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Abstract

We consider a directed search environment where capacity constrained sellers reach uncoordinated buyers through costly advertising while buyers observed all prices probabilistically. We show that: (i) the equilibrium advertising intensity has an inverted U-shape in market tightness, (ii) the equilibrium advertising intensity is higher under an auction mechanism than under posted pricing, and (iii) the equilibrium price and measure of informed buyers may be positively correlated even in large markets.

JEL Classification: E52, E63.
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1 Introduction

The presence of trading frictions have become increasingly recognized as being an important feature of many markets. The directed search paradigm has recently received an extensive recognition in the literature as a natural progression from

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the Diamond-Mortensen-Pissarides framework to capture these trading frictions.[1] While the matching technology proposed in the Diamond-Mortensen-Pissarides is exogenous and relates to market tightness, the directed search framework considers a strategic environment. Capacity constrained sellers typically post prices, and uncoordinated buyers make choices over sellers to trade with. This environment generates an endogenous matching function. The equilibrium matching function depends on market tightness as well as on the vector of prices posted by one side of the market. Another important feature of directed search, is that it can generate multilateral matches, allowing the study of a richer set of trading mechanisms than the traditional Generalized Nash Bargaining used in pairwise matching models.[2] In standard directed search models, two basic and very important assumptions are maintained: (i) posting price is costless, and (ii) buyers can perfectly observe all posted prices.

In practice posting prices is costly. Sellers often use marketing channels or public media to inform buyers about their posted prices and locations.[3] Moreover, the observability of advertisements (hereafter ads) sent by sellers is not perfect. In other words, not all ads sent by sellers reach all buyers.[4] Typically a buyer only observes a subset of all posted prices. Under these circumstances, what kind of pricing and advertising strategies will sellers adopt, if sellers can only reach buyers through costly advertising and buyers randomly observe prices? Does the equilibrium advertising intensity depend on the particular pricing mechanism used by sellers? Are the results from the directed search literature robust to costly advertising and imperfect information structures? How do equilibrium prices and advertising intensities vary with changes in market tightness? Are equilibrium advertising intensities efficient ?

In order to answer these questions, we consider an environment where sellers inform buyers about their prices through costly advertising while buyers stochastically receive ads from sellers. Sellers are capacity constrained and buyers are

[1] Canonical models of directed search include among others, Peters (1984, 1991, 2000), Montgomery (1991), McAfee (1993), Julien, King and Kennes (2000, 2008), Burdett, Shi and Wright (2001), Shi (2001, 2005) and Shimer (2005). Our findings also relate to *Competitive Search Models*, Simer (1996), Moen (1997), however we discuss on the potential limitations in Section 6 of the paper.

[2] The Diamond-Mortensen-Pissarides focus on pairwise matching matching and bargaining. Earlier decentralized purely random matching models also had this feature. Among others, see Rubinstein and Wolinsky (1995).

[3] Large expenditures on advertising are a commonly observed throughout many industries.

[4] This may be because of the limited attention span of buyers, their time constraints or simply the cost of reaching all consumers is too costly.

uncoordinated.[5] We first consider an economy populated by homogeneous sellers each trying to sell one unit of an homogenous goods to ex-ante identical buyers. Each seller chooses price (or reserve price if the sales mechanism is auction) and advertising intensity that determines the probability by which a particular buyer receives her ads. Since the advertising technology is stochastic, not all buyers receive the same number of ads. After each buyer receives the ads, she then chooses one seller to visit and trade with. Finally, if a seller meets buyers (possibly multiple), the good is sold at the posted price (under price-posting mechanism) or through ex-post bidding (under auction mechanism). In this environment we then characterize the symmetric advertising equilibrium in an economy with a finite number of buyers and sellers. Then, we use the limit property of the finite equilibrium to study the equilibrium properties of the large market economy where the numbers of buyers and sellers are infinite.

For the large market, we find that the equilibrium advertising intensity has an inverted U-shape with respect to market tightness (buyer-seller ratio). This inverted U-shape implies that, for the information to have a large measure of informed buyers (market transparency), the number of market participants needs to be relatively balanced. Specifically, if the buyer-seller ratio is relatively low (small consumer base), the intense price competition amongst sellers prevents any particular seller to invest in advertising as the additional benefit is very low. Conversely, if the buyer-seller ratio is relatively high (large consumer base), sellers do not have enough incentive to invest in advertising given the ease to match with at least one buyer. We show precisely how the equilibrium inverted U-shape results from sellers' profits maximizing, and from the properties of equilibrium expected revenue and cost. Schmalensee (1989) and Bagwell (2007) find that the inverted-U shape relation between advertising intensity and market concentration is a stylized fact in industrial organization. We are one of the first studies to provide a theoretical explanation for this empirical regularity.[6]

We also show that the equilibrium matching rate for a seller in the large economy is the same as in the traditional directed search framework in that it only depends on the buyer-seller ratio. In this sense, the equilibrium prices found in the standard models are robust against the imperfect observability and the costly advertising. This property also implies that the equilibrium price only depends on, and is strictly increasing in the buyer-seller ratio. Combining this result with our findings on equilibrium advertising intensity, the equilibrium price and number

[5]In the spirit of directed search, we keep these two assumptions throughout the paper.

[6]The findings also generate more empirical question and specifications. For instance, are the empirical findings on inverted U-shape robust to markets with capacity constrained sellers?

of informed buyers (market transparency) can have positive or negative correlation when changing the buyer-seller ratio. When the buyer-seller ratio is small, the equilibrium price and advertising intensity are both low. Price competition is very intense, and the resulting lower prices only generate small amount costly advertisements. Increasing the buyer-seller ratio shows induces a large positive marginal impact on expected revenue, justifying an increase in advertising intensity. This positive correlation between equilibrium price and measure of informed buyers, those who receive ads, has also been shown by Lester (2011) but for finite markets only.[7] Here we find this effect but in a large market for relatively low buyer-seller ratio and the number of informed buyers is endogenous. When the buyer-seller ratio is large, the equilibrium price is high, but advertising intensity is low. Increasing the buyer-seller ratio has a small positive marginal impact on expected revenue, and sellers' response is to lower advertising. This generates a negative correlation between equilibrium price and market transparency as per the conventional wisdom.

In this paper, we also compare the equilibria from two trading mechanisms: auction and price-posting. For the large market, we find that the equilibrium advertising intensity under auction is higher than the one under price-posting, and holds for any values of the buyer-seller ratio. Under an auction mechanism, the ex-post competition between buyers allows sellers to enjoy full surplus. Thus, sellers are concerned with the probability of attracting at least two buyers which gives higher incentive for sellers to advertise more. Under a price-posting mechanism, sellers only care about the probability of attracting at least one buyer. However, since the equilibrium price and probability of trade only depends on buyer-seller ratio, sellers have the same expected revenue under both mechanisms. Sellers will then have higher advertising levels when an auction is used as the trading protocol. Julien, Kennes and King (2001, 2002) showed that auction beats price-posting in the finite economy but sellers are indifferent between these two mechanisms in the large economy. Here clearly, sellers would be better off under price posting in large markets, and hence, our finding contrasts with the previous conclusions in the directed search literature with perfect observability of prices.

Finally, we analyze the efficiency properties of equilibria. We show that in general, efficiency cannot be achieved simultaneously at the intensive (advertising) and extensive margin (entry). Efficiency is possible whenever the market is balanced, that is, for equal number of buyers and sellers. Furthermore, depending

[7] Lester (2011) has exogenous number of informed and uniformed buyers about prices, for a range of informed buyers, increasing the fraction of informed buyers lead to higher equilibrium prices. However, his result only holds in finite markets not in large markets.

on parameter values, efficiency on one margin can only be achieved, either the intensive or the extensive.

The work closes to this paper is that of Menzio (2007) who studies the cheap-talk information transmission in a competitive matching market. A continuum of heterogeneous sellers send ads which contain the information about their productivity. Every buyer observes all ads and choose which seller to visit. Sellers do not commit to the information they send but can choose how informative their ads are. Menzio (2007) finds that the informativeness of advertising is "inverted U-shape" in market tightness. In our environment, sellers commit to the information they send and the advertising intensity affects the reach of buyers. We provide more details on the connection of our paper to the literature in Section 6.

The remainder of the paper is organized as follows. Section 2 describes the basic environment and specifies the information structures, players' beliefs and strategies as well as the equilibrium concept. In section 3, we characterize the equilibrium outcome in the finite economy, then we explore the limit properties of this equilibrium as an exact-equilibrium in the large economy. Section 4 characterizes the equilibrium under the alternative price-posting mechanism. We compare the equilibrium advertising intensities under different mechanisms. Section 5 studies the efficiency properties while section 6 relates the findings of our paper to the previous literature. The final section offers some concluding remarks.

2 The model

We consider an economy populated by M sellers and N buyers. We denote the sets of sellers and buyers by $\mathcal{M}$ and $\mathcal{N}$, respectively. The ratio of buyers to sellers, $\phi = N/M$, represents the corresponding market tightness.

Each seller owns one unit of the homogeneous good. The reservation value to sellers is normalized to be zero. Each buyer is ex-ante homogenous and demands one unit of the homogenous goods. Both sellers and buyers are risk-neutral. If a buyer purchases the good at price p, his net payoff is $1 - p$. If a seller sells at price p, her profit is simply p.

Sellers inform buyers about their prices through costly advertising while buyers stochastically receive ads from sellers. Buyers who receive at least one advertisement from sellers will be referred as informed buyers from now.

Game structure. When the market opens, sellers simultaneously and independently choose (reserve) prices and advertising intensities. All informed buyers then simultaneously and independently choose which seller to visit. The visiting decision is irreversible even if the buyer eventually does not obtain the goods. Buyers who are uninformed do not participate in the market.

If sellers and buyers meet, the seller uses a certain mechanism to allocate the good. We consider two mechanisms in this paper: first-price auction and price posting.[8] The auction mechanism captures the fact that in a wide range of markets, sellers have only limited ability to commit to the posted price. When there is only one buyer visiting a particular seller, the terms of trade is given by the posted reserve price. However, when there is more than one buyer visiting a particular seller, the seller can let the buyers compete for her good.

Advertising. The advertising technology we adopt is in the spirit of Butters (1977) and Grossman and Shapiro (1984) so that ads stochastically attract buyers. Ireland (1993) and Eaton, McDonald and Meriluoto (2010) allow for target advertising so that sellers can choose the fraction of buyers who will receive ads. In this paper we do not consider this technology as advertising has an important stochastic component. In our model, each seller chooses an advertising intensity $q \in [0, 1]$ so that each buyer has probability q to receive ads from this particular seller. Under stochastic advertising, a seller who spends lots of resources on advertisements might not be able to attract many buyers, though the possibility of this event is very small.

