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ABSTRACT. We prove two results on the generic determinacy of Nash equilibrium in voting games. The first one is for negative plurality games. The second one is for approval games under the condition that the number of candidates is equal to three. These results are combined with the analogous one obtained in De Sinopoli (2001) for plurality rule to show that, for generic utilities, three of the most well-known scoring rules, plurality, negative plurality and approval, induce finite sets of equilibrium outcomes in their corresponding derived games—at least when the number of candidates is equal to three. This is a necessary requirement for the development of a systematic comparison amongst these three voting rules and a useful aid to compute the stable sets of equilibria (Mertens, 1989) of the induced voting games. To conclude, we provide some examples of voting environments with three candidates where we carry out this this comparison.

KEYWORDS: Approval voting, Plurality voting, Negative plurality, Sophisticated voting, Mertens Stability.
JEL CLASSIFICATION NUMBERS: C72, D72.

1. INTRODUCTION

The Gibbard-Satterthwaite Theorem teaches us that we must limit the number of desirable properties that we can ask for to our voting systems. But collective decisions still need to be made whose outcome is legitimized by the participation in the decision process of all the individuals that will be affected by its outcome. This paper is a contribution to a positive research agenda that aims at understanding how the electoral system determines the political outcome.

An electoral system must be judged in terms of how it maps the constituency's preferences into the set of possible political outcomes. For this reason, it is important to describe the incentives created by the different voting games generated by the different electoral systems and to characterize, in as much detail as possible, their sets of equilibrium outcomes. The first step of this process is to qualify what

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we mean by *equilibrium*. Often voting games have unreasonable Nash equilibria that do not successfully capture plausible voting behavior. Farquharson (1969) suggested the *sophisticated voting principle*: reasonable equilibria must survive iterative deletion of dominated strategies. Within the more general framework of finite games, the literature on equilibrium refinements has proposed a number of other equilibrium concepts and rationality requirements.[1] It seems that *Mertens' stability* (Mertens, 1989) is the equilibrium concept that satisfies the most comprehensive list of desirable game theoretical properties, including stability against iterative deletion of dominated strategies. Therefore, it appears to be the most suitable tool to make equilibrium analysis in voting games.

Of course, the task of comparing voting procedures would be much more amenable if the set of (stable) equilibrium outcomes was always unique. Unfortunately, it is often the case that uniqueness can only be obtained after imposing restrictive assumptions that are not necessarily compelling in every voting situation. Thus, it seems that we have to put up with multiplicity of equilibria if we want to deal with a broader realm of voting environments and that we should, at best, hope for finiteness in the set equilibrium outcomes. However, this is again impossible if we do not put restrictions on the set of possible preference profiles that the electorate can have. We have to, at least, restrict attention to generic preferences to obtain an appealing terrain where we can analyze voting systems and make comparisons amongst them.

Indeed, De Sinopoli (2001) shows that, for generic plurality games, the set of Nash equilibrium outcomes is finite. (Under plurality, each voter votes for just one candidate, the candidate with the most votes wins the election and ties are broken randomly.) In this paper, we first obtain an analogous result for negative plurality.[2] (Under negative plurality, each voter casts a negative vote for just one candidate, the candidate with the least negative votes wins the election and ties are broken randomly.) Secondly, we prove that under approval voting and generic utilities the set of equilibrium distributions with at most three candidates in its support is finite. (Under approval voting, each voter casts a ballot that gives one and only one approval vote to as many candidates as she wants, the candidate with most approval votes wins the election and ties are broken randomly.) These results imply that, if utilities are generic, each of the stable sets of the voting games generated by plurality, negative plurality and approval (with three candidates) map into a unique outcome. Ideally, we would like to obtain the general result for approval for an arbitrary number of candidates. We hope that our detailed analysis of negative plurality and the partial result for approval help shed light to this general case.

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[1] See van Damme (1991) for an excellent review.

[2] For simplicity, we focus on winner-takes-all elections.

We then use these results to compare plurality, negative plurality and approval in some generic voting environments with three candidates. We first briefly consider those voting environments where the normal-form game induced by each voting system is dominance solvable. As we have already mentioned, stable sets satisfy iterated deletion of dominated strategies (and the specific order of deletion does not matter), i.e. each stable set contains a stable set of the game obtained by deleting dominated strategies. Therefore, proving that, for generic preferences, each stable set maps into a unique outcome implies that, generically, a dominance solvable game has a unique stable outcome.

Uniqueness of equilibrium outcomes is an important property in voting scenarios. It is argued by Myerson and Weber (1993) that the number of equilibrium outcomes has political significance because the larger the number of equilibria, the wider is the scope for focal manipulation by political leaders. Moreover, there already are results available that give sufficient conditions so that plurality, negative plurality and approval voting games are dominance solvable (e.g. Dhillon and Lockwood (2004) and Buenrostro et al. (in print)) that can therefore be read as sufficient conditions so that those voting games have a unique stable set that maps into a unique outcome. It follows that the set of voting games that are dominance solvable is quite relevant because they generate a unique stable outcome whose computation is very tractable. To get some feeling of which voting system generates a unique stable outcome "more often", in Table 1 we present some actual computations showing how many voting games generated by each voting system are dominance solvable.

