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STABILITY IN A THREE-SECTOR DYNAMIC GROWTH MODEL WITH ENDOGENOUS LABOR SUPPLY

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Abstract

Abstract. This paper explores the stability of the stationary state for a dynamic growth model with wealth and human capital accumulation. Knowledge is created through research and learning-by-doing, while the time allocation between labor and leisure is endogenized. We analyze the model in both its deterministic and stochastic versions. First, we describe the deterministic model and analyze the stationary state. Second, using the stationary state, we define the stochastic perturbation and study the mean and squared mean values of the system states for the linearized model. Third, we prove that for certain parameters, the stationary state is asymptotically stable both in the deterministic and the stochastic model. Finally, we perform the comparative dynamic analysis for the propensities to save and to enjoy leisure, the tax rates used to finance research, and the knowledge utilization efficiency.

Key words: labor, human capital, capital accumulation, economic growth.

JEL classification: C62, E22, O41

1 Introduction

The economic literature has long noted the importance of time allocation between labor and leisure (Ladrón-de-Guevara et al., 1997). Unfortunately, even a simple model of growth with labor dynamics tends to lead to analytically intractable dynamic systems (Barro and Sala-I-Martin, 1995; Gong et al., 2004; Blankenau and Simpson, 2004). These technical issues have long prevented the economic literature from exploring more complex models featuring

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endogenous labour. For example, Zhang (2009) proposed a rich three-sector growth model with endogenous labor and knowledge. However, due to the model's complexity, the dynamic properties of the system were examined only numerically. For certain initial values of the stock of capital and knowledge, the author also found the stationary state (equilibrium) to be unique and stable. In this paper, we look at a similar class of growth models with multiple sectors and endogenous labour. Our aim is to provide a more rigorous characterization of the dynamic properties of such models. Once established, the same methodology could be easily implemented in finance, demography and social sciences in general, allowing literature advances in these fields.

Up to our best knowledge, this is the first analysis that employs complex mathematical methods to study the stability of multi-sector dynamic economic systems to its full extent. The contribution of this paper is threefold. First, in an endogenous capital and knowledge dynamic growth model à la Zhang, we prove that the stationary state is always stable. In other words, the equilibrium is unique and stable for any initial values of the capital and knowledge stock. Second, we extend the analysis of labor supply dynamics in the economic growth theory with endogenous knowledge (Jones et al., 1993, Novales and Ruiz, 2002, Zhang, 2005), by developing the stochastic version of this model. The stochastic model uses the stationary state of the deterministic system and, based on this deterministic system, it considers a stochastic perturbation. We also provide a complete analysis of the dynamic properties of the stochastic model using the method in Lei and Mackey (2007), i.e., the mean and quadratic mean values of the state variables for the stochastic model liniarization. As for the deterministic model, we show the equilibrium to be stable, regardless of the initial levels of capital of knowledge. Third, we perform a comparative dynamic analysis for certain parameters of the stochastic model and derive the economic implications for the stock of capital and knowledge, as well as for labor and savings.

The paper is organized as follows. The remaining of Section 1 presents the relation to the economic growth literature. Section 2 describes the three-sector growth model with endogenous labor supply, savings and consumption, and examines the dynamic properties of the deterministic model. Section 3 defines the stochastic perturbation of the deterministic model and analyzes the squared mean values of the state variables. In section 4 we show the numerical simulations and prove that the stationary states of the deterministic and stochastic models are asymptotically stable. We also present results on the comparative dynamic analysis for the propensities to use leisure time and save, the tax rates that finances the research, as well as for the knowledge utilization efficiency rates. Section 5 concludes.

1.1 Relation to Literature

Our paper contributes to the broader literature on growth with endogenous capital and knowledge. The model builds on the main features of the Solow (1957), Arrow (1962), Uzawa (1965), and Uzawa-Lucas (1988) models. In 1957, Solow developed the neoclassical model of growth based on the assumption of diminishing returns of factors of production. Due to this assumption, the model predicted that per capita income would stabilize at the steady state, and sustainable economic growth would be impossible. This was implausible and further research that considered both physical and human capital emerged. First, Arrow (1962) introduced knowledge accumulation through learning-by-doing. His model assumed that an increase in capital can boost the stock of knowledge (public good in the economy) and enable a firm to produce more efficiently. The same positive effects for the production sector productivity were predicted by the Uzawa (1965) model, which formally introduced the knowledge sector that uses labor and the existing stock of knowledge to produce new knowledge.

Based on these models, Romer (1986) and Lucas (1988) laid the foundations of the endogenous growth theory. Romer (1986) used Arrow (1962) to develop a model with learning-by-doing and knowledge spillover. Lucas (1988) built on Romer (1986) and Uzawa (1965) and developed a two-sectors model with human capital defined to include both workers' skills and abstract knowledge. There is a final good sector, where the combination of physical and human capital results in a final good, which can be consumed or invested (changed into capital). Also, there is a sector of production and accumulation of capital, with production of physical and human capital occurring differently. Importantly, this model assumed non-diminishing returns in the production of human capital, because growth was driven by this sector. This unrealistic assumption led to include technological progress (Romer, 1987, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992, 1998).

Technological progress occurs through knowledge created as a result of R&D activity, and governments can significantly affect economic growth by advancing the knowledge through taxation, providing infrastructures, etc. Technology could make a combination of production factors more effective, and therefore the same amount of labor and capital could increase output. Alternatively, the same amount of output could be obtained with less labor or physical capital. Indeed, recent work on the decline of hours worked point to one underlying driving force, namely technological progress. Greenwood and Vandenbroucke (2005), and Aguiar and Hurst (2007) confirm that better technology is reflected in higher wage rate and more abundant goods consumption. This enhances the marginal value of leisure and induces more consumption of leisure, in spite of its higher price.

