output expansions and the average input contractions. The axiomatic properties of this measure are identical to those of the FGL index.[26]

The Measure of Inefficiency Proportions, as described in Cooper, Park, and Pastor [1999], is explicitly identical to I_{AUI} and hence does not need separate treatment. The Measure of Efficiency Proportions, proposed by Banker and Cooper [1994] for DEA technologies, can be defined for the class of technologies $\mathcal{T}$ by

$$I_{MEP}(x,y,T) = \max_{s,t} \left\{ 1 - \frac{1}{n+m} \left[\sum_i \frac{s_i}{x_i} + \sum_j \frac{t_j}{y_j + t_j} \right] \;\middle|\; \langle s, t \rangle \in \Gamma(x,y,T) \right\}. \tag{7.10}$$

Substitution for $s_i,\ i = 1,\ldots,n$, from (7.1) and for $t_j\ j = 1,\ldots,m$, from (7.2) and some algebraic manipulation indicates that this representation is equivalent to the I_{FGL} index.

Tone [2001] introduced a slacks-based measure (SBM) of efficiency in an additive DEA model. This index can be extended to an efficiency index defined for all technologies as follows:

$$E_{SBM}(x,y,T) = \min_{s,t} \left\{ \frac{1 - \frac{1}{n}\sum_i \frac{s_i}{x_i}}{1 + \frac{1}{m}\sum_j \frac{t_j}{y_j}} \;\middle|\; \langle s, t \rangle \in \Gamma(x,y,T) \right\}. \tag{7.11}$$

Again, a simple rearrangement of terms shows that $1 - E_{SBM}(x,y,T)$ is equivalent to the I_{PRS} index and hence does not need independent treatment.

8. Conclusion.

The results in Sections 5–7 are summarized in Figure 8. Two of the branchings, those labeled Theorem 1 and Theorem 2, reflect dilemmas posed by incompatibility of

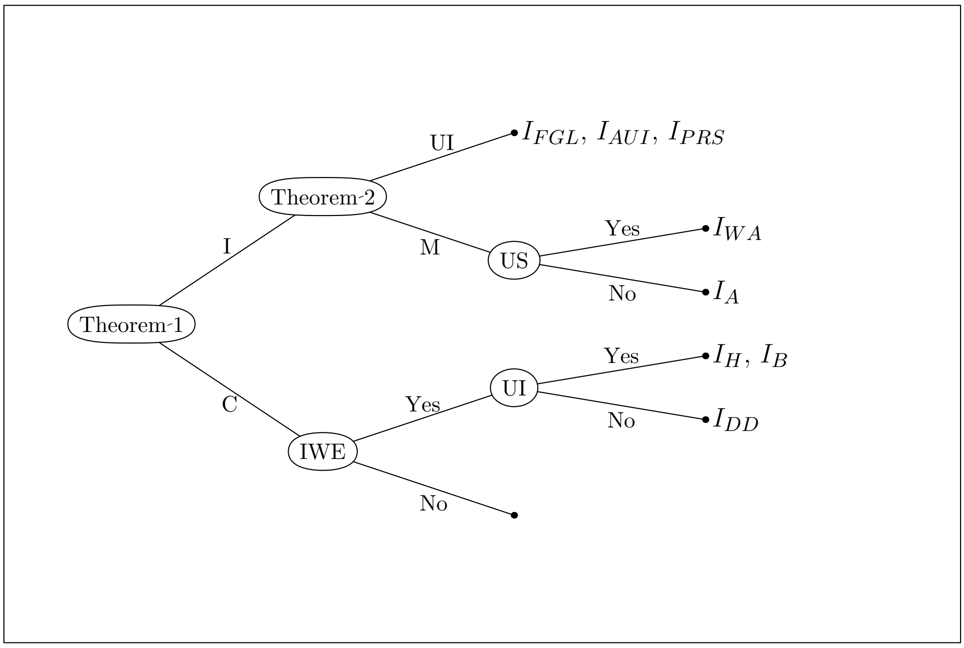
[24] This can be seen by following the same arguments as in the proofs of the properties of I_{FGL} in the Appendix.

[25] Pastor, Ruiz, and Sirvent define an efficiency index, which we invert to obtain an inefficiency index. Also, we use a different but equivalent parameterization.