Voting games can also be compared when they have multiplicity of equilibria by comparing the nature of the corresponding equilibrium outcomes. Using Poisson games, Myerson (2002) compares plurality, negative plurality and approval voting and finds that plurality generates too many discriminatory equilibria (equilibria where one candidate is not considered a serious contender) while negative plurality does not generate enough. Myerson (2002) argues that approval seems to provide a good balance between the two. We show how analogous results can be obtained using normal-form games and their stable sets of equilibria. We present some general voting environments where plurality generates discriminatory equilibria where a universally preferred candidate is not regarded as a serious contender. We also present voting environments where negative plurality generates too few discriminatory equilibria, further implying that in every equilibrium outcome, a universally disliked candidate is considered a serious contender. Again, approval voting solves those problems in the voting environment considered.

Approval voting seems to come out in an advantageous position from these comparisons. Hence, we submit it to further scrutiny by considering generic voting environments where either plurality or negative plurality seem to produce better results. In our view, these generic voting environments demonstrate that none of the scoring rules is unambiguously superior to the the others. In any case, these environments must also be considered even if one argues that approval voting is the best voting system available as these examples, and more of its kind, must be weighted by their relevance against the benefits of approval in any such argument.

In the next section we introduce the voting model in general terms. It can be easily specialized to approval, plurality and negative plurality voting. Section 3 proves the generic finiteness of Nash equilibrium *distributions* for negative plurality games. Section 4 proves a similar result for approval: for generic preferences the number of admissible Nash equilibrium distributions that elect with positive probability at most three candidates is necessarily finite. In Section 5 we introduce some basic properties of stable sets (Mertens, 1989) and combine them with the previous results to derive some results about the stable set of equilibria in plurality, negative plurality and approval voting games. These are used in the last section to study several simple examples that show some of the fundamental ways in which the set of equilibrium outcomes varies as we change the voting system.

2. The Voting Model

We consider an election with electorate $N \equiv \{1, \ldots, n\}$ and set of candidates $K \equiv \{1, \ldots, k\}$. Each voter $i \in N$ casts a ballot $v_i \in V_i \equiv \mathscr{V} \subset \mathbb{Z}^k$, where $\mathscr{V}$ is the set of ballots allowed by the electoral system. A ballot $v_i = (v_i^1, \ldots, v_i^k)$ is a vector with as many entries as candidates where, for every candidate $c \in K$, v_i^c is the number of votes given by voter i to candidate c.

An electoral system must specify the set $\mathscr{V}$ of permissible ballots and an election rule that selects a winning candidate from K for each ballot profile $v = (v_1, \ldots, v_n) \in \prod_{i=1}^n V_i \equiv V$. For example, in an election with three candidates, the set of possible ballots $\mathscr{V}_p$ allowed by plurality rule consists of four elements, namely, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 0, 0)$—the zero vector corresponds to abstention. The set of ballots $\mathscr{V}_a$ allowed by approval voting is obtained by enlarging the set ballots allowed by plurality rule with $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$.[3] In turn, Borda count provides the set $\mathscr{V}_b$ that consists of $(0, 1, 2)$, $(2, 0, 1)$, $(1, 2, 0)$, $(0, 2, 1)$, $(1, 0, 2)$, $(2, 1, 0)$ and $(0, 0, 0)$. Finally, the set of ballots available under negative plurality is $\mathscr{V}_{np} \equiv \{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (0, 0, 0)\}$.

It is reasonable to choose an election rule $p : V \to \Delta(K)$ that makes the candidates that obtain more support more likely to win. Given a voting profile $v \in V$, the set of winning candidates is

$$W(v) = \left\{ c \in K : \sum_{i=1}^{n} v_i^c \geq \sum_{i=1}^{n} v_i^d \text{ for all } d \in K \right\}. \tag{2.1}$$

[3] Given the election rule below, (1,1,1) is equivalent to abstention $(0, 0, 0)$.

And the probability $p(c \mid v)$ that candidate c wins the election if voters cast the ballot profile v is

$$p(c \mid v) \equiv \begin{cases} 0 & \text{if } c \notin W(v), \\ 1/\#W(v) & \text{if } c \in W(v). \end{cases}$$

Voter i's set of mixed strategies is $\Sigma_i \equiv \Delta(\mathscr{V})$. As usual, $\Sigma \equiv \prod_{i=1}^n \Sigma_i$ is the set of mixed strategy profiles. The probability attached to the ballot profile $v = (v_1, \ldots, v_n)$ by the mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ is $\sigma(v) \equiv \prod_{i=1}^n \sigma_i(v_i)$. Therefore, the probability that candidate c is elected when voters use the mixed strategy profile σ is $p(c \mid \sigma) \equiv \sum_{v \in V} \sigma(v) p(c \mid v)$.