Advertising, sending signals to buyers, is costly. The cost of advertising, $C(q)$, is convex so that $C'(q) > 0$, $C''(q) \geq 0$. Moreover, the cost also has the following properties $C(0) = 0, C(1) > 1, C'(0) = 0$ and $C'(1) = \infty$. The advertising cost does not depend on the market tightness ϕ.[9] The advertising technology is consistent with modern marketing channels such as portal-site advertising, television advertising and vacancy-posting on online matchmakers.[10]

Example. Consider a constant-reach, independent readership (CRIR) advertising technology. A seller can place ads in any set of websites. Assume that each website has a readership of x in the population. Then a fraction x of the population

[8]The auction mechanism was introduced into matching environment by Peters and Severinov (1997), Peters (1997) and Julien, Kennes and King (2000). The price-posting mechanism was discussed by Peters (1984, 1991, 2000) and Burdett, Shi and Wright (2001).

[9]In Butters (1977), the urn-ball matching advertising requires the probability of reaching a buyer to be decreasing in the total number of buyers, holding the resource devoted to advertising fixed. This advertising technology prevails in those traditional mailbox advertising and phone-call advertising where the consumer reach depends on the investment per capita.

[10]For instance, in labor markets, the advertising cost of a firm depends on the number of job-hunting sites it posts its ads but not the number of potential job seekers.

is exposed to an ad published in any given website. If a seller places ads in s websites, the probability that a given consumer will see none of these ads is then $(1-x)^s$. The reach of such an advertising campaign is $q = 1-(1-x)^s$. Equivalently, we can write the number of websites in which ads must be placed to achieve a reach of q as $s = \ln(1-q)/\ln(1-x)$. Suppose each website charges y for posting ads. Then, in order to reach each buyer with probability q, the seller must incur the following cost

$$C(q) = \frac{y \ln(1-q)}{\ln(1-x)}. \quad (1)$$

Note that this advertising technology satisfies all conditions we previously assumed.

Information structure. The number of sellers and buyers as well as the characteristics of the homogenous good are public knowledge. Thus although buyers might not know the exact number of market participants, they hold a rational estimate.

Before the market opens, a representative buyer lacks the following information: (i) the prices charged by each seller, (ii) where each seller is located and (iii) the advertising intensity of each seller. Restrictions (i) and (ii) on consumers' information set makes sellers' advertising essential. A trade can only take place after a buyer knows the location and price the seller is going to charge. If consumers a priori know sellers' location, they may conduct random search even if they do not receive any ad. We rule out this possibility.[11] Assumption (iii) is useful because despite the fact that consumers usually do not know the exact advertising input of each seller, the ability to perfectly predict the advertising level on the "off-equilibrium paths" can lead to the non-existence of pure-strategy equilibrium.

While all buyers are ex-ante homogeneous, the stochastic advertising endogenously creates heterogeneity among buyers. Buyers may receive a different number of ads. Moreover, these ads may come from different sellers. Therefore, we use information set $\mathcal{I}$ to describe the buyers' information prior to the selecting game. A buyer's information depends on how many ads a particular buyer receives. Each ad contains a seller's name, her location and the reserve prices she charges. We then have that $\mathcal{I} \in \mathbb{R}_+^M$, where the ith argument is the reserve price $r_i \in [0,1]$ charged by seller i. If a buyer does not receive ads from seller j, we assume the jth argument in her information set is infinity and this seller will never be selected. For example, if a buyer observes only seller 1's ads, her information set is $\{r_1, \infty, ..., \infty\}$. For any information set $\mathcal{I}$, we use $|\mathcal{I}| \leq M$ to denote the number of arguments in $\mathcal{I}$ which are non-equal to ∞. We refer $|\mathcal{I}|$ as the size of $\mathcal{I}$.

[11] Random search dilutes the essentiality of advertising and an autarky equilibrium may emerge.

Beliefs and strategies. When each seller can only provide limited supply, advertising intensity affects not only sellers' profits but also buyers' payoffs. A high advertising intensity tends to attract more buyers and therefore reduces the probability of trade for all attracted buyers. Let us denote $\mu : [0,1]^M \to [0,1]^M$ the buyers' beliefs. Note then that $\mu(\cdot)$ maps a buyer's information set to the set of all sellers' advertising intensities. Thus, $\mu_j(\cdot)$ denotes the conjectured level of seller j's advertising intensity. A belief system is said to be passive if $\mu_j(\mathcal{I}) = \hat{q}, \forall j \in \mathcal{M}, \mathcal{I} \in [0,1]^M$ for some $\hat{q} \in [0,1]$.

A seller's pure strategy is $(p,q) \in [0,1]^2$. A buyer's strategy is a mapping from his information set and belief to a distribution over the observed sellers: $\sigma(\mathcal{I},\mu)$ where $\sigma_j(\mathcal{I},\mu)$ is the probability to select seller j and $\sum_{k=1}^{M} \sigma_k = 1$. If seller j is not observed, $\sigma_j = 0$.

We impose two further restrictions on buyers' strategies:

Anonymity: If $r_j, r_k \in \mathcal{I}$ and $r_j = r_k$, then $\sigma_j = \sigma_k$.

Symmetry: If there exists a one-to-one correspondence $\eta : \mathcal{M} \to \mathcal{M}$ such that $r_j = r_{\eta(j)}$ where $r_j \in \mathcal{I}$ and $r_{\eta(j)} \in \mathcal{I}'$, then $\sigma_j(\mathcal{I},\mu) = \sigma_{\eta(j)}(\mathcal{I}',\mu)$ for all $j \in \mathcal{M}$.

These two restrictions are modified from the standard ones to accommodate imperfect observability.[12] Anonymity requires that if a buyer observes two sellers charging the same reserve price, she should choose them with the same probability. Symmetry requires that if a one-to-one correspondence can be constructed between two information sets $\mathcal{I}$ and $\mathcal{I}'$ so that r_j in $\mathcal{I}$ equals to its image, say r'_k in $\mathcal{I}'$, then the selecting probability over seller j when the information set is $\mathcal{I}$ equals to the selecting probability over seller k when the information set is $\mathcal{I}'$.

Equilibrium. Throughout this paper, we focus on an equilibrium where the advertising intensity is strictly positive. The formal definition of the equilibrium is given by Definition 1.

Definition 1 *A* Symmetric Advertising Equilibrium *(SAE) is a triplet* $\{(r^*,q^*), \sigma^*(\mathcal{I},\mu), \mu\}$ *such that:*

(i) Given buyers' common belief μ and the symmetric selecting strategies $\sigma^*(\mathcal{I},\mu)$*, all sellers use symmetric strategies* (r^*,q^*) *to solve (20);*

[12] In standard directed search models with perfect observability, *anonymity* requires buyers to select two sellers with the same probability if they charge the same price. *Symmetry* requires that two buyers have to use the same selecting strategy since they always observe the same set of prices.

(ii) A buyer with given information set $\mathcal{I}$ and belief μ, uses strategy $\sigma^*(\mathcal{I}, \mu)$ to maximize his expected payoff;

(iii) The equilibrium advertising intensity q^* coincides with the common passive belief $\hat{q}$;

(iv) The equilibrium advertising intensity is strictly positive: $q^* > 0$.

3 Advertising Equilibrium

We use backwards induction to solve for the equilibrium. Consider the final stage where the auction takes place. In a match with seller i, a buyer also observes the total number of buyers showing up at seller's i location. Her bidding strategy depends on the reserve price, r_i, and the number of buyers showing up, n_i. Since there is no information asymmetry in the bidding stage, simple reasoning, as in the Bertrand competition, gives the optimal bidding strategy

$$b(r_i, n_i) = \begin{cases} r_i & \text{if} \quad n_i = 1; \\ 1 & \text{if} \quad n_i > 1. \end{cases} \tag{2}$$

The profit that seller i receives is then given by

$$\pi_i(r_i, n_i) = \begin{cases} 0 & \text{if} \quad n_i = 0; \\ r_i & \text{if} \quad n_i = 1; \\ 1 & \text{if} \quad n_i > 1. \end{cases} \tag{3}$$

Notice that under an auction mechanism, multi-lateral matching leads to zero consumer surplus. Whenever there is more than one buyer selecting seller i, seller i extracts all the surplus from selling the good.

3.1 Buyers' decision

In the selecting stage, an informed buyer is trading off between the probability of trade and the price. A low price may attract more buyers and therefore increase the risk of not trading. Similarly, if a seller is suspected to have high advertising intensity (holding other factors constant), visiting this seller is less attractive. This is the case as this seller will tend to have more buyers competing for the good. When signals sent by sellers are imperfectly observed by buyers, her choice is more complicated than the standard directed search one. To better establish a buyer's selection problem, we first analyze a case where M sellers compete for two buyers, buyer 1 and 2, and extend this analysis to the N-buyer case later.

If buyer 1 receives ads *only* from seller i, she has no choice but to select seller i no matter what she thinks of buyer 2's choice. Now suppose buyer 1 receives at least two ads including the one from seller i. If buyer 1 is the only visitor at seller i, she can get $1 - r_i$. However, if buyer 2 also shows up at seller i, buyer 1 gets zero payoff. Thus, buyer 1's expected payoff of visiting seller i is given by

$$U_i^1 = (1 - r_i)\Pr[\text{being alone at seller } i]. \tag{4}$$

To assess the probability of being alone at seller i, buyer 1 needs to consider the following: (i) whether buyer 2 receives ads from seller i; if so, the number of sellers observed by buyer 2 (buyer 2's information set); and (ii) given buyer 2's information set, the selected strategy used by buyer 2. We can write the probability of being alone at seller i as follows

$$\Pr[\text{being alone at seller } i] = P_0 + P_+, \tag{5}$$

where P_0 is the probability that buyer 2 does not receive ads from seller i, and P_+ is the probability that buyer 2 observes r_i but chooses not to select seller i. Buyer 1's beliefs about seller i's advertising intensity is $\mu_i(\mathcal{I}_1)$, $P_0 = 1 - \mu_i(\mathcal{I}_1)$. To compute the probability P_+, we need to take into account all possible formations of buyer 2's information set. P_+ is given by

$$P_+ = \mu_i(\mathcal{I}_1)\left[1 - \sum_{\substack{\mathcal{I}_2 \text{ s.t.} \\ r_i \in \mathcal{I}_2}} \sigma_i(\mathcal{I}_2, \mu) \cdot \prod_{\substack{r_j \in \mathcal{I}_2 \\ j \neq i}} \mu_j(\mathcal{I}_1) \cdot \prod_{r_k \notin \mathcal{I}_2} (1 - \mu_k(\mathcal{I}_1))\right]. \tag{6}$$