2 The deterministic three-sector model

To fix notation, we first present the three-sector growth model with endogenous time allocation between labor and leisure à la Zhang. Second, we define the stationary state and obtain the characteristic equation needed to perform the stability analysis.

The economic system consists of three sectors: capital goods, consumption goods and research goods. Like in the Uzawa-Lucas two-sector growth model, we assume that consumption and capital goods are different commodities, produced in two different production sectors. There is only one capital good, which depreciates at a constant exponential rate δ_1, independent of the manner of use. Labor is considered constant and homogeneous. Capital and labor are smoothly substitutable for each other in each sector and are freely transferable from one sector to another.

The output in the two production sectors is obtained using knowledge, a single type of capital good and a single grade of labor. Knowledge growth occurs through learning-by-doing (in the production sectors) and through R&D activities (by the university). For the production of research goods, the university is financially supported through public funds raised by the government, by taxing the two production sectors.

The population consists of workers (agents working in the capital and consumption goods sectors) and scientists (agents working in the university). We denote by $j \in \{1, 2, 3\}$ the capital, consumption and research goods sectors respectively.

The output level of the productive sectors each point of time t is given by the following production function:

$$F_j(t) = A_j Z(t)^{m_j(t)} K_j(t)^{\alpha_j} N_j(t)^{\beta_j}, \text{ for } j = \{1, 2\} \text{ and } \alpha_j, \beta_j > 0, \ \alpha_j + \beta_j = 1,$$

where $A_j > 0$ is the labor augmenting technical progress, $Z(t) > 0$ captures the total knowledge stock at time t and m_j represents a measure of sector j's knowledge utilization efficiency.

The capital stock in the economy at time t is denoted by $K(t)$. The terms $K_j(t)$ and $N_j(t)$ are the capital stock and total labor time in sector j, $j \in \{1, 2, 3\}$. Markets are competitive (i.e., firms earn zero profits), and capital and labor earn their marginal products (interest rate $r(t)$, and wage rate $w(t)$, respectively). For an individual firm, $r(t)$ and $w(t)$ are given at each point of time.

The production sector j chooses $K_j(t)$ and $N_j(t)$ to maximize the profit. Optimality in the productive sectors requires that

$$r(t) + \delta_1 = \tau_{11} A_1(t) \alpha_1 Z(t)^{m_1} k_1(t)^{-\beta_1} = \tau_{22} A_2(t) \alpha_2 p(t) Z(t)^{m_2} k_2(t)^{-\beta_2}$$

and

$$w(t) = \tau_{11} A_1(t) \beta_1 Z(t)^{m_1} k_1(t)^{\alpha_1} = \tau_{22} A_2(t) \beta_2 p(t) Z(t)^{m_2} k_2(t)^{\alpha_2},$$

where

$$k_1(t) = \frac{K_1(t)}{N_1(t)}, \ k_2(t) = \frac{K_2(t)}{N_2(t)}, \ \tau_{11} = 1 - \tau_1, \ \tau_{22} = 1 - \tau_2$$

and τ_j is the tax rate on the productive sector j, $j \in \{1, 2\}$.

Using

$$f_j(t) = F_j(t)/N_j(t) = A_j Z(t)^{m_j} k_j(t)^{\alpha_j}, \ \ for \ j \in \{1, 2\},$$

the above expressions become

$$r(t) + \delta_1 = \tau_{11}\alpha_1 f_1(t)/k_1(t) = \tau_{22}\alpha_2 p(t) f_2(t)/k_2(t),$$

$$w(t) = \tau_{11}\beta_1 f_1(t) = \tau_{22}\beta_2 p(t) f_2(t). \tag{1}$$

On the other hand, households choose consumption and leisure time, as well as the amount of savings. There are several ways to finance consumption. First, households can use their per capita current income $y(t)$, from the interest payments $r(t)k(t)$ and the wage payments $w(t)T(t)$, as follows

$$y(t) = r(t)k(t) + w(t)T(t),$$

where $k(t)$ represents the wealth owned per capita and $T(t)$ is the total work time. Second, households can decumulate their wealth, up to the maximum level of wealth owned $k(t)$. Note that this implies a no-borrowing constraint, i.e., agents cannot borrow to finance their current consumption. Therefore, per capita disposable income of the household is defined as the sum of the current income and the wealth available for financing consumption $c(t)$ and savings $s(t)$,

$$y_1(t) = y(t) + k(t) = (1 + r(t))k(t) + w(t)T(t). \tag{2}$$

Therefore, the budget constraint is given by

$$y_1(t) = p(t)c(t) + s(t), \tag{3}$$

where $p(t)$ represents the price of consumption goods.

The time constraint is expressed by

$$T(t) + T_3(t) = T_0,$$

where $T_3(t)$ represents the leisure time and T_0 is the total available time (for work and leisure). Substituting this function into equations (2) and (3), we get

$$w(t)T_3(t) + p(t)c(t) + s(t) = y_2(t) = (1 + r(t))k(t) + w(t)T_0. \tag{4}$$

We assume that utility $U(t)$ depends on consumption $c(t)$, leisure time $T_3(t)$ and savings $s(t)$, as follows:

$$U(t) = c(t)^{\zeta_0} T_3(t)^{\sigma_0} s(t)^{\lambda_0}, \text{ with } \zeta_0, \sigma_0, \lambda_0 > 0. \tag{5}$$

where ζ_0, σ_0 and λ_0 represent the propensities to consume, enjoy leisure and own wealth respectively. For consumers, the wage rate $w(t)$ and the rate of interest $r(t)$ are given, while wealth $k(t)$ is given at the beginning of the period, before the decision occurs.