Within this framework, a voting system $(\mathscr{V}, p)$ together with a utility vector u defines a voting game $(\mathscr{V}, p, u)$. The utility vector specifies for each voter i and each candidate c, the utility $u_i(c)$ to voter i if candidate c gets elected. Therefore, once the electoral system is fixed, a voting game is given by a point in $u \in \mathscr{U} \equiv \mathbb{R}^{nk}$. The expected utility derived by voter i, if voters play according to the mixed strategy profile σ, is computed in the usual manner $U_i(\sigma) \equiv \sum_{c \in K} p(c \mid \sigma) u_i(c)$.

Given a voting game $(\mathscr{V}, p, u)$, a Nash equilibrium is a strategy profile σ such that for every voter i and every ballot v_i,

$$U_i(\sigma) \geq U_i(\sigma_{-i}, v_i).$$

The next section is concerned with negative plurality. Henceforth, for every $i \in N$ we fix the set of pure strategy profiles to be equal to

$$V_i \equiv \mathscr{V}_{np} \equiv \left\{ (v^1, \ldots, v^k) \in \{0, -1\}^k : \sum_{c \in K} v^c \in \{0, -1\} \right\}. \tag{2.2}$$

With slight abuse of notation we denote by c both candidate $c \in K$ and the ballot that gives a negative vote to candidate c. When that is the case, we say that a voter casts a negative vote against c or, simply, that she votes against c. The symbol 0 represents abstention. Therefore, we may write $\mathscr{V}_{np} = \{0\} \cup K$.

3. Generic Determinacy of Equilibria in Negative Plurality Games

In this section we show that, for generic negative plurality games, the set of probability distributions induced by Nash equilibria is finite. Here we use the term "generic subset" meaning that its complement is a closed, lower-dimensional, semi-algebraic subset of $\mathscr{U}$.[4] We say that a point is generic if it resides in a generic subset.

Before getting into the proof, we first point out one complication of the analysis of negative plurality. Even if more than one candidate wins with positive probability, abstention can be a best response for some voters. Note that the same is not true

[4] A set is semi-algebraic if it is defined by a finite system of polynomial inequalities. A function or a correspondence is semi-algebraic if its graph is a semi-algebraic set. Every set and correspondence defined in this paper is semi-algebraic.

with plurality or approval because voting for the most preferred candidate among those who win with positive probability always yields a strictly larger payoff than abstention (as long as utilities are generic).

Example 1. Consider a negative plurality voting game with set of voters $N = \{1,2,3,4\}$ and set of candidates $K = \{a,b,c,d\}$. Writing voter i's utility vector as $u_i = (u_i(a), u_i(b), u_i(c), u_i(d))$, voter i's preferences are given by:

$$u_1 = (0,0,-1,0), \quad u_2 = (4,3,0,6), \quad u_3 = (6,3,0,4), \quad u_4 = (0,-\varepsilon,-1,0).$$

where $\varepsilon > 0$ is a suitable small number. A Nash equilibrium of this voting game is $\sigma = (c, \frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}b + \frac{1}{2}d, 0)$. Under this Nash equilibrium, candidates a and d win with probability $3/8$, and candidate b wins with probability $1/4$. Note that for voter 4, voting against b, her least preferred candidate among those who win with positive probability, is not a best response. The reason is that if voter 4 votes against b then candidate c, her least preferred candidate overall, wins with positive probability. Moreover, this is a generic example. Every game in a neighborhood has a close by Nash equilibrium with the same characteristics.

Nevertheless, we must point out that both abstention and voting against candidate c are best responses for voter 4 and that abstaining is always a dominated strategy (by voting against the least preferred candidate overall) when voting is costless.[5]

Thus, in a Nash equilibrium, a voter may find it optimal to abstain even in close races. It should also be clear that voting against a candidate that wins with zero probability is "similar" to abstention in the sense that, once we fix the behavior of the rest of the voters, it does not affect the probability distribution over winning candidates.

Taking this caveat into account, we focus on Nash equilibria where more than one candidate wins with positive probability. Formally, if $p(\sigma) = (p(c \mid \sigma))_{c \in K}$ denotes the probability distribution on candidates induced by the strategy profile σ, the set of *nondegenerate equilibria* is defined as:

Definition 1. The Nash equilibrium σ is *nondegenerate* if $p(\sigma)$ is not a vertex of $\Delta(K)$. In other words, $p(c \mid \sigma) < 1$ for every $c \in K$.

Given that the set of probability distributions where only one candidate wins with positive probability is necessarily finite, it is enough to prove that the set of equilibrium distributions induced by nondegenerate equilibria is finite.