[26] Again, this can be seen by following the same arguments as in the proofs of the properties of I_{FGL} in the Appendix.

axioms. The first branching (Theorem 1) partitions indexes that satisfy indication from those that satisfy continuity in either quantities or technologies. Theorem 2 partitions the indexes that satisfy indication into those that satisfy monotonicity (the additive and the weighted additive indexes) and those that satisfy unit independence (the FGL index and similar indexes). The final branching on the upper tree at the US node reflects the dominance of the weighted additive index over the (unweighted) additive index owing to the satisfaction of unit scalability by the former. The branching at the IWE node of the lower tree reflects a choice between a property called indication of weak efficiency (IWE) and monotonicity.[27] The indexes satisfying IWE are partitioned into those that satisfy unit independence (the hyperbolic and Briec indexes) and the one that does not (the directional distance index). (The RAM index and the generalized hyperbolic index are omitted from the diagram because the former cannot be adapted to the general set of technologies $\mathcal{T}$ and the latter satisfies neither indication nor continuity.) Each of the nodes in this schematic reflects trade-offs among the indexes in terms the axiomatic structure outlined in Section 4.

Figure 8: Choices Among Inefficiency Indexes.



We therefore reach the following conclusions:

- The directional distance inefficiency index is dominated by the hyperbolic and Briec inefficiency indexes (which satisfy a stronger unit invariance property).
- Owing to the incompatibility of the indication and continuity axioms and the in-

[27] See Russell [1987] for a definition of indication of weak efficiency. All inefficiency indexes of which we are aware satisfy this property so we have not emphasized it here.

compatibility of the monotonicity and unit independence axioms, trade-offs remain among the other four indexes:

(i) If identification and unit independence are essential, choose either the FGL or the PRS inefficiency index (or one of the alternatives that is axiomatically equivalent).

(i) If identification and monotonicity are essential, choose the weighted additive index.

(i) If continuity and unit independence are essential, choose the hyperbolic or Briec inefficiency index.

Appendix: Proof of Theorem 3.

While numerous results on the properties of various indexes have been reported in the literature, few have pertained to general technologies. Many of the proofs, moreover, exploit the special properties of restricted technologies, especially those in the DEA class. We therefore present proofs of each of the claims in Theorem 3, even in some cases where similar results have been reported in the literature. Our de novo approach has the added advantage of presenting proofs that apply to several indexes rather than needlessly exploiting the special properties of each index.

The proof is divided into four parts: (i) indication, (ii) monotonicity, (iii) unit independence, and (iv) continuity.

i. Indication.

It is clear that the only candidates for θ in the definition of (I) are the minimal values in the ranges of the efficiency index mappings: 0 for I_{DD}, I_B, I_{FGL}, I_A, and I_{WA} and 1 for I_H.

That I_H, I_{DD}, and I_B fail to satisfy (I) is obvious from Figures 1, 2, and 3, respectively.[28] In each case, the production vector $\langle \bar{x}'', \bar{y}'' \rangle$ is inefficient but $I(\bar{x}'', \bar{y}'', T) = \theta$.

To show that I_{FGL} satisfies indication, first suppose that $\langle x, y \rangle \in T$, with $\langle x, y \rangle \gg 0$, is not efficient so that there exists a production vector $\langle x', y' \rangle \in T$ satisfying $\langle x', y' \rangle \gg 0$ and $\langle x', -y' \rangle < \langle x', -y' \rangle$. Then their exists an α and a β satisfying $0 < \alpha \leq 1^{[n]}$ and $0 < \beta \leq 1^{[m]}$ and either $\alpha < 1^{[n]}$ or $\beta < 1^{[m]}$ such that $x' = \alpha \otimes x$ and $y' = y \oslash \beta$. Since $\langle \alpha, \beta \rangle \in \Omega(x, y, T)$, we have

$$I_{FGL}(x, y, T) \geq 1 - \frac{\sum_i \alpha_i + \sum_j \beta_j}{n + m} > 0. \tag{8.1}$$

Next suppose that $I_{FGL}(x, y, T) > 0$. Then either $0 < \alpha < 1^{[n]}$ or $0 < \beta < 1^{[m]}$, so that there exists a point $\langle x', y' \rangle \in T$, with $x' = \alpha \otimes x$ and $y' = y \oslash \beta$, satisfying $\langle x', y' \rangle \in T$, $\langle x', y' \rangle \gg 0$, and $\langle x', -y' \rangle < \langle x', -y' \rangle$. Therefore, (x, y) is inefficient, and I_{FGL} satisfies indication.

We next sketch the analogous proof for satisfaction of (I) by I_{WA}; satisfaction of this axiom by I_A follows as a special case with $u = 1^{[n]}$ and $v = 1^{[m]}$. If $\langle x, y \rangle$ is inefficient, there exists a $\langle s, t \rangle \in \Gamma(x, y, T)$ such that either $0^{[n]} < s$ or $0^{[n]} < t$, which implies that $I_{WA}(x, y, T) > 0$. If $I_{WA}(x, y, T) > 0$, there exists an $\langle s, t \rangle \in \Gamma(x, y, T)$ satisfying either $s > 0^{[n]}$ or $t > 0^{[n]}$, in which case $\langle x, y \rangle$ is inefficient.

[28] And follows from Theorem 1 and the proofs of continuity of these indexes below.

ii. Monotonicity.