Seller i is the only potential deviator. Therefore, the reserve price charged by seller i is the only possible observed price which is different from buyer 1's expected level. The conjecture about other sellers' advertising intensities should not be affected by the deviation of seller i. Thus these are held at some fixed level $\hat{q}$ in the symmetric setting. In addition, the anonymity and symmetry assumptions imply that buyer 2's strategy only depends on the size of her information set. Thus, equation (6) can be simplified to

$$P_+ = \mu_i(\mathcal{I}_1)\left[1 - \sum_{\substack{\mathcal{I}_2 \text{ s.t.} \\ r_i \in \mathcal{I}_2}} \sigma_i(\mathcal{I}_2, \mu) \cdot \hat{q}^{|\mathcal{I}_2|-1} \cdot (1 - \hat{q})^{M-|\mathcal{I}_2|}\right]. \tag{7}$$

From the 2-buyer case, it is clear that buyer 1's expected payoff depends on her own beliefs about the deviator's advertising intensity and buyer 2's strategy. Note that the term in the bracket of (7) can be understood as the expected probability that buyer 2 does not visit seller i conditional on receiving ads from seller i. To further simplify the exposition, we define the second term in the bracket of (7) as Λ_i which can be understood as any other buyer's information set. In a symmetric environment, Λ_i is determined by the deviating price r_i, the non-deviating price r and the belief system μ. Thus

$$P_+ = \mu_i(\mathcal{I}_1)\left[1 - \Lambda_i(r_i, r, \mu)\right]. \tag{8}$$

We are now ready to extend buyer 1's reasoning to a N-buyer setting. We still analyze from buyer 1's perspective and assume she receives ads from the deviator, seller i. Let $\Lambda_i^j(r_i, r, \mu)$ denote buyer 1's assessment of buyer j's choice, conditional on buyer j receive ads from seller i. The probability that buyer 1 is the only one who receives seller i's ads is $P_0 = (1 - \mu_i(\mathcal{I}_1))^{N-1}$. In the N-buyer case, P_+ is interpreted as the probability of the event where someone other than buyer 1 receives seller i's ads but none of them select seller i. From now on, we omit the arguments in Λ_i. To compute this aggregate probability P_+, the following events and the corresponding probabilities to be considered are the following:

1. only one other buyer observes r_i, but does not visit i which occurs with probability $\mu_i(\mathcal{I}_1)\left(1 - \mu_i(\mathcal{I}_1)\right)^{N-2} \sum_{j \in \mathcal{N} \setminus \{1\}} (1 - \Lambda_i^j)$;

2. two other buyers observe r_i, but neither of them visit i which occurs with probability $(\mu_i(\mathcal{I}_1))^2\left(1 - \mu_i(\mathcal{I}_1)\right)^{N-3} \sum_{j,k \in \mathcal{N} \setminus \{1\}} (1 - \Lambda_i^j)(1 - \Lambda_i^k)$;

n. ....;

N-1. all other $N-1$ buyers observe r_i, but none of them visit i which occurs with probability $(\mu_i(\mathcal{I}_1))^{N-1}(1 - \Lambda_i^2) \cdots (1 - \Lambda_i^N)$.

Buyer 1 does not observe other buyers' information set. Therefore, any two buyers are viewed as the same from the point of view of buyer 1. That is $\Lambda_i^j = \Lambda_i^k = \Lambda_i, \forall\, j, k \neq i$. Under symmetry, a buyer's selecting strategy does not depend on the specific information this buyer receives but the size of his information set. This is the case given that the deviation of seller i is observed. That is

$$\forall \mathcal{I}, \mathcal{I}' \text{ such that } r_i \in \mathcal{I}, \mathcal{I}' \text{ and } |\mathcal{I}| = |\mathcal{I}'|, \text{then } \sigma_i(\mathcal{I}, \mu) = \sigma_i(\mathcal{I}', \mu). \tag{9}$$

Given the fact that the selection strategy only depends on the size of information sets, all events included in Λ_i can be further classified into $M-1$ cases. The expression of Λ_i is then

$$\Lambda_i(r_i, r, \mu) = \sum_{k=0}^{M-1} \binom{M-1}{k} \cdot \hat{q}^k \cdot (1-\hat{q})^{M-k-1} \cdot \sigma_i(\mathcal{I}, \mu \mid |\mathcal{I}| = k+1) \tag{10}$$

where $\sigma_i(\mathcal{I}, \mu \mid |\mathcal{I}| = k+1)$ is the conditional probability of selecting seller i when the information set $\mathcal{I}$ ($r_i \in \mathcal{I}$) and the size of the information set is $k+1$. Thus we can conclude that the probability that buyer 1 is alone at seller i, conditional on that he observes r_i and holds the belief μ, is

$$P_0 + P_+ = \sum_{n=0}^{N-1} \binom{N-1}{n} \cdot (\mu_i(\mathcal{I}_1))^n \cdot (1-\mu_i(\mathcal{I}_1))^{N-1-n} \cdot (1-\Lambda_i)^n = \left(1-\mu_i(\mathcal{I}_1)\Lambda_i\right)^{N-1}. \tag{11}$$

Then, the expected payoff from visiting seller i is

$$U_i = (1-r_i)\left(1-\mu_i(\mathcal{I})\Lambda_i\right)^{N-1}. \tag{12}$$

From now on, as in the directed search literature, we focus on the subgames where buyers use mixed strategies when selecting sellers. Similarly as we have done in the previous analysis, define Λ as the expected probability that a seller visits a seller other than i contingent on him receiving the seller's ads. A buyer who receives both deviator and non-deviator's ads uses mixed strategies only if visiting these sellers yields the same expected payoffs. That is

$$(1-r_i)\left(1-\mu_i(\mathcal{I})\Lambda_i\right)^{N-1} = (1-r)\left(1-\hat{q}\Lambda\right)^{N-1}, \tag{13}$$

where r is the symmetric reserve price charged by all non-deviators. If a buyer observes $m > 1$ prices, his strategy must satisfy

$$\sigma_i(\mathcal{I}, \mu) + \sum_{k \neq i, k \in \mathcal{I}} \sigma_k(\mathcal{I}, \mu) = 1 \ \forall \mathcal{I} \text{ with } |\mathcal{I}| > 1. \tag{14}$$

Since non-deviators are assumed to choose the same reserve price, they should be treated equally when a buyer decides her visiting strategy. We then must have

$$\sigma_k(\mathcal{I}, \mu) = \frac{1-\sigma_i(\mathcal{I}, \mu)}{m-1}, \ \forall r_k \in \mathcal{I} \text{ with } |\mathcal{I}| = m. \tag{15}$$

When we combine (15) and the indifference condition (13), we derive Lemma 1 which is a market clearing condition.

Lemma 1 *The conditional expected probabilities of selecting the deviator, Λ_i and the conditional expected probabilities of selecting a non-deviator, Λ, satisfy the following condition:*

$$\mu_i(\mathcal{I})\Lambda_i(r_i, r, \mu) + (M-1)\hat{q}\Lambda(r_i, r, \mu) = 1 - (1 - \mu_i(\mathcal{I}))(1-\hat{q})^{M-1} \tag{16}$$

All proofs can be found in the appendices.

The relationship between the conditional expected probability, Λ_i and Λ is intuitive. If we multiply both sides of (16) by the number of buyers, N, the left-hand side of (16) is the expected number of active buyers in the market while the right-hand side is the expected number of informed buyers. In our model, any informed buyer will visit a seller, because visiting is costless. A buyer becomes active if she shows up at some seller's auction. Equation (16) says that the expected number of active buyers in the market is equal to the expected number of informed buyers. It is worth noting that $\mathcal{I}_1$ appears in both sides of (16). That is, the market clearing condition is a subjective concept and holds for every buyer who observes the deviating reserve price r_i.

Combine the indifference condition (13) and the market clearing condition (16), we can explicitly express Λ_i as

$$\mu_i(\mathcal{I}_1)\Lambda_i \;=\; 1 - \left[\frac{(M-1) + (1-\mu_i(\mathcal{I}_1))(1-\hat{q})^{M-1}}{(M-1)\left(\frac{1-r_i}{1-r}\right)^{\frac{1}{N-1}} + 1}\right]. \tag{17}$$

This probability in general hinges on the observed reserve prices r_i and r and the belief about the deviator's advertising intensity μ_i and the belief about the non-deviator's advertising intensity, $\hat{q}$, which in turn depends on r_i and r.

3.2 Sellers' choice

In this section, we analyze sellers pricing and advertising choices. As seller i is the only potential deviator, we focus on her incentive to deviate from the equilibrium reserve price and advertising intensity. Seller i knows her own choice on r_i and q_i, takes other sellers' choices r_{-i} and q_{-i} as given, and also foresees buyers' belief and choice in the selection game. Thus, seller i's maximization problem is

$$\max_{r_i, q_i} \Pi_i = r_i \cdot \Pr[n_i = 1] + 1 \cdot \Pr[n_i \geq 2] - C(q_i).$$

It is clear that seller i extracts all the surplus only when there is more than one buyer selecting her. The seller can earns the reserve price when only one buyer selects her. Therefore, two probabilities, $\Pr[n_i = 1]$ and $\Pr[n_i \geq 2]$ need to be computed. One thing worth mentioning is that sellers use the same reasoning as buyers. To seller i, the conditional probability that a buyer visits her given that a buyer receives her ads is Λ_i. The difference is that seller i knows her true advertising intensity q_i and takes other sellers' symmetric advertising intensities q as given in a simultaneous move game. We now show that the two probabilities above have simple representations. For $n_i = 0$, the following events are included:

00. no buyer receives ad from seller i with probability $(1 - q_i)^N$;

01. one buyer receives ad from seller i, but he does not visit seller i with probability $Nq_i(1 - q_i)^{N-1}(1 - \Lambda_i)$;

0n. ....;

0N. all buyers receive ad from seller i, but none of them visit seller i with probability $(q_i)^N(1 - \Lambda_i)^N$.

Adding the probabilities of these events yields:

$$\Pr[n_i = 0] = \sum_{k=0}^{N} \binom{N}{k} [q_i(1 - \Lambda_i)]^k (1 - q_i)^{N-k} = (1 - q_i\Lambda_i)^N . \tag{18}$$

For $n_i = 1$, the following events are included:

11. 1 buyer receives seller i's ad and he visits seller i with probability $Nq_i(1 - q_i)^{N-1}\Lambda_i$;

12. 2 buyers receive seller i's ad but only one of them visits seller i with probability $N(N - 1)(q_i)^2(1 - q_i)^{N-2}[2\Lambda_i(1 - \Lambda_i)]$;

1n. ....;

1N. N buyers receive seller i's ad but only one of them visits seller i with probability $(q_i)^N N\Lambda_i(1 - \Lambda_i)^{N-1}$.