For any strategy profile σ we let $\mathscr{C}(\sigma) \equiv \{v : \sigma(v) > 0\}$ denote the *carrier* of σ. Note that $\mathscr{C}(\sigma)$ has a product structure. Any strategy profile σ with carrier C

[5] If voting is costly and that cost is small enough then σ is also Nash equilibrium of the resulting game. This feature should be taken into account when interpreting turnout if the electoral rule is negative plurality.

satisfies $p(c \mid \sigma) < 1$ for every $c \in K$ if and only if $\#\left(\bigcup_{v \in C} W(v)\right) > 1$. This implies that we can meaningfully say that a carrier C is nondegenerate. Given a nondegenerate carrier C, we can construct the set of candidates that cannot win if voter i abstains. Every ballot in C_i that consists of a negative vote against one of such candidates is analogous to abstention. That set of ballots is denoted $\mathscr{A}_i(C)$. In symbols, $\mathscr{A}_i(C) = C_i \setminus \bigcup_{v_{-i} \in C_{-i}} W(v_{-i})$.

Insofar as we aim to establish a result that holds for generic utilities, we can restrict the analysis to utility vectors where no player is indifferent between two candidates. The set of all such utility vectors is denoted $\tilde{\mathscr{U}}$. The set $\tilde{\mathscr{U}}$ is obtained removing a finite number of lower-dimensional hyperplanes from $\mathscr{U}$ and its closure coincides with $\mathscr{U}$.

Assumption 1. For every voter $i \in N$ and every pair of candidates $c, d \in K$ we have $u_i(c) \neq u_i(d)$.

For the time being, we fix a negative plurality voting game $u \in \tilde{\mathscr{U}}$, a nondegenerate carrier C and a Nash equilibrium σ such that $\mathscr{C}(\sigma) = C$. Take an arbitrary ballot profile $v^* \in C$ that satisfies $v_i^* \in \mathscr{A}_i(C)$ whenever $\mathscr{A}_i(C) \neq \varnothing$ (otherwise v_i^* is an arbitrary element of C_i). For each $i \in N$, let $\hat{K}_i \equiv C_i \setminus \left(\mathscr{A}_i(C) \cup \{v_i^*\}\right)$. For each voter $i \in N$ and each pure strategy $c \in \hat{K}_i$, the following equality holds:

$$\sum_{d \in K} p(d \mid \sigma_{-i}, c) u_i(d) = \sum_{d \in K} p(d \mid \sigma_{-i}, v_i^*) u_i(d).$$

Subtracting from both sides voter i's expected utility if she abstains and letting $\pi(d \mid \sigma_{-i}, c) \equiv p(d \mid \sigma_{-i}, c) - p(d \mid \sigma_{-i}, 0)$, we can rewrite the previous equality as:

$$\sum_{d \in K} \pi(d \mid \sigma_{-i}, c) u_i(d) = \sum_{d \in K} \pi(d \mid \sigma_{-i}, v_i^*) u_i(d). \tag{3.1}$$

Rearranging (3.1), for each voter i and each ballot $c \in \hat{K}_i$ we obtain:

$$- \sum_{d \in \hat{K}_i} \left[\pi(d \mid \sigma_{-i}, c) - \pi(d \mid \sigma_{-i}, v_i^*)\right] u_i(d) = \sum_{d \notin \hat{K}_i} \left[\pi(d \mid \sigma_{-i}, c) - \pi(d \mid \sigma_{-i}, v_i^*)\right] u_i(d). \tag{3.2}$$

Therefore, for each voter $i \in N$ we have $\hat{k}_i \equiv \#\hat{K}_i$ equalities. Suppose that we know the values assumed by u_i over candidates in $K \setminus \hat{K}_i$. We call this vector u_i^*. We can interpret the $\hat{k}_i$ equalities as a system of $\hat{k}_i$ equations in $\hat{k}_i$ unknowns; the set of unknowns being the values assumed by u over candidates in $\hat{K}_i$. Let us call this vector of unknowns u_i^o so that $u_i = (u_i^o, u_i^*)$. We let X_i^C denote the $\hat{k}_i \times \hat{k}_i$ matrix of coefficients of this system of equations. Hence, the (c, d)-th entry of X_i^C is

$$X_i^C(c, d) = -\pi(d \mid \sigma_{-i}, c) + \pi(d \mid \sigma_{-i}, v_i^*). \tag{3.3}$$

It is also convenient to denote by Π_i^C the matrix whose (c, d)-th element is $-\pi(d \mid \sigma_{-i}, c)$.

Lemma 1. *The following assertions hold:*

(i) Every nondiagonal element of Π_i^C *is weakly negative.*

(ii) Every diagonal element of Π_i^C *is strictly positive.*

(iii) Every row in Π_i^C *adds up to some weakly positive number.*

(iv) Every element of the vector $(\pi(c \mid \sigma_{-i}, v_i^*))_{c \in \hat{K}_i}$ *is weakly positive.*

Proof. Part (i) merely states that a negative vote against candidate c can never decrease the probability that some other candidate $d \neq c$ gets elected.