Maximizing $U(t)$ in (5) subject to the budget constraint yields

$$\begin{aligned} p(t)c(t) &= \zeta y_2(t), \\ w(t)T_3(t) &= \sigma y_2(t), \\ s(t) &= \lambda y_2(t), \end{aligned} \tag{6}$$

where

$$\zeta = \rho\zeta_0, \quad \sigma = \rho\sigma_0, \quad \lambda = \rho\lambda_0, \quad \rho = 1/(\sigma_0 + \zeta_0 + \lambda_0).$$

To obtain the capital accumulation equation, note that each period, the change in the household's wealth is given by

$$dk(t)/dt = s(t) - k(t). \tag{7}$$

Denoting by N_0 the total household population, the output of the consumption goods sector is consumed according to

$$c(t)N_0 = F_2(t), \tag{8}$$

while the output of the capital goods sector is equal to the sum of net savings and capital stock depreciation,

$$[S(t) - K(t)] + \delta_1 K(t) = F_1(t), \tag{9}$$

where $S(t) = s(t)N_0$.

Following Arrow's learning-by-doing model and Uzawa-Lucas's education model, we assume the knowledge is produced in the productive sectors, as well as in the university sector. Total knowledge therefore evolves according to:

$$dZ(t)/dt = \tau_{01}F_1(t)/\left(Z(t)^{\varepsilon_1}\right) + \tau_{02}F_2(t)/\left(Z(t)^{\varepsilon_2}\right) + \tau_{03}Z(t)^{\varepsilon_3}K_3(t)^{\alpha_3}N_3(t)^{\beta_3} - \delta_3 Z(t), \tag{10}$$

in which δ_3 is the depreciation rate of knowledge, and $\tau_{0j}, \varepsilon_j, \alpha_3$ and β_3 are non-negative ($j \in \{1, 2, 3\}$). The parameter ε_3 can be either positive (an increasing knowledge stock may imply the university using more effectively the existing research to create new knowledge) or negative (an increasing knowledge stock may make discovery of new knowledge difficult). The first two terms in equation (10) represent the contribution to knowledge through learning-by-doing in the capital and consumption sectors, respectively. The last two terms capture the contribution to knowledge creation through research, net of the knowledge stock depreciation.

Finally, the research carried by the university is funded by the government through taxation. The tax is levied on the value of the two productive sectors output,

$$Y_3(t) = \tau_1 F_1(t) + \tau_2 p(t) F_2(t),$$

and used by the university to pay the interest $(r(t) + \delta_1)K_3(t)$ for the capital employed and the wage $w(t)N_3(t)$ for scientists,

$$(r(t) + \delta_1)K_3(t) + w(t)N_3(t) = Y_3(t). \tag{11}$$

Conditional on obtaining the research funds $Y_3(t)$ the university chooses the amount of equipment $K_3(t)$ and the number of scientists $N_3(t)$ that maximize the output (i.e., the contribution to knowledge growth):

$$\begin{aligned}&Max \quad \tau_{03}Z(t)^{\varepsilon_3}K_3(t)^{\alpha_3}N_3(t)^{\beta_3},\\&s.t.\ Y_3(t) = (r(t)+\delta_1)K_3(t) + w(t)N_3(t).\end{aligned}$$

Optimality in the research sector requires that

$$\begin{aligned}K_3(t) &= \alpha_4 Y_3(t)/(r(t)+\delta_1),\\N_3(t) &= \beta_4 Y_3(t)/w(t),\end{aligned}$$

where

$$\alpha_4 = \alpha_3/(\alpha_3+\beta_3),\ \ \beta_4 = \beta_3/(\alpha_3+\beta_3).$$

As expected, note that the amount of capital and labor university employs into research increases with tax income, and therefore with the output of the two productive sectors, and decreases with the price of the two production factors.

As full employment of capital and labor is assumed, we have

$$\begin{aligned}&K_1(t)+K_2(t)+K_3(t) = K(t) = k(t)N_0,\\&N_1(t)+N_2(t)+N_3(t) = T(t)N_0.\end{aligned}$$

The dynamics of the economy is characterized by the following differential equations system in $k_1(t)$ and $Z(t)$:

$$\begin{aligned}dk_1(t)/dt &= G_1(k_1(t),Z(t)),\\dZ(t)/dt &= G_2(k_1(t),Z(t)),\end{aligned} \tag{12}$$

where $G_1(k_1(t),Z(t))$ and $G_2(k_1(t),Z(t))$ are unique functions of $k_1(t)$ and $Z(t)$ at any point of time, given by

$$G_1(k_1(t), Z(t)) = k_1(t) \left(\frac{d_1}{k_1(t)} + \frac{\alpha d_2}{f_1(t)} \right)^{-1} \left(\begin{array}{c} \frac{d_{11} f_1(t)}{k_1(t) N_0 \Lambda(t)} - \frac{\frac{d_{11}\sigma_1 f_1(t)}{N_0 k_1(t)} + \sigma\delta - \frac{\lambda\tau_{11}\beta_1 T_0 f_1(t)}{\Lambda(t)}}{\left(\sigma - \frac{\lambda\tau_{11}\beta_1 N_0}{d_{12}}\right)\Lambda(t)} - \\ - \frac{\delta_1}{\Lambda(t)} - m_1 d_2 \frac{G_2(k_1(t),Z(t))}{f_1(t) Z(t)} \end{array} \right), \quad (13)$$