Adding the probabilities of these events yields:

$$\Pr[n_i = 1] = q_i\Lambda_i \sum_{k=0}^{N-1} \binom{N}{k+1} \binom{k+1}{1} [q_i(1 - \Lambda_i)]^k (1 - q_i)^{N-k-1} = Nq_i\Lambda_i (1 - q_i\Lambda_i)^{N-1} . \tag{19}$$

Given the derived probabilities, seller i's maximization can be summarized by

$$\max_{r_i,q_i} \Pi_i = 1 - (1-r_i)Nq_i\Lambda_i (1-q_i\Lambda_i)^{N-1} - (1-q_i\Lambda_i)^N - C(q_i) \\ \text{s.t.} \quad \text{Equation (17).} \tag{20}$$

Seller i is maximizing her expected profit while her choice is constrained by buyers' response in the selecting game.

Define $\tau = 1-(1-q^*)^M$ as the equilibrium probability that a buyer is informed. Proposition 1 gives the conditions for the existence and uniqueness of a symmetric advertising equilibrium (SAE) in a finite economy with auction mechanism.

Proposition 1 *In a finite market where all sellers use first-price auction, a symmetric advertising equilibrium exists. The equilibrium reserve price and advertising intensity* (r^*, q^*) *are characterized by the following first-order conditions:*

$$r^* = \frac{(\phi - \frac{1}{M})\tau}{(M-1)+(\phi-1)\tau}, \tag{21}$$

$$C'(q^*) = r^*\frac{\phi\tau}{q^*}\left(1-\frac{\tau}{M}\right)^{N-1} + (1-r^*)\frac{\phi(\phi-\frac{1}{M})\tau}{q^*}\left(1-\frac{\tau}{M}\right)^{N-2}. \tag{22}$$

When $N \geq M+1$*, the equilibrium is unique.*

When the communication between sellers and buyers becomes perfect, that is $q^* \to 1$, the equilibrium reserve price r^* converges to the reserve price in standard directed search models for finite markets.[13] To have a clearer idea regarding the equilibrium advertising intensity, we use CRIR cost function introduced in section 2 to plot the level set, or iso-advertising curves, and a three-dimensional graph of $q^* = q(M,N)$ in Figure 1. Figure 1 shows that when the number of seller M is fixed the advertising intensity is increasing with the number of buyers N initially, but eventually decreasing.

3.3 Competitive matching equilibrium

Following the directed search literature we extend the finite number of participants analysis with to the large market environment by imposing that, M and N, approach infinity while the market tightness ϕ is fixed.

[13]See Julien, Kennes and King (2000) for more on this.

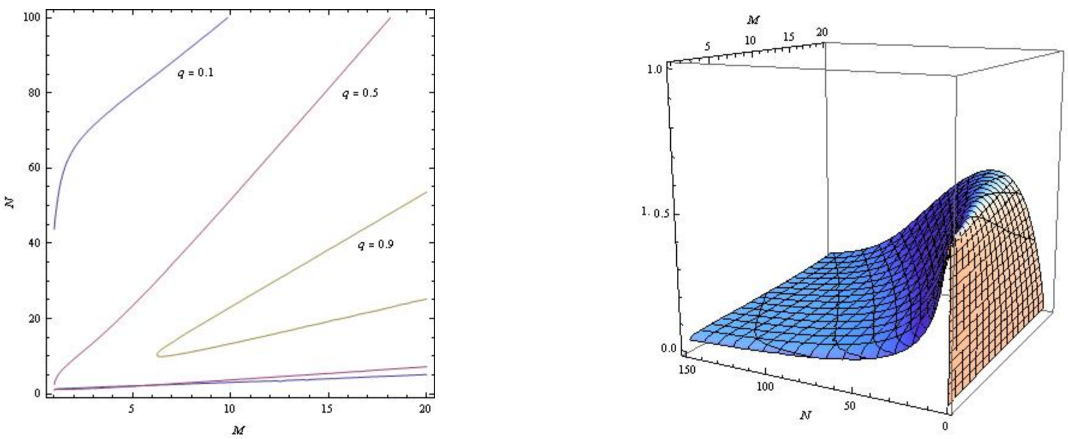

Figure 1: Equilibrium advertising intensities when $x = 0.6$, $y = 0.4$

To determine the large market equilibrium, we take the limit of the sellers' payoff functions and construct their profit maximization problem, and compute the equilibrium reserve price and advertising intensity in this large market. In this environment, the limit of the solution in finite market coincides with the solution to the limit payoff function.

Consider the payoff function in (20) and take the limit when $M, N \to \infty$, which yields

$$\Pi = r\frac{q_i\phi}{\hat{q}}e^{-\frac{q_i\phi}{\hat{q}}} + (1 - e^{-\frac{q_i\phi}{\hat{q}}} - \frac{q_i\phi}{\hat{q}}e^{-\frac{q_i\phi}{\hat{q}}}) - C(q_i). \tag{23}$$

Notice that we do not need to consider the constraint in (20) anymore as the large market property eliminates the competition effect between buyers. To choose the optimal reserve price, equation (23) shows that seller i is doing a linear programming problem. So r^* should take value 0 or 1. However, the convergence property and equation (21) implies $r^* = 0$ for any $q^* \in (0, 1)$. For the equilibrium advertising intensity we take the derivative with respect to q_i in (23) and set $q_i = \hat{q}$. We then have the following condition

$$\phi^2 e^{-\phi} = q^* C'(q^*). \tag{24}$$

It is also easy to check that (22) converges to (24) the convergence of (21) that $r^* = 0$.

Proposition 2 *In the large market where $M, N \to \infty$ and $N/M = \phi$, the equilibrium reserve price $r^* = 0$ and the equilibrium advertising intensity is characterized by*

$$\phi^2 e^{-\phi} = q^* C'(q^*).$$

The equilibrium price and advertising intensity in the large market are the limits of the corresponding ones in the finite market.

The equilibrium condition (24) can be rewritten as follows

$$\frac{\phi^2 e^{-\phi}}{q} = C'(q). \tag{25}$$

The left hand side (LHS) of equation (25) is the marginal revenue of advertising while the right hand side (RHS) is the marginal cost of advertising. Since the reserve price converges to zero in the large market, a seller makes positive profit only when there are at least two buyers selecting her. The marginal revenue thus coincides with the additional probability of attracting more than two buyers by increasing advertising intensity. This additional probability decreases as the advertising intensity increases. When sellers increase advertising, more consumers become (partially) informed and obtain access to the market. Also, the inframarginal consumer is aware of more sellers and put less probability on visiting each seller in their information set. The former effect unambiguously increases the revenue of the seller and always dominates the later effect. However, the marginal benefit diminishes as the prevailing advertising level increases in the market since the number of inframarginal buyers is large.

A primary focus of the standard directed search model with perfect signals is the correlation between the equilibrium price and the market tightness. With auctions we show that reserve price goes to zero in the large market and therefore has no correlation with the market tightness. The imperfect observability of signals in our model gives rise to another interesting question: how does the equilibrium advertising intensity change when market tightness increases? Or in other words, does market transparency improve if there are more buyers available? Proposition 3 shows that the equilibrium advertising intensity is inverted U-shape in market tightness. The advertising intensity is increasing when the number of buyers is small relative to the number of sellers. After reaching a certain level of market tightness, the advertising intensity starts to decrease.

Proposition 3 *The equilibrium advertising intensity q^* is of inverted U-shape in the market tightness ϕ. In particular, when $\phi < 2$, $dq^*/d\phi > 0$ and $\phi > 2$, $dq^*/d\phi < 0$.*

For illustrative purposes let us consider a linear cost structure where $C'(q) = 1$. When the number of buyers is relatively small, sellers are less willing to invest in advertising as the probability of attracting buyers is small. Let us now consider an

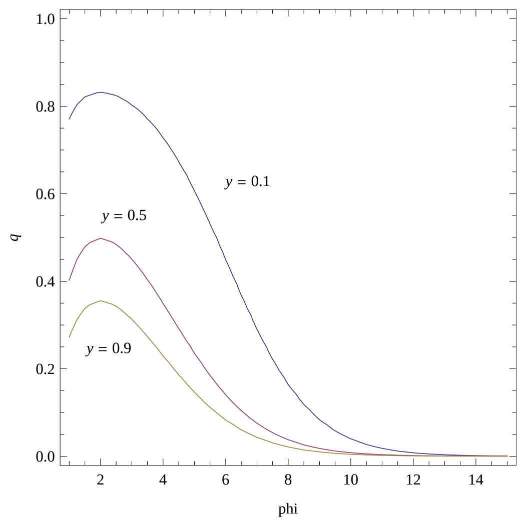

Figure 2: Relation between q^* and ϕ when $x = 0.6$ and $y = 0.1, 0.5, 0.9$.

increase of market tightness ϕ so that there are relative more buyers than sellers. Having more buyers drastically increases the probability of making sales. Since the stock of informed buyers for each firm is small, increasing advertising does not significantly affect the profit from the marginal visitor. Hence, sellers increase advertising level to inform more buyers in response to the change. When the number of available buyers reach a certain level and the advertising intensity is sufficiently high, maintaining the original advertising intensity yields a marginal profit lower than the marginal cost. A seller only needs to attract two buyers. When there is a very large number of available buyers, the marginal benefit of advertising is low. Thus, a buyer reduces her advertising level after she is almost assured that two buyers will visit her. Figure 2 plots the equilibrium advertising intensities in the large market when cost function of advertising is CRIR. It is clear from Figure 2 that advertising has an inverted U-shape.

4 Alternative Sales Mechanism: Price-Posting

When the trading protocol is an auction, sellers only partially commit to the posted price. The ex-post bidding enables sellers to fully extract the surplus whenever there is more than one buyer visiting a particular seller. However, in many markets, after-matching haggling is not allowed and sellers fully commit to the posted price. This section explores these types of markets and determine how robust are the results under the auction mechanism.

In this new environment, whenever there is local excessive demand (more than one buyer visiting a seller), the allocation problem is solved by an equal-chance rationing rule. Thus, both the seller and buyer can obtain positive surplus. This kind of price-posting mechanism has been thoroughly analyzed in Peters (1984, 2000) and Burdett, Shi and Wright (2001) in environments where signals are costless and perfectly observable to all buyers. By considering imperfect observability and costly advertising, this paper asses the robustness of the results in the directed search literature.

4.1 Advertising equilibrium

Sellers simultaneously choose the advertising intensity q and price p. A seller will commit to her posted price no matter how many buyers select her. Then all informed buyers independently choose which seller to visit. Finally, if n buyers select a seller, a rationing rule assigns each buyer a probability of $1/n$ obtaining the good. The advertising technology and belief formation are the same as in the auction mechanism case.