To prove part (ii) we need to show that a negative vote against a candidate always decreases the probability that she wins the election. If $c \in \hat{K}_i$ there exists a ballot profile $v_{-i} \in C_{-i}$ such that c is the candidate that collects the least number of negative votes under $(v_{-i}, 0)$. Since C_{-i} has a product structure, starting from the ballot profile v_{-i} and changing one coordinate at a time we can obtain another ballot profile $v'_{-i} \in C_{-i}$ where some other candidate $c' \neq c$ obtains the least number of negative votes under $(v'_{-i}, 0)$. During this transition we must go through some ballot profile $v''_{-i} \in C_{-i}$ such that, under $(v''_{-i}, 0)$, candidate c either obtains the same number of negative votes as some other winning candidate or wins the election outright but collecting just one negative vote less than the next candidate. Given that every ballot profile in C_{-i} receives positive probability under σ_{-i}, we immediately obtain $p(c \mid \sigma_{-i}, c) < p(c \mid \sigma_{-i}, 0)$.

Part (iii) follows because the decrease in the probability $\pi(c \mid \sigma_{-i}, c)$ that candidate c gets elected when player i votes negatively for c is necessarily equal to the increase in probability that candidate c is not elected. That is,

$$-\pi(c \mid \sigma_{-i}, c) = \sum_{d \in K \setminus \{c\}} \pi(d \mid \sigma_{-i}, c) \geq \sum_{d \in \hat{K}_i \setminus \{c\}} \pi(d \mid \sigma_{-i}, c),$$

with strict inequality whenever $\pi(d' \mid \sigma_{-i}, c) > 0$ for some candidate $d' \notin \hat{K}_i$.

Finally, if $v_i^* \notin \mathscr{A}_i(C)$ then Part (iv) is a consequence of the argument given in the the proof of Part (i). Note that if $v_i^* \in \mathscr{A}_i(C)$ then $\pi(c \mid \sigma_{-i}, v_i^*) = 0$ for every c. □

If we want to use the system of equations (3.2) to find out u_i^o we need to show that the matrix X_i^C is nonsingular. To this end, we use the following result proved in Ostrowski (1955, p. 97).

Lemma 2. *Let* Π *be an* $n \times n$ *M-matrix and let* $\pi = (\pi_1, \ldots, \pi_j, \ldots, \pi_n)$ *be a non negative vector. The determinant of the* $n \times n$ *matrix* X *whose* (i, j)*-th element is given by* $X_{ij} = \Pi_{ij} + \pi_j$ *is strictly positive.*

By virtue of Lemma 1(iv) we only need to show that Π_i^C is an M-matrix. M-matrices can be characterized as square matrices with nonpositive nondiagonal elements whose (real) eigenvalues are all strictly positive. Lemma 1(i) says that every nondiagonal element of Π_i^C is nonpositive. We now proceed to showing that every eigenvalue of Π_i^C is strictly positive.

The Gershgorin Circle Theorem (Gershgorin, 1931) tells us that every eigenvalue of a square matrix $A = (a_{cd})$ can be found in one of the closed disks $D(a_{cc}, R_c)$ with center a_{cc} and radius $R_c = \sum_{d \neq c} |a_{cd}|$. Therefore, Lemma 1(i)–(ii) imply that every eigenvalue of Π_i^C lies in some closed disk with center $-\pi(c \mid \sigma_{-i}, c)$ and radius $\sum_{d \in \hat{K}_i} \pi(d \mid \sigma_{-i}, c)$. As a consequence, Lemma 1(ii)–(iii) establish that every eigenvalue of Π_i^C is weakly positive. In order to prove that every eigenvalue is strictly positive we now show that Π_i^C is nonsingular.

Lemma 1(i)–(iii) show that Π_i^C is a *dominant diagonal* matrix. Recall that a matrix $A = (a_{cd})$ is dominant diagonal if $|a_{cc}| \geq \sum_{d \neq c} |a_{cd}|$ for every row c. Price (1951) gives the following bound on the determinant $|A|$ of a dominant diagonal matrix:

$$\prod_c \left(|a_{cc}| - \sum_{d > c} |a_{cd}| \right) \leq |A|. \tag{3.4}$$

Now we can prove:

Lemma 3. *The matrix Π_i^C in nonsingular and, therefore, the matrix X_i^C is also nonsingular.*

Proof. Reorder the rows and columns of Π_i^C so that columns (rows) corresponding to more preferred candidates appear before columns (rows) corresponding to less preferred candidates. With this reordering of the matrix, if $-\pi(c \mid \sigma_{-i}, c) = \sum_{d > c} \pi(d \mid \sigma_{-i}, c)$ then the decrease in the probability that candidate c is elected is equal to the increase in the probability that candidates worse than c (according to voter i's preferences) win the election. This provides a contradiction because, using Assumption 1, voter i's utility from voting against c would be strictly lower than under abstention. Consequently, $-\pi(c \mid \sigma_{-i}, c) > \sum_{d > c} \pi(d \mid \sigma_{-i}, c)$ for every candidate c.

In light of Lemma 1(i)–(ii) we can apply equation (3.4) to Π_i^C knowing that every term on the left-hand side is strictly positive. Therefore, Π_i^C is nonsingular and, given that we already established that every eigenvalue of this matrix is weakly positive, Π_i^C is also an M-matrix. We can now apply Lemma 2 to conclude that X_i^C is nonsingular. □

Therefore, if for each voter i we know u_i^* then we can reconstruct the entire vector of utilities u using the strategy profile σ and the system of equations (3.2). This allows us to construct a continuous function from the set of Nash equilibria with carrier C to the set of utility vectors $\tilde{\mathscr{U}}$. The generic determinacy of the set of Nash equilibria is a direct consequence of applying the following result to such a function.