$$\begin{aligned} G_2(k_1(t), Z(t)) = {} & \tau_{01} f_1(t) N_1(t)/Z(t)^{\varepsilon_1} + \tau_{02} f_2(t) N_2(t)/Z(t)^{\varepsilon_2} - \delta_3 Z(t) + \\ & + \tau_{03}\alpha_6^{\alpha_3}\beta_6^{\alpha_3} k_1(t)^{\alpha_3} (N_1(t) + \tau p_0 N_2(t))^{(\alpha_3+\beta_3)} Z(t)^{\varepsilon_3}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} & \alpha_5 = \frac{\alpha_2\beta_1}{\alpha_1\beta_2},\ \alpha_6 = \frac{\alpha_4\tau_1}{\tau_{11}\alpha_1}, \\ & \tau_3 = \frac{\tau_{11}A_1}{\tau_{22}A_2} \left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2} \left(\frac{\beta_1}{\beta_2}\right)^{\beta_2},\ \tau_4 = \tau_2/\tau_1, \\ & \beta_5 = \frac{\tau_1\beta_4}{\tau_{11}\beta_1},\ p_0 = \frac{\tau_{11}\beta_1}{\tau_{22}\beta_2}, \\ & D_1 = \frac{N_0}{(1+\alpha_6)(1+\beta_5\tau_4 p_0) - (1+\beta_5)(\alpha_5+\alpha_6\tau_4 p_0)}, \end{aligned}$$

$$\begin{aligned} & d_{11} = (1+\beta_5\tau_4 p_0) D_1,\ d_{12} = (\alpha_5 + \alpha_6\tau_4 p_0) D_1, \\ & d_{21} = (1+\beta_5) D_1,\ d_{22} = (1+\alpha_6) D_1, \\ & d_3 = \left(\frac{d_{22} p_0 \lambda/\zeta + d_{12}}{d_{12}\sigma/N_0 - \lambda\tau_{11}\beta_1} \right),\ d_1 = \frac{d_{11} d_3 \sigma/N_0 - d_{11} - d_{21} p_0 \lambda/\zeta}{d_3 \lambda\tau_{11}\beta_1 T_0}, \\ & d_2 = \frac{d_3\sigma\delta - \delta N_0}{d_3\lambda\tau_{11}\beta_1 T_0}, \\ & f_1(t) = A_1 Z(t)^{m_1} k_1(t)^{\alpha_1},\ f_2(t) = A_2 Z(t)^{m_2} k_1(t)^{\alpha_2}, \\ & \Lambda(k_1(t), Z(t)) = \Lambda(t) = \frac{1}{d_1/k_1(t) + d_2/f_1(t)}, \\ & T(t) = \frac{\frac{\sigma d_{11}\Lambda(k_1(t),Z(t))}{N_0 k_1(t)} + \frac{\sigma\delta\Lambda(k_1(t),Z(t))}{f_1(t)} - \lambda\tau_{11}\beta_1 T_0}{d_{12}\sigma/N_0 - \lambda\tau_{11}\beta_1}, \\ & N_1(t) = d_{11}\Lambda(k_1(t), Z(t))/k_1(t) - d_{12} T(t), \\ & N_2(t) = -d_{21}\Lambda(k_1(t), Z(t))/k_1(t) + d_{22} T. \end{aligned} \quad (15)$$

Let (k_{10}, Z_0) be a stationary state of the system (12), solution for the system

$$\begin{cases} G_1(k_1, Z) = 0 \\ G_2(k_1, Z) = 0 \end{cases}. \tag{16}$$

The linearization of system (12) in (k_{10}, Z_0) is:

$$\begin{aligned} du_1(t)/dt &= a_{11}u_1(t) + a_{12}u_2(t), \\ du_2(t)/dt &= a_{21}u_1(t) + a_{22}u_2(t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} a_{11} &= \tfrac{\partial G_1(k_1,Z)}{\partial k_1}|(k_{10}, Z_0), a_{12} = \tfrac{\partial G_1(k_1,Z)}{\partial Z}|(k_{10}, Z_0), \\ a_{21} &= \tfrac{\partial G_2(k_1,Z)}{\partial k_1}|(k_{10}, Z_0), a_{22} = \tfrac{\partial G_2(k_1,Z)}{\partial Z}|(k_{10}, Z_0). \end{aligned}$$

The characteristic equation of system (16) is

$$x^2 - (a_{11} + a_{22})x + a_{11}a_{22} - a_{12}a_{22} = 0. \tag{18}$$

If the roots of the above equation have a negative real part, then the stationary state is asymptotically stable. We perform the stability analysis in Section 4.

3 The stochastic three-sector model

Interestingly, a stochastic model can have a stable stationary state even if the corresponding deterministic model does not. In this section we, develop the stochastic version of the model presented previously and obtain the characteristic equation of the new system. As before, the stationary state is asymptotically stable if the roots of this equation have a negative real part.

Let $(\Omega, F(t), P)$ be a given probability space, and $B(t) \in \mathbb{R}$ be a scale Wiener process defined on Ω, having independent stationary Gauss increments with $B(0) = 0$ and $E(B(t)B(s)) =$

$min(t, s)$, where E denotes the mathematical expectation. The sample trajectories of $B(t)$ are continuous, nowhere differentiable and have infinite variation on any finite time interval (Kloeden and Platen, 1995). We are interested in the effect of the noise perturbation of the deterministic system (12).