Let us first consider a finite number of sellers and buyers. A seller always makes profits p if she makes a sale. A buyer obtains surplus $1 - p$ if she buys the good at price p. Now consider the buyers' selection game. As in the previous section we focus on symmetric equilibrium and treat seller i as the single potential deviator. Then we only need to consider the problem faced by an informed buyer who receives seller i's ads. Call this agent "buyer 1". If seller i is the only seller that buyer 1 observes, the problem is trivial as buyer 1 has no other choices but selecting seller i. We assume buyer 1 observes at least another non-deviating seller's ads. Buyer 1's trade-off in visiting seller i is between the price and the probability of trade. As discussed in the previous section, the probability that any other buyer showing up at seller i is given by $\mu_i(\mathcal{I})\Lambda_i$, where μ_i the conjecture on seller i's advertising intensity and Λ_i is the conditional probability to select seller i. Thus, buyer 1 should believe the probability of trading with seller i is given by

$$\Omega_i = \frac{1 - (1 - \mu_i(\mathcal{I}_1)\Lambda_i)^N}{N\mu_i(\mathcal{I}_1)\Lambda_i}. \tag{26}$$

Notice that the numerator of equation in (26) is the expected number of sales of seller i, while the denominator is the expected queue length at seller i. Since each buyer selecting seller i has equal chance to obtain the good, the probability of getting the good from seller i is just the expected number of sales of seller i divided by the expected queue length at seller i. Then, buyer 1's expected payoff from visiting seller i is given by

$$U_i = \frac{(1-p_i)\left[1-(1-\mu_i(\mathcal{I}_1)\Lambda_i)^N\right]}{N\mu_i(\mathcal{I}_1)\Lambda_i}. \tag{27}$$

Similarly, we can write the expected payoff from visiting a non-deviating seller, seller j, conditional on receiving ads from seller j. Seller j charges symmetric equilibrium prices p and buyers' conjecture of her advertising intensity is $\hat{q}$. The expected payoff of visiting seller j is then

$$U_j = \frac{(1-p)\left[1-(1-\mu_j(\hat{q}\Lambda)^N\right]}{N\hat{q}\Lambda}. \tag{28}$$

Since all sellers other than seller i are following a symmetric equilibrium strategy, then the expected payoff from visiting any non-deviating seller, U, equals to U_j. Whenever buyer 1 receives at least two ads including seller i's, she is assumed to use a mixed strategy when selecting over all observed sellers. The induced indifference condition is then given by

$$U_i = U. \tag{29}$$

Combining this indifference condition with Lemma 1, we can solve for Λ_i in terms of $\mu_i(\mathcal{I}_1)$,$\hat{q}$, p_i and p. However, in order to solve for the equilibrium condition, we do not need the explicit expression of Λ_i.

Let us now we study the seller's pricing and advertising strategies. When sellers commit to the posted prices, they only care about the probability of attracting at least one visitor. This is the case as her profits do not change even if more than one buyer visits her. For the potential deviator seller i, the profit maximization problem is given by

$$\max_{q_i,p_i} p_i\left[1-(1-q_i\Lambda_i)^N\right] - C(q_i) \;\text{ s.t. }\; U = U_i.$$

As in the last section, seller i's choice is constrained by buyers selecting behavior. Define $\rho = 1 - q^*\Lambda = 1 - (\tau/M)$ as the equilibrium probability that a buyer does not choose a particular seller. Proposition 4 characterizes the symmetric advertising equilibrium (SAE) in the finite market.

Proposition 4 *When the trading mechanism is price-posting in a finite economy, a symmetric advertising equilibrium (SAE) exists. The equilibrium price and advertising intensity are jointly determined by*

$$p^* = \frac{M(1-\rho^N) - \rho^{N-1}(1-\rho)MN}{M(1-\rho^N) - N\rho^{N-1}(1-\rho)}, \tag{30}$$

$$\phi\tau\rho^{N-1}p^* - q^*C'(q^*) = 0. \tag{31}$$

It is worth noticing that the situation whereby every seller chooses $(q, p) = (0, 0)$ is not an equilibrium outcome even if it satisfies the previous conditions. Note that when all other sellers choose $(q, p) = (0, 0)$, a seller can choose a positive price p and an arbitrarily small but positive advertising intensity q so that some buyers will be informed. These buyers will select this seller with probability 1 as this seller is the only seller in their information sets. Thus, the marginal revenue of the deviator is strictly positive. This is a profitable deviation from $(q, p) = (0, 0)$ given the fact that $C'(0) = 0$. When every seller chooses $q = 1$ so that $\rho = 1 - (1/M)$, the market becomes perfectly transparent in the sense that every buyer observes all posted prices in the market. Rearranging some terms, condition (30) can be written as follows

$$p^* = \frac{(M-1) - (\frac{M-1}{M})^{M\phi}(M-1+M\phi)}{(M-1) - (\frac{M-1}{M})^{M\phi}(M-1+\phi)}. \tag{32}$$

This is the equilibrium price derived in the standard directed search model with price-posting.[14]

Similarly as in the last section with the auction mechanism, we explore the large market property of the equilibrium price and advertising intensity. We are especially interested in the relation between equilibrium advertising intensity and market tightness, and also the comparison between the equilibrium advertising intensities under auction mechanism and under price-posting. Proposition 5 characterizes the large market equilibrium with price-posting and shows that the inverted U-shape result still holds.

Proposition 5 *When the trading mechanism is price-posting in the large market, the unique equilibrium price and advertising intensity are characterized by*

$$p^* = 1 - \frac{\phi}{e^{\phi} - 1};, \tag{33}$$

$$q^*C'(q^*) = \frac{\phi e^{-\phi}(1 - e^{-\phi} - \phi e^{-\phi})}{1 - e^{-\phi}}. \tag{34}$$

[14]See Burdett, Shi and Wright, 2001, page 1071, equation 15.

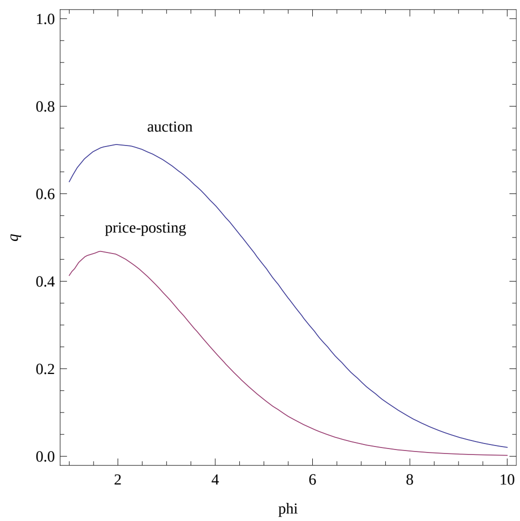

Figure 3: Comparison of equilibrium advertising intensities when $x = 0.2$ and $y = 0.6$.

The equilibrium advertising intensity q^* is inverted U-shape in market tightness ϕ.

Notice that the equilibrium price in the large economy coincides with the equilibrium price in the standard directed search model, which is independent of the equilibrium advertising intensity. As in the auction mechanism, when the market tightness increases, the equilibrium advertising intensity increases initially and eventually decreases. From Proposition 5, we also see that the market tightness corresponding to the maximum advertising ($\phi = 1.669$) is smaller than the one under auction ($\phi = 2$).

Proposition 6 *Compared to the equilibrium advertising intensity in a market with auctions, the equilibrium advertising intensity q^* with price-posting market is always lower for all market tightness ϕ.*

Thus, Proposition 6 shows that sellers invest more in advertising under an auction mechanism. When advertising has a CRIR cost function, Figure 3 plots the equilibrium advertising intensities under both mechanisms.

In the large-market, the probability that a seller has at least one visitor is given by

$$\Pr[n \geq 1] = \lim_{M,N\to\infty} \left[1 - (1 - q^*\Lambda^*)^N\right] = 1 - e^{-\phi}. \tag{35}$$

The probability that a seller has at least two visitor is given by

$$\Pr[n \geq 2] = \lim_{M,N\to\infty} \left[1 - (1 - q^*\Lambda^*)^N - Nq^*\Lambda^*(1 - q^*\Lambda^*)^{N-1}\right] = 1 - e^{-\phi} - \phi e^{-\phi}. \quad (36)$$

Then, the equilibrium revenue for a seller under auction mechanism is

$$\Pi^A = 1 \cdot \Pr[n \geq 2] = 1 - e^{-\phi} - \phi e^{-\phi}. \quad (37)$$

The equilibrium revenue for a seller under price-posting mechanism is

$$\Pi^P = p^* \cdot \Pr[n \geq 1] = 1 - e^{-\phi} - \phi e^{-\phi}. \quad (38)$$

Therefore, sellers have the same expected profits regardless which mechanism, auction or price-posting, is used. Following Proposition 6 we can establish the following corollary.

Corollary 1 *Compared to the market that has a price-posting mechanism, sellers in using auctions have strictly lower expected profit.*

4.2 Price and informed buyers

A large number of papers in the search literature, including directed search models (Lester, 2011), noisy search models (Burdett and Judd, 1983), sequential models with homogeneous products (Stahl, 1989) and sequential search models with differentiated products (Anderson and Renault, 2000), have studied the relationship between the equilibrium prices and measures of informed buyers. The conventional wisdom suggests that prices should be lower the larger the proportion of informed buyers. It is often argued that higher number of informed buyers intensifies competition amongst sellers, thus reducing prices. This argument is derived in Salop and Stiglitz (1977), Varian (1980), Burdett and Judd (1983), Stahl (1989) among others. The key features that deliver the conventional wisdom are: (i) products are homogeneous, (ii) uncertainty is only regarding prices, (iii) sellers do not suffer from capacity constraints, and (iv) the market structure tends to be oligopolistic.

Two exceptions worth mentioning are those of Anderson and Renault (2000) and Lester (2011). Anderson and Renault (2000) analyze a duopoly sequential search model with differentiated products where uninformed buyers do not know their match values. In this environment uninformed buyers create a positive

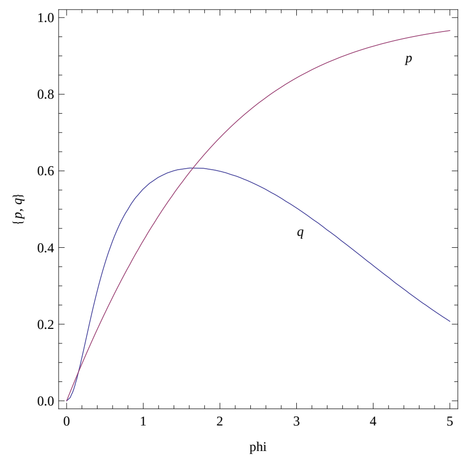

Figure 4: Co-movement of p^* and q^* when $x = 0.6$ and $y = 0.8$.

externality so that the price goes down when more uninformed buyers enter the market. Lester (2011) on the other hand focuses on homogeneous products. In Lester's environment uninformed buyers can still participate through random search while facing capacity constrained sellers. Informed buyers are directed by posted prices. The author shows that in a finite market, an increase in the fraction of informed buyers can cause prices to decrease, remain constant, or even increase. These results critically depend on parameter values capturing the fraction of uninformed buyers. However, in the large market, prices never increase when the number of informed buyers increases.