Lemma 4. *Let $f : X \to Y$ be a continuous semi-algebraic function. If* $\dim(X) \leq \dim(Y)$ *then, for generic $y \in Y$, $f^{-1}(y)$ is a finite or empty set.*

Lemma 4 follows from the Generic Local Triviality Theorem (Bochnak et al., 1998) and it is taken from Govindan and Wilson (2001). We now have all the necessary ingredients to prove:

Theorem 1. *For generic negative plurality games, the set of probability distributions on candidates induced by Nash equilibria is finite.*

Proof. If only one candidate can win under the carrier C (i.e. if $\#\left(\bigcup_{v\in C} W(v)\right) = 1$) then the set of equilibrium distributions induced by Nash equilibria with carrier C is necessarily finite. Thus, let the carrier C be nondegenerate.

Recall that given the carrier C, the set of ballots equivalent to abstention for voter i is $\mathscr{A}_i(C)$. We write $K_i(C) = C_i \setminus \mathscr{A}_i(C)$ for the subset of remaining ballots in C_i. Letting $K(C) = K_1(C) \times \cdots \times K_n(C)$ and $\mathscr{A}(C) = \mathscr{A}_1(C) \times \cdots \times \mathscr{A}_n(C)$ we can decompose the set Σ^C of mixed strategy profiles with carrier C as $\Sigma^C = \Sigma^{K(C)} \times \Sigma^{\mathscr{A}(C)} \times \{\mathbf{0}\}$, where $\Sigma^{K(C)}$ and $\Sigma^{\mathscr{A}(C)}$ are the obvious subspaces of Σ^C and $\mathbf{0}$ is the zero vector of appropriate dimension.

The graph of the Nash equilibrium sub-correspondence that contains only Nash equilibria with carrier C is

$$GNE^C \equiv \left\{(\sigma, u) \in \Sigma^C \times \tilde{\mathscr{U}} : \sigma \in \mathrm{NE}(u)\right\}.$$

Write $NE^{K(C)}$ for the projection of the set GNE^C on $\Sigma^{K(C)} \times \tilde{\mathscr{U}}$. There exists a correspondence $H^C : NE^{K(C)} \to \Sigma^{\mathscr{A}(C)}$ defined by

$$H^C(\sigma^k, u) = \left\{\sigma^a \in \Sigma^{\mathscr{A}(C)} : (\sigma^k, \sigma^a, \mathbf{0}, u) \in GNE^C\right\}.$$

The correspondence H^C is semi-algebraic with nonempty values. Hence, it admits a semi-algebraic selection function h^C (Schanuel et al., 1991, Section 2). Now write Gh^C for the graph of the function h^C and call E^C the projection of Gh^C on Σ^C and on those coordinates of $\tilde{\mathscr{U}}$ where the subvector $u^* = (u_1^*, \ldots, u_n^*)$ lives. Lemma 3 implies that there is a continuous function $f^C : E^C \to \tilde{\mathscr{U}}$ mapping (u^*, σ) into $u = (u^o, u^*)$. The function f^C is also semi-algebraic. Since $\dim(\tilde{\mathscr{U}}) = nk$, in view of Lemma 4, the only thing remaining to show is $\dim(E^C) \leq nk$.

Proposition 2.8.7 in Bochnak et al. (1998) implies $\dim(Gh^C) = \dim(NE^{K(C)})$. Furthermore, for each voter i, if $\mathscr{A}_i(C) = \varnothing$ then $\dim(\Sigma_i^{K(C)}) = \#C_i - 1 = \hat{k}_i$. If otherwise $\mathscr{A}_i(C) \neq \varnothing$ then $\dim(\Sigma_i^{K(C)}) = \#K_i(C) = \hat{k}_i$. We conclude that $\dim(Gh^C) \leq \sum \hat{k}_i + nk$. Consequently,

$$\dim\left(E^C\right) \leq \dim\left(Gh^C\right) - \sum_{i\in N} \hat{k}_i \leq \sum_{i\in N} \hat{k}_i + nk - \sum_{i\in N} \hat{k}_i = nk.$$

Applying Lemma 4 to the function $f^C : E^C \to \tilde{\mathscr{U}}$ shows that for generic games $u \in \tilde{\mathscr{U}}$ the set of Nash equilibria with carrier C such that the distribution of weights in $\mathscr{A}(C)$ is determined by the function h^C is finite. Given that the probability distribution on candidates induced by the profiles $(\sigma^k, \sigma^a, \mathbf{0})$ and $(\sigma^k, \tilde{\sigma}^a, \mathbf{0})$ coincide for every two $\sigma^a, \tilde{\sigma}^a \in H^C(\sigma^k, u) \subset \Sigma^{\mathscr{A}(C)}$, the set probability distributions induced by

Nash equilibria with nondegenerate carrier C is finite. The desired result follows from the finiteness in the number of possible carriers. □

4. Approval Voting and Three Candidates

This section deals with the generic determinacy of equilibria in approval games. Namely, if utilities over candidates are generic, we show that: (1) when the number of candidates is equal to three, every Nash equilibrium that induces a completely mixed strategy over candidates is *regular* (Harsanyi, 1973); and that (2) the set of equilibrium distributions that put a strictly positive weight to at most three candidates is finite. Whether a similar results holds in the general case with more than three candidates remains an open problem.