The noise perturbation on the system (12) is captured by the following system of stochastic differential equations:

$$\begin{aligned} dk_1(t) &= G_1(k_1(t), Z(t))dt + \sigma_1(k_1(t) - k_{10})dB(t), \\ dZ(t) &= G_2(k_1(t), Z(t))dt + \sigma_2(Z(t) - Z_0)dB(t), \end{aligned} \tag{19}$$

where $k_1(t) = k_1(t, \omega)$, $Z(t) = Z(t, \omega)$, $\omega \in \Omega$, $\sigma_1, \sigma_2 > 0$ and (k_{10}, Z_0) is a stationary state of the system (12).

Following Lei and Mackey (2007) and Mircea et al. (2011), we analyze the second moments of the solutions for system (19). Linearizing these expressions around the stationary state (k_{10}, Z_0), we obtain that:

$$\begin{aligned} du_1(t) &= (a_{11}u_1(t) + a_{12}u_2(t))dt + \sigma_1 u_1(t)dB(t), \\ du_2(t) &= (a_{21}u_1(t) + a_{22}u_2(t))dt + \sigma_2 u_2(t)dB(t). \end{aligned} \tag{20}$$

To examine the stability of the second moment of $u_1(t)$ and $u_2(t)$ for the linear stochastic differential equations in (20) we use Ito's rule. Let

$$\begin{aligned} R_{11}(t, s) &= E(u_1(t)u_1(s)), \\ R_{12}(t, s) &= E(u_1(t)u_2(s)), \\ R_{21}(t, s) &= E(u_2(t)u_1(s)), \\ R_{22}(t, s) &= E(u_2(t)u_2(s)). \end{aligned}$$

From the system (20) it follows that :

$$dR_{11}(t,t)/dt = (2a_{11} + \sigma_1^2)R_{11}(t,t) + 2a_{12}R_{12}(t,t),$$
$$dR_{12}(t,t)/dt = (a_{11} + a_{22} + 2\sigma_2)R_{12}(t,t) + a_{12}R_{22}(t,t) + a_{21}R_{11}(t,t),$$
$$dR_{22}(t,t)/dt = (2a_{22} + \sigma_2^2)R_{22}(t,t) + 2a_{21}R_{12}(t,t). \tag{21}$$

The characteristic equation of the system (21) is

$$(2x-2a_{11}-\sigma_1^2)(2x-2a_{22}-\sigma_2^2)(2x-a_{11}-a_{22}-\sigma_1\sigma_2)-2a_{12}a_{21}(4x-2a_{11}-2a_{22}-\sigma_1^2-\sigma_2^2) = 0. \tag{22}$$

If $\sigma_1 = \sigma_2 = \sigma$, equation (22) becomes

$$(2x - a_{11} - a_2 - \sigma^2)(4x^2 - 4x(a_{11} + a_{22} + \sigma^2) + (2a_{11} + \sigma^2)(2a_{22} + \sigma^2) - 4a_{12}a_{21}) = 0.$$

If the roots of the above equation have a negative real part, then the stationary state is asymptotically stable in quadratic mean. We perform the roots analysis in the next section.

4 Calibration

Due to its complexity, the dynamics of system (16) is difficult to study theoretically. Using Maple 12, we perform numerical simulations using the parameter values presented in Table 1 and find that the equilibrium value is given by $(k_{10}, Z_0) = (3.388, 1.016)$.

Table 1. Parameter Values

$N_0 = 20,$	$T_0 = 1,$	$A_1 = 0.8,$	$A_2 = 0.7,$
$m_1 = 0.3,$	$m_2 = 0.25,$	$\alpha_1 = 0.3,$	$\alpha_2 = 0.32,$
$\alpha_3 = 0.3,$	$\beta_3 = 0.4,$	$\tau_1 = 0.04,$	$\tau_2 = 0.04,$
$\tau_{01} = 0.01,$	$\tau_{02} = 0.01,$	$\tau_{03} = 0.05,$	$\delta_1 = 0.08,$
$\delta_2 = 0.04,$	$\varepsilon_1 = 0.2,$	$\varepsilon_2 = 0.2,$	$\varepsilon_3 = 0.3,$
$\zeta_0 = 0.08,$	$\sigma_0 = 0.2,$	$\lambda_0 = 0.35.$	

5 Numerical simulations

In this section we show the numerical simulations, for both the deterministic and stochastic models, that prove the stability of the stationary states. Also, we perform the dynamic analysis for the propensity to save and enjoy leisure, taxation and knowledge efficiency utilization.

5.1 Stability analysis

For the deterministic case, Figure 1 plots $G_1(k_1, Z) = 0$ and $G_2(k_1, Z) = 0$. Equation (18) has the roots $x_1 = -0.6835483991e-1$ and $x_2 = -.2867712542$, and therefore the stationary states are asymptotically stable. Our solution confirms Zhang's (2009) findings that the stationary state of the dynamic system in (16) is unique and stable, but in our model this hold true independently of any initial conditions regarding the values of k_1 and Z. Moreover, Figure 2 - 4 below presents the trajectories $(t, k_1(t))$, $(t, k_2(t))$ and $(t, Z(t))$. As it can be seen, the economy is converging to the equilibrium values.