Relative to Lester (2011) Anderson and Renault (2000), our analysis differs in that the market number of informed buyers is endogenous. Corollary 2 shows that when market number of informed buyers is endogenized, the equilibrium price and number of informed buyers can positively co-move even for large markets.

Corollary 2 *If advertising has a CRIR cost function then values for market tightness such that $\phi \leq 1.669$, we find that p^* and q^* jointly increase as ϕ increases. When market tightness is such that $\phi > 1.669$, then p^* increases but q^* decreases as ϕ increases.*

This result implies that when the market tightness is low, larger number of informed buyers is associated with higher prices. Moreover, when the market tightness is high, a larger number of informed buyers is associated with a lower price.

5 Efficiency

An important issue in the advertising literature is whether there exists over- or under-advertising in equilibrium. In his seminal paper, Butters (1977) had a striking finding that the equilibrium advertising intensity is always socially optimal. Subsequently, Stegeman (1991) and Stahl (1994) reported that the equilibrium advertising is inadequate. All these models consider sellers with unlimited capacity. An important application of the framework presented here are labor markets where sellers (firms) have only limited capacities (vacancies). In this section, we discuss the normative perspective of advertising in the presence of capacity constraint.

5.1 Advertising efficiency without free-entry

We first consider a social planner whose objective is to maximize social welfare. The social planner can potentially control both the advertising intensity and the buyers' entry.[15] Buyers' entry is costly. Each buyer incurs a sunk cost $k > 0$ to enter the market. Within the context of the labor market, the entry cost can be thought as costly education or an investment on skills.

After entering, each buyer may or may not receive ads from sellers. Once a buyer receives at least one advertisement and therefore becomes informed, the social planner can suggest him a seller in his information set to visit. However, when making the recommendation, the social planner is also constrained by the symmetry and anonymity constraints. This restriction on welfare maximization is consistent with the notion of constrained efficiency in directed search literature. Finally, we assume that the social planner cannot affect buyers' belief about advertising intensities.

We assume that the auction mechanism is used in the market. Once a seller is successfully matched with buyers, one unit of surplus is realized. The social planner's problem is then to maximize each seller's matching probability while taking both advertising cost and entry cost into account. We first consider the case where the social planner can only control the advertising intensities and the market tightness is fixed at ϕ. The social planner's maximization problem is given

[15]The entry from seller (firm) side can be discussed in a fully specified labor market model. After entry, firms post wages and send advertisements. An informed worker apply to all firms he observes. Then each firm who receives at least one application chooses a work to offer. If a worker receives multiple offers, he holds auction and let firms bid for his service.

by

$$\max_{q}\ (1-e^{-\frac{q\phi}{q^*}})-C(q). \tag{39}$$

Notice that here the passive belief is assumed to be the one consistent with the equilibrium choice, $\mu_j(\mathcal{I})=q^*,\forall j,\mathcal{I}$. The first-order condition yields

$$\frac{\phi}{q^*}e^{-\frac{q^\dagger\phi}{q^*}}=C'(q^\dagger), \tag{40}$$

where $q^\dagger$ is the efficient level of advertising.

Recall that the equilibrium advertising intensity q^* under auction mechanism satisfies

$$\frac{\phi^2e^{-\phi}}{q^*}=C'(q^*). \tag{41}$$

Proposition 7 compares the equilibrium level of advertising to the efficient level under auction mechanism.

Proposition 7 *When $\phi>1$, there exists excessive advertising, i.e. $q^*>q^\dagger$. When $\phi<1$, there exists inadequate advertising, i.e. $q^*<q^\dagger$. When $\phi=1$, the equilibrium advertising is also efficient, i.e. $q^*=q^\dagger$.*

5.2 Advertising efficiency with free-entry

Now we analyze the case where the buyer incurs an entry cost $k\in(0,1)$ to be able to participate in the market. The social planner can control both advertising intensities and the number of buyers entering the market. Since the entry is from the buyer side, the total surplus can be calculated on the per seller basis. The social planner's problem is then to choose q and ϕ to do the following maximization:

$$\max_{\phi,q}\ (1-e^{-\frac{q\phi}{q^*}})-C(q)-\phi k. \tag{42}$$

The first-order conditions are

$$\frac{q^\dagger}{q^*}e^{-\frac{q^\dagger\phi^\dagger}{q^*}}=k, \tag{43}$$

$$\frac{\phi^\dagger}{q^*}e^{-\frac{q^\dagger\phi^\dagger}{q^*}}=C'(q^\dagger), \tag{44}$$

where $q^\dagger$ and $\phi^\dagger$ are the efficient levels of advertising intensity and market tightness.

Now consider an equilibrium with free-entry. After entry, a buyer's expected payoff is e^{ϕ} if the market tightness is ϕ. The free-entry condition forces zero profit to the potential entrants. That is

$$e^{-\phi^*} = k. \tag{45}$$

where ϕ^* denotes the equilibrium market tightness with free entry. The equilibrium condition of advertising is

$$(\phi^*)^2 e^{-\phi^*} = q^* C'(q^*). \tag{46}$$

Thus, sellers' choice in a free-entry equilibrium, (ϕ_a^*, q_a^*), is characterized by (45) and (46). Notice that equation (45) uniquely determines ϕ^*. Since the RHS of the equilibrium advertising condition is strictly increasing, q_a^* is also uniquely determined. Now we can see if the equilibrium outcome (ϕ_a^*, q_a^*) can replicate social planner's solution $(\phi^\dagger, q^\dagger)$ which satisfies (44) and (43).

Proposition 8 *In a free entry equilibrium, the market tightness and advertising intensity will be both efficient only when $k = 1/e$.*

6 Relation to the Literature

The advertising intensities in our model can in general be classified into the search literature with explicit and endogenous search intensity. In the random matching environment, several authors have discussed the role played by search effort exerted by either sellers or buyers to form matches. For example, Shimer (2004), in a dynamic setting, modeled the search intensity as a worker's choice of the number of simultaneous applications to make. Viianto (2009) considered a similar search intensities from buyer side and showed the non-monotonic relationship between the market tightness and the equilibrium search intensity. More closely, Kaas (2010) considered an urn-ball matching with costly and continuous search intensities chosen by sellers. However, the author did not focus on the relationship between the search intensity and the market tightness.

We differ from previous work in that search is not random and informed buyers are directed by sellers' ads. Advertising, as one kind of search intensity, does not obviously increase the probability of being matched in equilibrium as we have shown. In contrast, high advertising intensity may lower the attractiveness of a particular seller as the informed buyer expects more intense local competition.

Besides the random matching literature, there has also been substantial effort being devoted in incorporating search intensities into directed search models. Albrecht, Gautier and Vroman (2006) extend the standard directed search model to a situation where workers can send multiple applications and buyers who receive multiple offers run an auction to sell his service. Galenianos and Kircher (2009) consider a similar setting but firms post wages and commit to them. Lester (2010) studies a directed search model where each firm decides how many vacancies to create. A firm with more vacancies attracts more applications. Each of these departures from the standard model are one way to explicitly model search intensities while still allow search to be directed. However, in these models the search intensities are usually discrete variables. In contrast in our environment, we model advertising intensity as the probability of reach which is continuous and can alternatively be understood as a seller's expenditure on searching. One of the advantages of using continuous variables to model search intensity is the ease by which the equilibrium can be analytically characterized. Further, discrete-choice models are less convenient for estimation purposes, so that a continuous measure for search intensities with differentiable cost function may be more appealing (such an application can be found in Wolthoff, 2011).

In all of the previous papers, search intensities can be measured by monetary expenditure. The more money an agent spends, the more applications a worker can send or the more vacancies a firm can create. Menzio (2007) studies the cheap-talk information transmission in a competitive matching market. A continuum of heterogeneous sellers send ads which contain information regarding their productivity. Every buyer observes all ads and choose which seller to visit. Sellers do not commit to the information they send but can choose how informative their ads are. Menzio found that unless the market tightness is not too small or not too large, there exists an equilibrium where the content of communication is positively correlated with the actual productivity. Otherwise the equilibrium features only uninformative advertising. In other words, Menzio's model shows that the informativeness of advertising has an inverted U-shape in market tightness. Moreover, buyers search randomly in equilibrium when the market tightness is either too high or too low. In our model, sellers commit to the information they send. But advertising intensity in our model affects the reach of buyers. The inverted U-shape advertising intensity we derive gives an interesting analogy to Menzio's under a different setting of advertising. Informed buyers' search are always directed for any market tightness as we exclude the possibility for uninformed buyers to search randomly.

Our results also relate to the literature of competitive search that are now widely used in labor markets and search based models of money (see Shimer 1996, Moen, 1997 and Rocheteau and Wright, 2005). Competitive search models differ in that sellers are assumed to take the maximum expected utility buyers can obtain from selecting other sellers as given. Peters (2000) refers to this maximum expected utility as "market utility" and the sellers are then constrained by the so-called "market utility property". In directed search as we consider in this paper, sellers have explicit market power and the deviation by a seller affects the expected utility a buyer can get from other sellers. However, it is well known that in the limit, the market power vanishes and the allocations of competitive search and directed search are equivalent.[16] In our model, under the assumption buyers' beliefs are passive about advertising, the observed price deviation gives rise to the standard trade-off on price and matching probability. Thus, as the market gets large, the "market utility property" still holds in our analysis and our results applies to the competitive search environments.

7 Conclusion

In this paper we consider a directed search model where buyers can not perfectly observe prices posted by where sellers who need to use costly advertising to reach buyers. We show that the advertising intensities are lower if there are too many sellers or too many buyers, and higher if the number of sellers and number of buyers are balanced. In a large market, the equilibrium matching rate for a seller depends only on the market tightness. In this sense, the standard directed search models are robust to having sellers send imperfect costly signals. We also compare different trading protocols and find that equilibrium advertising intensities are higher under auctions than under price-posting. Most interestingly, when the market informativeness is endogenized, the equilibrium price and the equilibrium measure of informed buyers can positively co-move even in the large market.

References

[1] Albrecht, J., P. Gautier and S. Vroman (2006): "Equilibrium Directed Search with Multiple Applications", *Review of Economic Studies*, Vol. 73(4), 869-891.

[16] Galenianos and Kircher (2010) provide an explicit analysis of this equivalence.

[2] Anderson, S. and R. Renault (2000): "Consumer Information and Firm Pricing: Negative Externalities from Improved Information", *International Economic Review*, Vol. 41(3), 721-742.

[3] Bagwell, K. (2007): "The Economic Analysis of Advertising", *Handbook of Industrial Organization*, Vol. 3, 1701-1844, North-Holland, Amsterdam.

[4] Burdett, K., S. Shi and R. Wright (2001): "Pricing and Matching with Frictions", *Journal of Political Economy*, Vol. 109, 1060-1085.