We start this section fixing the set of candidates K with $\#K = 3$, the set of voters N, and a utility vector u that satisfies Assumption 1. Each player set's of pure strategies is equal to $\mathscr{V}_a$ (see Section 2). Take a strategy profile σ whose carrier $C = \mathscr{C}(\sigma)$ satisfies $K = \bigcup_{v \in C} W(v)$. We denote as $B_i = \mathrm{PBR}_i(\sigma)$ player i's set of pure best responses against σ_{-i} and let $B = \prod_{i \in N} B_i$. Suppose that $C \subset B$ so that σ is a Nash equilibrium. We describe how B looks like. To that end, we write $c^*(i)$ to denote voter i's top-ranked candidate and $c(i)$ to denote voter i's second-ranked candidate. We also let v_i^* be the ballot that approves both $c^*(i)$ and $c(i)$. With slight abuse of notation, we denote by $c^*(i)$ both the candidate and the ballot that only approves candidate $c^*(i)$.

Lemma 5. *Take an approval voting game where $\#K = 3$ and Assumption 1 holds. Let $C = \mathscr{C}(\sigma)$, $B = \mathrm{PBR}(\sigma)$ and $C \subset B$. If $K = \bigcup_{v \in C} W(v)$ then either $B_i = \{c^*(i)\}$ or $B_i = \{c^*(i), v_i^*\}$.*

Proof. Since $K = \bigcup_{v \in C} W(v)$ every candidate wins with positive probability under σ. No player is indifferent between two candidates, therefore, in equilibrium, every voter approves her most preferred candidate and does not approve her least preferred one. □

Let $\hat{N} = \{i \in N : \#B_i = 2\}$. If $i \in \hat{N}$ we can write

$$\sum_{d \in K} p(d \mid \sigma_{-i}, c^*(i)) u_i(d) = \sum_{d \in K} p(d \mid \sigma_{-i}, v_i^*) u_i(d)$$

and rearranging

$$\left[p(c(i) \mid \sigma_{-i}, v_i^*) - p(c(i) \mid \sigma_{-i}, c^*(i))\right] u_i(c(i)) = \\ - \sum_{d \neq c(i)} \left[p(d \mid \sigma_{-i}, v_i^*) - p(d \mid \sigma_{-i}, c^*(i))\right] u_i(d).$$

Candidate $c(i)$ wins with positive probability under σ so, using a similar logic to the proof of Lemma 1(ii), it must be the case that $p(c(i) \mid \sigma_{-i}, v_i^*) > p(c(i) \mid \sigma_{-i}, c^*(i))$. Thus, if we know the utility derived by voter i from her top- and bottom-ranked candidates then we can recover the utility that she derives from her second-ranked candidate.[6]

Analogously to the previous section, the graph of the Nash equilibrium sub-correspondence that contains only Nash equilibria with carrier C and set of pure best responses B is

$$GNE^{C,B} \equiv \left\{(\sigma, u) \in \Sigma^C \times \tilde{\mathscr{U}} \,:\, B = \mathrm{PBR}(\sigma) \text{ and } C \subset B\right\}.$$

Let us decompose the utility vector $u = (\hat{u}, u^o)$ so that $\hat{u} = (u_i(c(i)))_{i\in\hat{N}}$. Write $\hat{\mathscr{U}}$ and $\tilde{\mathscr{U}}^o$ for the projections of $\tilde{\mathscr{U}}$ on the corresponding coordinates so that $\hat{u} \in \hat{\mathscr{U}}$ and $u^o \in \tilde{\mathscr{U}}^o$. Project $GNE^{C,B}$ on the strategy space and on $\tilde{\mathscr{U}}^o$. If $E^{C,B}$ is such a projection there is a continuous function $f^{C,B} : E^{C,B} \to \tilde{\mathscr{U}}$ that takes u^o and a Nash equilibrium strategy and reconstructs the whole utility vector $u = (\hat{u}, u^o)$. We can now prove:

Proposition 1. *For generic approval voting games with three candidates every Nash equilibrium that induces a completely mixed probability distribution on the set of candidates is regular.*

Proof. Suppose first that $C \subset B$ and $C \neq B$. In such a case, the equilibrium is not regular because it is not quasi-strict (see van Damme (1991, Corollary 2.5.3)). Noting that the strict inclusion of C in B implies $\sum_{i\in N}[\#C_i - 1] < \#\hat{N}$, we can use Theorem 2.8.8 in Bochnak et al. (1998) to show that

$$\dim\left(f^{C,B}\left(E^{C,B}\right)\right) \leq \dim\left(E^{C,B}\right) = \sum_{i\in N}\left[\#C_i - 1\right] + nk - \#\hat{N} < nk.$$

That is, the set of approval games (such that $\#K = 3$ and utilities are in $\tilde{\mathscr{U}}$) with equilibria that is not quasi-strict is a lower-dimensional semi-algebraic set.