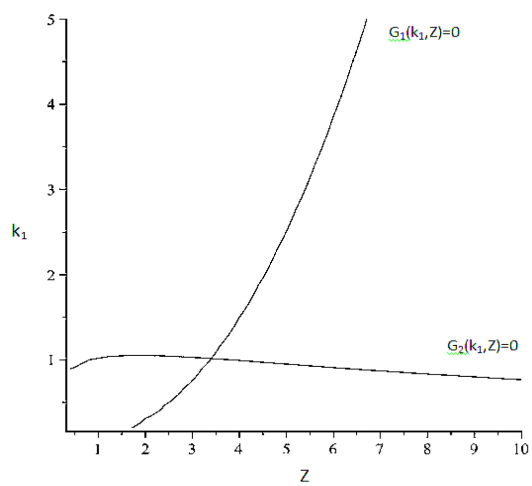
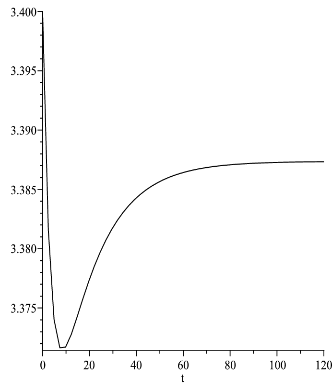
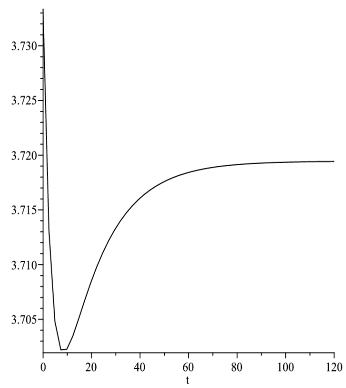


Fig 1. The trajectories $G_1(k_1, Z) = 0$ and $G_2(k_1, Z) = 0$





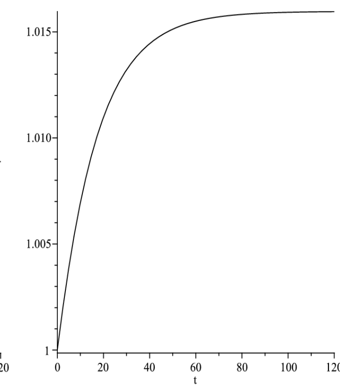
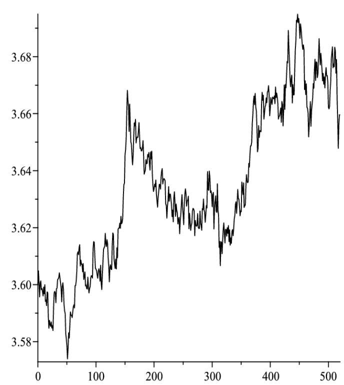
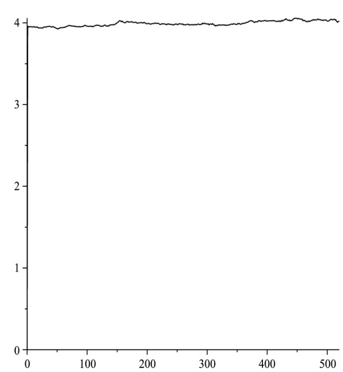


Fig 2. The trajectory (t, k_1) Fig 3. The trajectory (t, k_2) Fig 4. The trajectory (t, Z)

For the stochastic case, the characteristic equation (22) with $\sigma_1 = \sigma_2 = 0.3$ has the roots $x_1 = -0.2417712541$, $x_2 = -0.1325630470$ and $x_3 = -0.2335483990$. The squared mean values of the state variables $k_1(t, \omega)$ and $Z(t, \omega)$ are therefore asymptotically stable. The trajectories $(t, k_1(t, \omega))$, $(t, k_2(t, \omega))$, $(t, Z(t, \omega))$ are obtained using the Euler stochastic method and are shown in Figure 5 - 7.





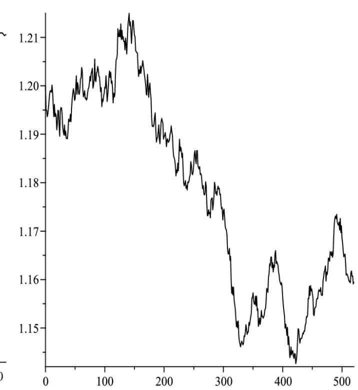


Fig 5. The trajectory (t, k_1) Fig 6. The trajectory (t, k_2) Fig 7. The trajectory (t, Z)

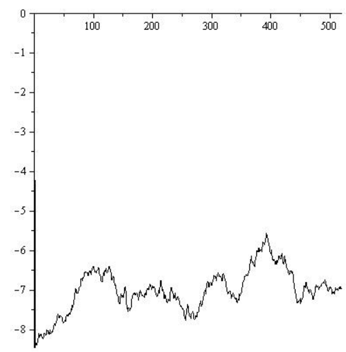
5.2 Counterfactual Experiments

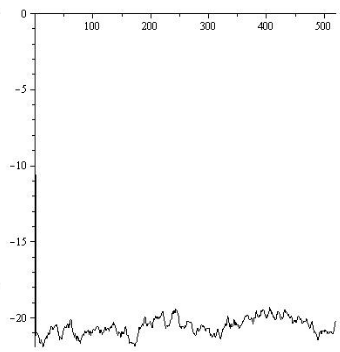
Using counterfactual simulations, we now analyze how changes in different parameter values may affect the economic system.