[5] Burdett, K. and K. Judd (1983): "Equilibrium Price Dispersion", *Econometrica*, Vol. 51(4), 955-969.

[6] Butters, G. (1977): "Equilibrium Distribution of Sales and Advertising Prices", *Review of Economic Studies*, Vol. 44(3), 465-91

[7] Diamond, P. (1982): "Aggregate Demand Management in Search Equilibrium", *Journal of Political Economy* Vol. 90(5), 881-94.

[8] Eaton, C., I. McDonald and L. Meriluoto (2001): "Existence Advertising, Price Competition and Asymmetric Market Structure", *The B.E. Journal of Theoretical Economics*, Vol. 10(1) (Contributions), Article 39.

[9] Galenianos, M. and P. Kircher (2009): "Directed Search with Multiple Job Applications", *Journal of Economic Theory*, Vol. 144(2), 445-471.

[10] Galenianos, M. and P. Kircher (2012): "On the Game-Theoretic Foundations of Competitive Search Equilibrium", *International Economic Review*, Vol. 53(1), 1-21.

[11] Grossman, G. and C. Shapiro (1984): "Informative Advertising with Differentiated Products", *Review of Economic Studies*, Vol. 51(1), 63-81.

[12] Ireland, P. (1993): "The Provision of Information in a Bertrand Oligopoly", *Journal of Industrial Economics*, Vol. 41(1), 61-76.

[13] Julien, B., J. Kennes and I. King (2000): "Bidding for Labor", *Review of Economic Dynamics,* Vol. 3, 619-649.

[14] Julien, B., J. Kennes and I. King (2001): "Auctions and Posted Prices in Directed Search Equilibrium", *Topics in Macroeconomics,* Vol. 1(1), Article 1.

[15] Julien, B., J. Kennes and I. King (2008): "Bidding for Money", *Journal of Economic Theory,* Vol. 142(1), 196-217.

[16] Kaas, L. (2010): "Variable Search Intensity with Coordination Unemployment", *The B.E. Journal of Macroeconomics,* Vol. 10(1), Article 31.

[17] Lester, B. (2010): "Directed search with multi-vacancy firms", *Journal of Economic Theory,* Vol. 145(6), 2108-2132.

[18] Lester, B. (2011): "Information and Prices with Capacity Constraints", *American Economic Review,* Vol. 101(4), 1591-1600.

[19] McAfee, P. (1993): "Mechanism Design by Competing Sellers", *Econometrica,* Vol. 61(6), 1281-1312.

[20] Menzio, G. (2007): "A Theory of Partially Directed Search", *Journal of Political Economy,* Vol. 115, 748-769.

[21] Moen, E. (1997): "Competitive search equilibrium", *Journal of Political Economy,* Vol. 105(2), 385-411.

[22] Montgomery, J. (1991): "Equilibrium Wage Dispersion and Interindustry Wage Differentials" *Quaterly Journal of Economics,* Vol. 106(1), 163-179.

[23] Mortensen, D. and C. Pissarides (1994): "Job Creation and Job Destruction in the Theory of Unemployment", *Review of Economic Studies,* Vol. 61(3), 397-415.

[24] Peters, M. (1984): "Bertrand Equilibrium with Capacity Constraints and Restricted Mobility", *Econometrica* Vol. 52(5), 1117-1127.

[25] Peters, M. (1991): "Ex Ante Pricing in Matching Games: Non Steady States", *Econometrica* Vol. 59(5), 1425-1454.

[26] Peters, M. (2000): "Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search", *Journal of Economic Theory* Vol. 95(2), 139-168.

[27] Peters, M. and S. Severinov (1997): "Competition Among Sellers who offer Auctions instead of Prices", *Journal of Economic Theory* Vol. 75(1), 141-179.

[28] Robert, J. and D. O. Stahl (1993): "Informative Price Advertising in a Sequential Search Model", *Econometrica*, Vol. 61, 657-86.

[29] Rocheteau, G. and R. Wright (2005): "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium", *Econometrica*, Vol. 73, No. 1, pp. 175-202.

[30] Rubinstein, A. and A. Wolinsky (1985): "Equilibrium in a Market with Sequential Bargaining", *Econometrica*, Vol. 53(5), 1133-1150.

[31] Schmalensee, R. (1989) "Inter-industry Studies of Structure and Performance", in: R. Schmalensee and R. Willig, eds., *Handbook of Industrial Economics*, 952-1009, North-Holland, Amsterdam.

[32] Shi, S. (2001): "Frictional Assignment. Part I. Efficiency", *Journal of Economic Theory*, Vol. 98(2), 232-260.

[33] Shi, S. (2005): "Frictional Assignment, Part II: Infinite Horizon and Inequality", *Review of Economic Dynamics*, Vol. 8(1), 106-137.

[34] Shimer, R. (1996): "Contracts in a Frictional Labor Market", *manuscript*, MIT.

[35] Shimer, R. (2005): "The Assignment of Workers to Jobs in an Economy with Coordination Frictions", *Journal of Political Economy*, Vol. 113(5), 996-1025.

[36] Stahl, D. (1989): "Oligopolistic Pricing with Sequential Consumer Search", *American Economic Review* Vol. 79(4), 700-712.

[37] Varian, H. (1980): "A Model of Sales", *The American Economic Review*, Vol. 70, 651-659.

[38] Viianto, L. (2009): "Directed Search with Endogenous Search Effort", *working paper*.

[39] Wolthoff, R. (2011): "Applications and Interviews. A Structural Analysis of Two-Sided Simultaneous Search", *Working Papers*, Fondazione Eni Enrico Mattei.

8 Appendix I

Proof of Lemma 1.
By definition, λ is the expected probability that a buyer selects a non-deviator contingent on receiving this seller's ads. A buyer is believed to observe the deviator's ads with probability $\mu_i(\mathcal{I})$. Therefore, Λ can be written as

$$\begin{aligned}\Lambda = \mu_i(\mathcal{I}_1) \sum_{k=0}^{M-2} \left[\binom{M-2}{k} \cdot (\hat{q})^k \cdot (1-\hat{q})^{M-2-k} \cdot \frac{1-\sigma_i(\mathcal{I},\mu)}{k+1} \right] \\ + (1-\mu_i(\mathcal{I})) \sum_{k=0}^{M-2} \left[\binom{M-2}{k} \cdot (\hat{q})^k \cdot (1-\hat{q})^{M-2-k} \cdot \left(\frac{1}{k+1} \right) \right].\end{aligned} \tag{47}$$

The size of some other buyer's information set, $\mathcal{I}$, is given by $k+2$ if r_i is observed and $k+1$ otherwise. The first term is the expected probability that a buyer is believed to observe the deviating price r_i and some other non-deviating prices r including seller l's, but selects seller l. The second term is the expected probability that a buyer is believed not to observe r_i but some other prices r including l's, and finally selects seller l. When a buyer observes the k prices in total including the deviating price, he selects a non-deviator with probability $(1-\sigma_i(\mathcal{I},\mu))/(k-1)$. Otherwise, he selects a non-deviator with probability $1/k$. Use the following fact for the second term

$$\sum_{k=0}^{M-2} \left[\binom{M-2}{k} \cdot (\hat{q})^k \cdot (1-\hat{q})^{M-2-k} \cdot \frac{1}{k+1} \right] = \frac{1-(1-\hat{q})^{M-1}}{(M-1)\hat{q}}, \tag{48}$$

and the fact $\sigma_i(\mathcal{I},\mu) = 1$ when r_i is the only element in $\mathcal{I}$, we obtain

$$\begin{aligned}
\Lambda = {} & \mu_i(\mathcal{I}_1)\sum_{k=0}^{M-2}\left[\binom{M-2}{k}\cdot(\hat{q})^k\cdot(1-\hat{q})^{M-2-k}\cdot\frac{1}{k+1}\right] \\
& -\mu_i(\mathcal{I}_1)\sum_{k=0}^{M-2}\left[\binom{M-2}{k}\cdot(\hat{q})^k\cdot(1-\hat{q})^{M-2-k}\cdot\frac{\sigma_i(\mathcal{I},\mu)}{k+1}\right] \\
& -\mu_i(\mathcal{I}_1)\sum_{k=0}^{M-2}\left[\binom{M-2}{k}\cdot(\hat{q})^k\cdot(1-\hat{q})^{M-2-k}\cdot\frac{1}{k+1}\right]+\frac{1-(1-\hat{q})^{M-1}}{(M-1)\hat{q}} \\
= & -\mu_i(\mathcal{I}_1)\sum_{k=0}^{M-2}\left[\binom{M-2}{k}\cdot(\hat{q})^k\cdot(1-\hat{q})^{M-2-k}\cdot\frac{\sigma_i(\mathcal{I},\mu)}{k+1}\right]+\frac{1-(1-\hat{q})^{M-1}}{(M-1)\hat{q}} \\
= & -\mu_i(\mathcal{I}_1)\left[\frac{1}{(M-1)\hat{q}}\sum_{k=0}^{M-1}\left[\binom{M-1}{k}\cdot(\hat{q})^k\cdot(1-\hat{q})^{M-1-k}\cdot\sigma_i(\mathcal{I},\mu)\right]-\frac{(1-\hat{q})^{M-1}}{(M-1)\hat{q}}\right] \\
& +\frac{1-(1-\hat{q})^{M-1}}{(M-1)\hat{q}} \\
= & -\frac{\mu_i(\mathcal{I}_1)\Lambda_i}{(M-1)\hat{q}}+\frac{1-(1-\mu_i(\mathcal{I}_1))(1-\hat{q})^{M-1}}{(M-1)\hat{q}}.
\end{aligned}$$

Note that in the transformation from the second equation to the third equation, we use the fact that $\sigma_i(\mathcal{I},\mu) = 1$ if $k = 0$. Rearrange the equation, the result in Lemma 1 follows. ■

Proof of Proposition 1.
Consider the potential deviator seller i's maximization problem (20). The first-order condition w.r.t r_i is given by

$$\frac{\partial(q_i\Lambda_i)}{\partial r_i} = \frac{-q_i\Lambda_i(1-q_i\Lambda_i)}{(Nq_i\Lambda_i-1)(1-r_i)+(1-q_i\Lambda_i)}. \tag{49}$$

A symmetric equilibrium requires that seller i does not have a profitable deviation from the equilibrium strategy (r^*, q^*). Under a passive belief setting, $\mu_i\mathcal{I}$ is not affected by the deviating reserve price r_i. Also, buyers' belief should be correct on the equilibrium path, i.e., $\mu_i(\mathcal{I}) = \hat{q} = q^*$. Substitute in these expressions, we obtain

$$q_i\Lambda_i|_{r_i=r^*,q_i=q^*} = \frac{1-(1-q^*)^M}{M}.$$

Take the total derivative of (17) and substitute in r_i, q_i, we derive

$$\frac{\partial(q_i\Lambda_i)}{\partial r_i}\Big|_{r_i=r^*,q_i=q^*} = \frac{-(M-1)^2-(M-1)\left(1-q^*\right)^M}{M^2(N-1)(1-r^*)}.$$