Failure of I_H, I_{DD}, and I_B to satisfy (M) is obvious from Figures 1, 2, and 3. In each figure, starting at $\langle x'', y'' \rangle$ and increasing x while holding y at y'' must leave the respective index unchanged, since $\bar{y}''$ (as well as y'') is unchanged.[29]

To establish (WM) for I_H, note that $\langle x/\lambda, \lambda y \rangle \in T$ and $\langle x, -y \rangle > \langle x', -y' \rangle$ implies, using free disposability, that $\langle x'/\lambda, -\lambda y' \rangle \in T$; hence $I_H(x, y, T) \leq I_H(x', y', T)$. Proofs of (WM) for I_{DD} and I_B are analogous.

Figure 7 provides an example showing that I_{FGL} is not monotonic. For the technology with level set $L(y)$ and the production vectors (x, y) and (x', y) with $x < x'$, a simple calculation shows that $I_{FGL}(x, y, T) = I_{FGL}(x', y', T) = (1 + m)/(2 + m)$ (assuming y is efficient in output space).

To prove that I_{FGL} satisfies weak monotonicity, let $\langle x, -y \rangle < \langle x', -y' \rangle$ and note that $\Omega(x, y, T) \subset \Omega(x', y', T)$. The definition of E_{FGL} in (3.5) implies that $E_{FGL}(x', y', T) \leq E_{FGL}(x, y, T)$ so that $I_{FGL}(x, y, T) \leq I_{FGL}(x', y', T)$

To prove that I_{WA} (and I_A) satisfies monotonicity, let $\langle x, -y \rangle < \langle x', -y' \rangle$. Let $\langle s, t \rangle$ be the solution to (3.10) for $\langle x, y \rangle$ so that $I_{WA}(x, y, T) = \sum_i s_i + \sum_j t_j$. Define $\langle \bar{x}, \bar{y} \rangle = \langle x, y \rangle + \langle s, t \rangle$ so that $\langle \bar{x}, \bar{y} \rangle \in T$. Define $\langle s', t' \rangle = \langle x', y' \rangle - \langle \bar{x}, \bar{y} \rangle$ and note that $\langle s', t' \rangle \in \Gamma(x', y', T)$ and $\langle s', t' \rangle > \langle s, t \rangle$. Therefore,

$$I_{WA}(x', y', T) \geq \sum_i s'_i + \sum_j t'_j > \sum_i s_i + \sum_j t_j = I_{WA}(x, y, T), \tag{8.2}$$

which proves the result.

iii. Unit independence.

Proofs that (UI) is satisfied are straightforward and nearly identical for I_H, I_B, and I_{FGL}. We therefore write out a formal proof only for I_H. Given a transformation of the units of x and y, we have

$$\begin{aligned} I_H(\hat{x}, \hat{y}, \widehat{T}) &= \max \left\{ \lambda \mid \langle \hat{x}/\lambda, \lambda y \rangle \in \widehat{T} \right\} \\ &= \max \left\{ \lambda \mid \left\langle W_x x/\lambda, \ \lambda W_y y \right\rangle \in \widehat{T} \right\} \\ &= \max \left\{ \lambda \mid \left\langle W_x^{-1}(W_x x)/\lambda, \ \lambda W_y^{-1}(W_y y) \right\rangle \in T \right\} \\ &= I_H(x, y, T). \end{aligned} \tag{8.3}$$

It is straightforward to demonstrate that I_{DD}, I_A and I_{WA} violate (UI).[30] With I_{DD}, the direction g is defined independently of the units of measurement so that a change of

[29] Failure of I_H and I_B to satisfy (M) also follows from Theorem 2 and the proofs of (UI) for these indexes below.

[30] Salnykov and Zelenyuk [2005] prove that I_{DD} violates (UI).

units effectively changes the direction. In I_A and I_{WA}, the definition of slack variables independently of the units of measurement ensures the violation of (UI).[31]

As I_A contains no parameters to adjust for unit changes, it fails to satisfy (US) as well. Both I_{DD} and I_{WA}, however, have sufficient parameters to satisfy (US). Balk [1998] has proved this for the I_{DD}; we provide a proof for I_{WA}. Consider a transformation of the units, $\hat{x} = W_x x$ and $\hat{y} = W_y y$ and define the parameter transformation

$$\langle \hat{u}, \hat{v} \rangle = \langle W_x^{-1} u, W_y^{-1} v \rangle. \tag{8.4}$$