Consider first the case in which we increase the propensity to use leisure time σ_0, by 5 percentage points (from 0.2 to 0.21). All the parameters, except σ_0 remain the same as in Table 1 above. We use the relative estimation error method employed in statistics to calculate the relative rate of change in $k_1(t,\omega)$, $k_2(t,\omega)$ and $Z(t,\omega)$ due to changes in parameter values. We denote the rates of change by $\Delta k_1(t,\omega)$, $\Delta k_2(t,\omega)$ and $\Delta Z(t,\omega)$ respectively,

$$\Delta x(n,\omega) = \frac{(x(t,\omega,\sigma_{01}) - x(t,\omega,\sigma_{00}))}{x(t,\omega,\sigma_{00})} * 100, \text{ where } x = \{k_1, k_2, Z\}.$$

and show their simulated paths in Figure 8 - 10 below. As expected, we see that an increase in the propensity to use leisure time will harm the economic performance. With less labor in the economy, output decreases, making capital, labor time and, consequently, knowledge diminish. For all three sectors, this entails a one-off negative effect on prices, namely a drop in interest rates, wage and consumption price, but no significant long term effects.





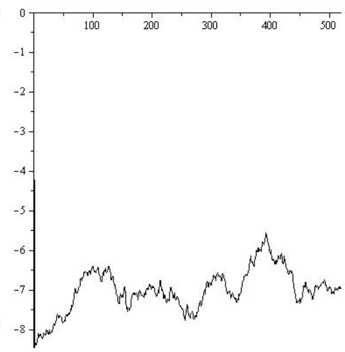
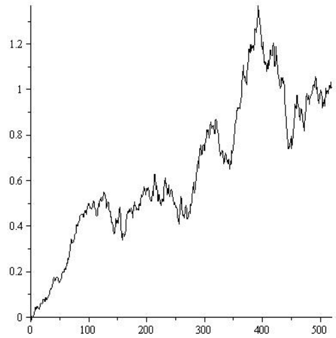
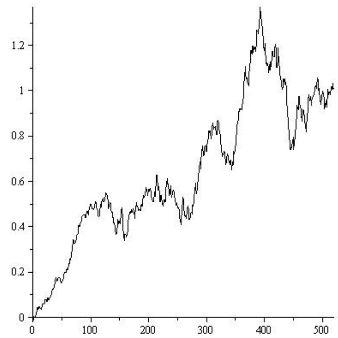


Fig 8. The trajectory $(t, \Delta k_1)$ Fig 9. The trajectory $(t, \Delta k_2)$ Fig 10. The trajectory $(t, \Delta Z)$

Second, we examined the effects of a change in the propensity to save λ_0 on the economic system. We allow the propensity to save to increase from 0.35 to 0.37 and plot the results in Figure 11 - 13. Unsurprisingly, increasing the saving propensity has positive effects on both capital and consumption sectors, as the prices for consumption goods increase and interest rate decreases. Overall, more savings will boost wealth, labor and eventually output. Since savings are not directly taxed, the effect of an increase in the propensity to save on knowledge will pass through the capital and consumption sectors. With higher capital in both productive sectors, the stock of knowledge will initially benefit, but as economy stabilizes, it will become smaller with respect to the benchmark model (i.e., the model with parameter values presented in Table 1).





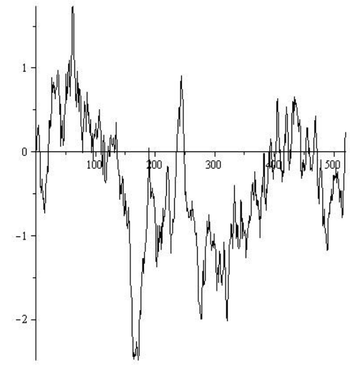
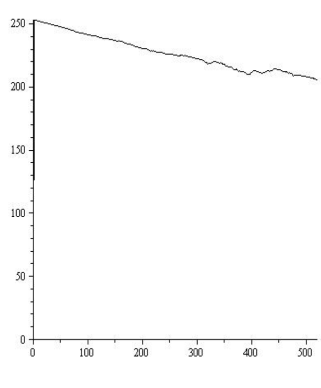
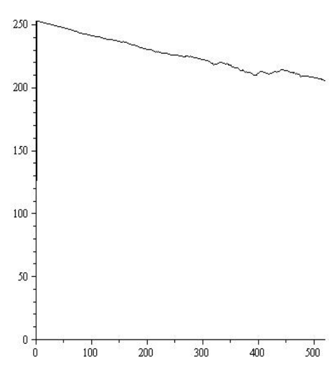


Fig 11. The trajectory $(t, \Delta k_1)$ Fig 12. The trajectory $(t, \Delta k_2)$ Fig 13. The trajectory $(t, \Delta Z)$

Third, consider what will happen to the economy if the government encourages research by increasing the tax rate on the two production sectors $\tau_1 = \tau_2$ from 4 to 5 percentage points. The effects are plotted in Figure 14 - 16 below. Levying a higher tax to finance research has a strongly positive effect on both productive sectors as interest, consumption prices and wages increase in the long term. Capital stock is significantly higher in the first periods, after which the series start to mildly converge to the benchmark model. Interestingly, the input factors for the knowledge productions are also increased, but, as in the previous case, this does not lead to a monotonic increase in knowledge with respect to the benchmark model. Our results confirm de Hek's (2006) finding that taxing the capital sector induces the economy to shift resources towards human capital production and leisure. With respect to the time allocated to leisure, this substitution effect is balanced by the corresponding income effect. With no income tax, taxing only physical capital makes human capital accumulation even more attractive, and this allows the long-run growth rate to raise.