Substitute the above two expressions into the first-order condition, the equilibrium reservation price r^* can then be written as a function in term of q^*

$$r^* = \frac{(N-1)\left[1-(1-q^*)^M\right]}{M(M-1)+(N-M)\left[1-(1-q^*)^M\right]} \tag{50}$$

To derive the equilibrium advertising rate, differentiate the objective function w.r.t q_i. Notice that Λ_i does not depend on q_i. We obtain

$$r_iN\Lambda_i\left(1-q_i\Lambda_i\right)^{N-1}+(1-r_i)N(N-1)q_i\Lambda_i^2\left(1-q_i\Lambda_i\right)^{N-2}=C'(q_i) \tag{51}$$

Let $\mu_i(\mathcal{I}) = \hat{q} = q^*$ and $r_i = r = r^*$. Substituting $q_i\Lambda_i|_{r_i=r^*,q_i=q^*} = \frac{1-\left(1-q^*\right)^M}{M}$ into this first-order condition, the equilibrium advertising rate q^* is characterized by

$$\begin{pmatrix} r^*\frac{N[1-(1-q^*)^M]}{Mq^*}\left[1-\frac{1-(1-q^*)^M}{M}\right]^{N-1}+ \\ (1-r^*)\frac{N(N-1)[1-(1-q^*)^M]^2}{M^2q^*}\left[1-\frac{1-(1-q^*)^M}{M}\right]^{N-2}\end{pmatrix}=C'(q^*). \tag{52}$$

Let $\tau = 1-(1-q)^M$ and $\phi = N/M$, this condition further simplifies to

$$r^*\frac{\phi\tau}{q^*}\left(1-\frac{\tau}{M}\right)^{N-1}+(1-r^*)\,\frac{\phi(\phi-\frac{1}{M})\tau}{q^*}\left(1-\frac{\tau}{M}\right)^{N-2}=C'(q^*). \tag{53}$$

Existence: Note that the RHS of equation (53) is continuous over [0,1]. Furthermore, we have $\lim_{q\to 0}\tau/q = M$ and $\lim_{q\to 1}\tau/q = 1$. The RHS goes to $\phi(\phi-1/M)M$ when q goes to 0 and it goes to $\phi(1-1/M)^{N-2}(r+\phi-r\phi-1/M)$ when q goes to 1, where $r = (\phi-1/M)/(M+\phi-2)$. Therefore, the RHS of (53) is bounded. Then, $C'(q)\geq 0$ and $C'(1)=\infty$ guarantee that a solution q^* exists.

Uniqueness: We rewrite the RHS of condition (53) as

$$r(q)A(q)+[1-r(q)]B(q).$$

where $A(q) = \frac{\phi\tau}{q}\left(1-\frac{\tau}{M}\right)^{N-1}$ and $B(q) = \frac{\phi(\phi-\frac{1}{M})\tau}{q}\left(1-\frac{\tau}{M}\right)^{N-2}$. Differentiate it, we have

$$r'(q)[A(q)-B(q)]-r(q)A'(q)+[1-r(q)]B'(q).$$

The second and third terms are both strictly negative when $M \geq 1$ and $N > 2$. To see this, notice that both τ/q and $1 - (\tau/M)$ are decreasing in q. Especially, $d(\tau/q)/dq = \left[(1-q)^M + Mq(1-q)^{M-1} - 1\right]/q^2 < 0$ because the numerator is negative when $0 < q < 1$. In the first term, $[A(q) - B(q)]$ is non-positive if

$$M + 1 - \tau - N \geq 0.$$

Since $\tau \in [0, 1]$, this is indeed true if $N \geq M + 1$. Moreover, given that $r'(q)$ is positive, the RHS of condition (53) is negative if $M + 1 - \tau - N \geq 0$. For strictly increasing $C'(q)$, $C'(0) = 0$ and $C'(1) = +\infty$, there will be an unique intersection between $C'(q)$ and $r(q)A(q) + [1 - r(q)]B(q)$, and hence the equilibrium advertising intensity is unique. ■

Proof of Proposition 3.
Differentiate both sides of (24) in ϕ, we have

$$\phi e^{-\phi}(2 - \phi) = C''(q)q + C'(q)\frac{dq}{d\phi}.$$

Then,

$$\frac{dq}{d\phi} = \frac{q^2 \phi e^{-\phi}(2 - \phi)}{C''(q)q + C'(q)}.$$

The result follows immediately. ■

Proof of Proposition 4.
The first-order conditions w.r.t p_i and q_i are given by

$$1 - (1 - q_i\Lambda_i)^N + p_i N q_i (1 - q_i\Lambda_i)^{N-1} \frac{d\Lambda_i}{dp_i} = 0, \tag{54}$$

$$p_i N \Lambda_i (1 - q_i\Lambda_i)^{N-1} - C'(q_i) = 0. \tag{55}$$

From the indifference condition (29) and Lemma 1, we obtain

$$\frac{d\Lambda}{dp} = \frac{(1-\rho)(1-\rho^N)}{q(1-p)\left[\frac{M}{M-1}(1-\rho^N) - \frac{MN}{M-1}\rho^{N-1}(1-\rho)\right]}, \tag{56}$$

where $\rho = 1 - q\Lambda = 1 - (\tau/M)$. Substitute $d\Lambda/dp$ into (54) and set $q_i = q^*$ and $p_i = p$. Also the correct but passive belief on the equilibrium path requires $\mu_i(\mathcal{I}) = q^*, \forall \mathcal{I}$. we get the equilibrium price p^* in term of q^*

$$p^* = \frac{M(1-\rho^N) - \rho^{N-1}(1-\rho)MN}{M(1-\rho^N) - N\rho^{N-1}(1-\rho)}.$$

Then, (55) can be written as

$$p^* N\Lambda^* \left(1 - q^*\Lambda^*\right)^{N-1} = C'(q^*). \tag{57}$$

When $q \to 0$, $\Lambda \to 1$ and $p \to 1$. Thus, the LHS of (57) converges to N. When $q \to 1$, $\Lambda \to 1/M$ and p converges to a positive number

$$p^s = \frac{(M-1) - (\frac{M-1}{M})^{M\phi}(M-1+M\phi)}{(M-1) - (\frac{M-1}{M})^{M\phi}(M-1+\phi)} \in (0,1).$$

The LHS of (57) then converges to ϕp^s. Since the marginal cost function $C'(q) > 0$, a interior solution $q^* \in (0,1)$ exists. ■

Proof of Proposition 5.
Take the limits of (30) and (31), we obtain (33) and (34). Totally differentiate (34) w.r.t ϕ and rearrange, we have

$$\frac{dq}{d\phi} = \frac{e^{-\phi} f(\phi)}{\left[C''(q)q + C'(q)\right](1-e^{\phi})^2}, \tag{58}$$

where

$$f(\phi) = 1 + (1-\phi)\left[e^{2\phi} + \phi - 2e^{\phi}(1+\phi)\right]. \tag{59}$$

The sign of $dq/d\phi$ then depends solely on the sign of $f(\phi)$ as both $e^{-\phi}$ and $C''(q)q + C'(q)$ are positive. When ϕ is small, $(1-\phi)\left[e^{2\phi} + \phi - 2e^{\phi}(1+\phi)\right]$ is bounded below 1. However, after ϕ reaches 1.669, $f(\phi)$ becomes positive as $e^{2\phi}$ increases much faster than $2(1-\phi)e^{\phi}$ and $1-\phi$ is negative. This makes $f(\phi)$ negative. Then, $dq/d\phi > 0$ when $\phi \leq 1.669$ and $dq/d\phi < 0$ when $\phi \geq 1.669$. The advertising intensity is of inverted-U shape in the market tightness ϕ. ■

Proof of Proposition 6.
Rearrange equations (24) and (34) so that the RHS of both equations are $qC'(q)$,

and the LHS of (24) is then $\phi^2 e^{-\phi}$ while the LHS of (34) is $\frac{\phi e^{-\phi}(1-e^{-\phi}-\phi e^{-\phi})}{1-e^{-\phi}}$. For the same level of q, the LHS of (24) is larger than the LHS of (34) if

$$\phi \geq \frac{(1-e^{-\phi}-\phi e^{-\phi})}{1-e^{-\phi}}.$$

This is equivalent to

$$\phi - 1 - e^{-\phi} \geq 0.$$

First notice that, when $\phi = 0$, this weak inequality holds as equality. Also, $\phi - 1 - e^{-\phi}$ is continuous in ϕ. Since

$$\frac{d}{d\phi}(\phi - 1 - e^{-\phi}) = 1 - e^{-\phi} > 0,$$

we know that the inequality holds for any $\phi > 0$. Since $qC'(q)$ is strictly increasing in q, the equilibrium advertising intensity under auction mechanism is strictly higher for any ϕ. ■

Proof of Corollary 1.
Sellers' expected revenues are the same under price-posting and auction, but Proposition 6 shows sellers invest more in advertising under auction, the result then follows. ■

Proof of Corollary 2.
Notice that the equilibrium price $p^* = 1 - \frac{\phi}{e^{\phi}-1}$ is strictly increasing in ϕ. Combine this with Proposition 5, the result follows. ■

Proof of Proposition 7.
Define $\varphi = q^{\dagger}/q^*$. First suppose $\phi > 1$ and $\varphi > 1$ hold. Then we must have $e^{-\varphi\phi} < \phi e^{-\phi}$ as $e^{-\varphi\phi}$ is decreasing in φ. Compare the LHS of (44) and (41), we must have

$$C'(q^{\dagger})q^{\dagger} < C'(q^*)q^*.$$

Since $C'(q)q$ is strictly increasing in q, we have $q^{\dagger} < q^*$. This contradicts with $\varphi = q^{\dagger}/q^* > 1$. Then, when $\phi > 1$, $q^{\dagger} < q^*$. Similarly, we can prove that when $\phi < 1$, $\varphi > 1$. Finally, when $\phi = 1$, q^* must equal to $q^{\dagger}$, given that $C'(\cdot)$ is non-deceasing. ■

Proof of Proposition 8.

The simplest way to check the claim in Proposition 8 is to set $q^* = q^\dagger$ and see if there exists a single ϕ^* which satisfies equation (43), (44), (45) and (46). When $q^* = q^\dagger$, (44) coincides with (46) if and only if $\phi = 1$. When $\phi = 1$, (43) can be satisfied only when $k = 1/e$. Therefore, $q^* = q^\dagger$, $\phi^* = \phi^\dagger$ only when $k = 1/e$. ■