Then

$$\begin{aligned}
I_{WA}\Big(\hat{x}, \hat{y}, \widehat{T}, \langle \hat{u}, \hat{v} \rangle \Big) &= \max_{s,t} \Big\{ \hat{u} \cdot s + \hat{v} \cdot t \mid \langle \hat{x} - s, \hat{y} + t \rangle \in \widehat{T} \Big\} \\
&= \max_{s,t} \big\{ W_x^{-1} u \cdot s + W_y^{-1} v \cdot t \mid \big\langle W_x^{-1}(\hat{x} - s),\ W_y^{-1}(\hat{y} + t) \big\rangle \in T \big\} \\
&= \max_{s,t} \big\{ u \cdot W_x^{-1} s + v \cdot W_y^{-1} t \mid \big\langle W_x^{-1}\hat{x} - W_x^{-1} s,\ W_y^{-1}\hat{y} + W_y^{-1} t \big\rangle \in T \big\} \qquad (8.5) \\
&=: \max_{s',y'} \big\{ u \cdot s' + v \cdot t' \mid \langle x - s', y + t' \rangle \in T \big\} \\
&= I_{WA}\big(x, y, T, \langle u, v \rangle\big).
\end{aligned}$$

iv. Continuity.

The failure of I_{FGL}, I_A, and I_{WA} to satisfy continuity in either quantities or technologies is demonstrated by the examples in Figures 5 and 6.[32]

To prove joint continuity of I_H, consider a sequence $\{x^\nu, y^\nu, T^\nu\}$ that converges to $\{x^o, y^o, T^o\}$. To simplify notation, let $\lambda^\nu = I_H(x^\nu, y^\nu, T^\nu)$, $\lambda^o = I_H(x^o, y^o T^o)$, $w^\nu = \langle x^\nu/\lambda^\nu, \lambda^\nu y^\nu \rangle$, and $w^o = \langle x^o/\lambda^o, \lambda^o y^o \rangle$. Obviously, λ^ν is bounded from below (at 1). Given $x^\nu \to x^o$ and $T^\nu \to T^o$, it follows that $P(x^\nu) \to P(x^o)$. By assumption, $P(x^o)$ is bounded; hence, for an arbitrary $\delta > 0$, there exists a ν' such that $P(x^\nu)$ is a subset of the cube $\{y \in \mathbf{R}^n_+ \mid y_k \leq \delta \ \forall\ k\}$ for all $\nu > \nu'$. Moreover, for arbitrary ϵ, there exists a ν'' such that $y^\nu \in N_\epsilon(y^o)$ for all $\nu > \nu''$. Consequently, $\lambda^\nu y^\nu$ is bounded for all $\nu > \max\{\nu', \nu''\}$

The strategy is to show that, for arbitrary ϵ, there exists a $\hat{\nu}$ such that

$$\lambda^\nu < \lambda^o + \epsilon \quad \forall\ \nu > \hat{\nu} \tag{8.6}$$

and

$$\lambda^\nu > \lambda^o - \epsilon \quad \forall\ \nu > \hat{\nu}. \tag{8.7}$$

31 In any event, these facts follow from Theorem 2 and the satisfaction of (M) by these two indexes.

32 Failure of I_A and I_{WA} to satisfy continuity also follows from Theorem 1 and satisfaction of (I) by these indexes.

To prove (8.6), suppose that $\lambda^\nu \geq \lambda^o + \epsilon$ for infinitely many ν and some $\epsilon > 0$. As λ^ν is bounded for sufficiently large ν, this sequence has a convergent subsequence: $\lambda^{\nu_k} \to \bar{\lambda} > \lambda^o$. This in turn implies that $w^{\nu_k} \to \bar{w} = \langle x^o/\bar{\lambda}, \bar{\lambda} y^o \rangle$. Since $w^{\nu_k} \in T^{\nu_k}$ for all ν_k and $T^\nu \to T^o$, we have $\bar{w} \in T^o$. Along with the definition of λ^o, this implies that $\bar{\lambda} \leq \lambda^o$, a contradiction.

To prove (8.7), suppose that $\lambda^\nu < \lambda^o + \epsilon$ for infinitely many ν and some $\epsilon > 0$. As λ^ν is bounded for sufficiently large ν, this sequence has a convergent subsequence: $\lambda^{\nu_k} \to \bar{\lambda} < \lambda^o - \epsilon$. As $T^\nu \to T^o$, this in turn implies that $u^{\nu_k} := \langle x^{\nu_k}/\lambda^{\nu_k}, \lambda^{\nu_k} y^{\nu_k} \rangle$, a boundary point in T^{ν_k}, converges to $\bar{u} = \langle \bar{x}, \bar{y} \rangle = \langle x^o/\bar{\lambda}, \bar{\lambda} y^o \rangle$, a boundary point in T^o. Moreover, $\bar{\lambda} < \lambda^o$ implies $\bar{x} > x^o$ and $\bar{y} < y^o$. Since $\bar{u}$ is a boundary point of T^o, this violates the free disposability assumption.

Proofs of joint continuity of I_{DD} and I_B are virtually identical to the proof of joint continuity of I_H and hence are left to the reader. ■

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