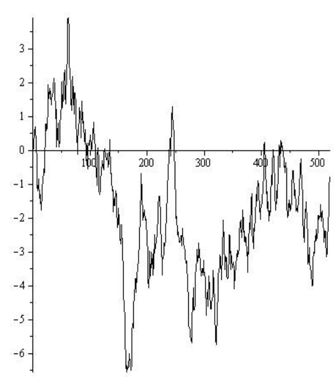
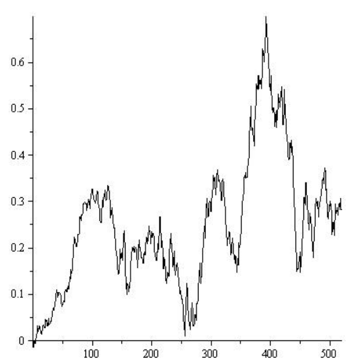
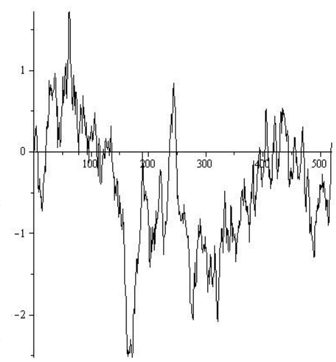


Fig 14. The trajectory $(t, \Delta k_1)$ Fig 15. The trajectory $(t, \Delta k_2)$ Fig 16. The trajectory $(t, \Delta Z)$

Fourth, an efficient knowledge utilization implies that the input knowledge domain matches perfectly the output knowledge requirements, with no under-utilization of knowledge. Moreover, this input-output relationship involving the knowledge is dynamic: knowledge is continually being created, extended, perfected, and it also depreciates. We are interested in the effects of an increase in the knowledge utilization efficiency in the capital sector m_1 from 0.3 to 0.35. Figure 17 - 19 show, as expected, a positive effect on the stock of capital and knowledge. This leads to an increase in wages, which in turn has two effects. On the one hand, agents will boost their work time, while on the other hand, this will harm the consumption sector as the price of consumption goods rises also.





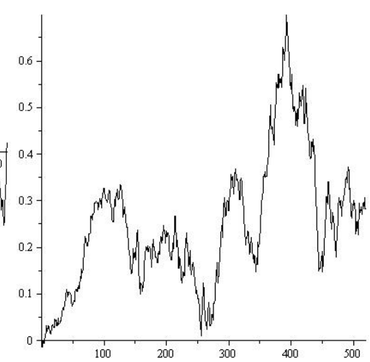
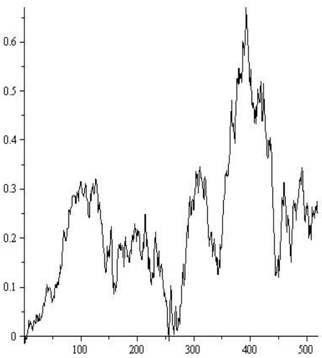
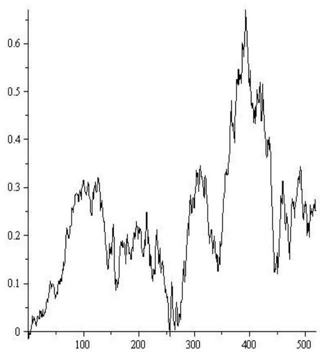


Fig 17. The trajectory $(t, \Delta k_1)$ Fig 18. The trajectory $(t, \Delta k_2)$ Fig 19. The trajectory $(t, \Delta Z)$

Finally, consider increasing the knowledge utilization efficiency in the consumption sector m_2 from 0.25 to 0.27. The results are illustrated in Figure 20 - 22. A more effective utilization of the knowledge in the consumption sector leads to a capital stock increase in both the capital and consumption sector. However, since the existing knowledge is used more efficiently, this entails that the rate of producing new knowledge can decrease without negative effects on output. As a result, we see that after an initial period of net increase, the stock of knowledge drops below what it would have been if the consumption sector would not have become more efficient in using the knowledge. Overall, we can observe a positive effect on the economy in the long run, and a similar effect on knowledge as an increase in propensity to save by roughly 6 percentage points.





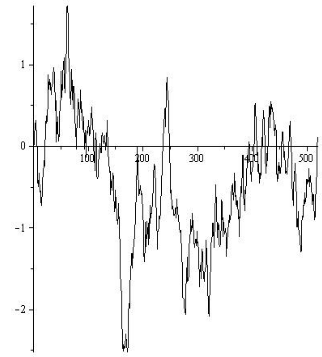


Fig 20. The trajectory $(t, \Delta k_1)$ Fig 21. The trajectory $(t, \Delta k_2)$ Fig 22. The trajectory $(t, \Delta Z)$

6 Concluding remarks

This paper proposes a dynamic model with wealth and human capital accumulation and analyzes it in both the deterministic and stochastic versions. First, we described the deterministic model and analyzed the stationary state. Second, we defined the stochastic perturbation, using the stationary state and we analyzed the mean and squared mean values for the system states. For certain parameter values, we showed that the stationary state is asymptotically stable in both the deterministic and stochastic model. Numerical simulations were performed to determine the trajectories of certain variables that characterize the system. We also conducted a comparative dynamic analysis for the propensities to save and to use leisure time, the tax rates that finance research, as well as for the knowledge utilization efficiency parameters. As expected, we found that increasing the propensity to use leisure time has negative economic consequences, both on the short and long term. On the other hand, a higher propensity to save has a positive economic effect, as does an increase in the research support through higher taxation. Finally, we find that a higher knowledge utilization efficiency seems to have a positive impact on the capital goods sector, but mixed effects on the consumption and knowledge patterns.

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