Thus, let $C = B$ so that the equilibrium is quasi-strict and, therefore, regular if and only if the Jacobian of the map $F(\cdot \mid v_i^*) : \Sigma \to \mathbb{R}^{7n}$ defined by

$$\begin{aligned} F^{v_i}(x \mid v_i^*) &= x_i(v_i)\Big[U_i(x_{-i}, v_i) - U_i(x_{-i}, v_i^*)\Big], \text{ for all } v_i \in K \setminus v_i^*, \text{ for all } i \in N, \\ F^{v_i^*}(x \mid v_i^*) &= \sum_{v_i\in V} x_i(v_i) - 1, \text{ for all } i \in N, \end{aligned}$$

and evaluated at $x = \sigma$ is nonsingular.[7] But this Jacobian is nonsingular if and only if the matrix

$$\left.\frac{\partial f^{C,B}(x, u^o)}{\partial x}\right|_{x=\sigma}$$

is nonsingular.

[6] In this case the system of equations is quite simple. For each voter whose set of pure best responses has two elements we only have one equation and one unknown.

[7] Note that with three candidates every voter has seven different pure strategies in the approval voting game.

The semi-algebraic version of Sard theorem (Bochnak et al., 1998, Theorem 9.5.2) ensures that the set of critical values of $f^{C,B}$ is a lower-dimensional semi-algebraic set, thus, completing the proof. □

Take now an arbitrary set of candidates K and a Nash equilibrium σ that induces a probability distribution that gives positive probability to just three candidates, say, c_1, c_2 and c_3. Construct a three-candidate approval game by choosing those three candidates. Interpreting ballots under approval as subsets of candidates, we define the strategy profile σ' of the three-candidate game by $\sigma'_i(v'_i) = \sum_{\{v_i : v'_i \subset v_i\}} \sigma_i(v_i)$ for every $i \in N$.[8] It is not difficult to see that σ' is a Nash equilibrium of the three-candidate approval game. It follows that:

Corollary 1. *For every generic approval voting game the set of probability distributions on three or fewer candidates induced by Nash equilibria is finite.*

Proof. The discussion leading to the corollary shows the result when just three candidates receive positive probability. Given that approval coincides with plurality when $\#K = 2$, a similar argument applied to the result in De Sinopoli (2001) proves the case where two candidates receive positive probability. To conclude, we only note that the set of degenerate distributions on candidates is necessarily finite. □

Remark 1. Analogously to Proposition 1, if voters have generic utilities over candidates then one can prove that every Nash equilibrium of the negative plurality game that induces a completely mixed probability distribution on the set of candidates K is a regular equilibrium (although this time, this is true regardless of the cardinality of K). However, we cannot derive generic finiteness of Nash equilibria in negative plurality games from such a result. On one hand, given a Nash equilibrium σ of a negative plurality game with $k - 1$ candidates, a straightforward extension of σ to a negative plurality game with k candidates would induce a different outcome because the kth candidate receives no negative vote. On the other hand, a Nash equilibrium where $\tilde{k}$ candidates win with positive probability may be supported by the existence of another candidate that wins with probability zero and that, with positive probability, receives just one negative vote more than the winning candidates. The Nash equilibrium analyzed in Example 1 is of this sort.

Remark 2. As we have already mentioned, a general proof of the generic finiteness of equilibrium distribution in approval voting games would be desirable. The problem that we encounter when we want to apply a proof along the lines of the ones presented above is that when the number of candidates is equal to k the number of

[8] The strategy σ' is well defined. Candidates c_1, c_2 and c_3, and only them, win with positive probability under σ. Hence, if σ is an equilibrium, every voter approves at least one of them in every pure strategy that is played with positive probability.

pure strategies is equal to $2^k - 1$. Classifying strategies into equivalence classes, as we do in negative plurality with abstention and voting against a candidate that has no chance of winning, could potentially alleviate the problem. However, even if we solved this dimensionality problem, the matrix that we would obtain in place of the one in equation (3.3) would not necessarily decompose into a dominant diagonal matrix.

5. Strategic Stability in Voting Games

In the next section, we analyze the voting games generated by plurality, negative plurality and approval using the notion of stability developed by Mertens (1989). The resulting stability concept is set valued, meaning that a set of equilibria (potentially generating a continuum of probability distributions on outcomes, see Govindan and McLennan (2001)) can be considered as equivalent members of the same solution. Mertens' stable sets satisfy a number of desirable properties that make them an appealing concept to analyze, among others, voting games. We need the following properties (cf. Mertens (1989)):

(α) Every game has a stable set.
(β) Stable sets are connected sets of normal form perfect equilibria.
(γ) A stable set contains a stable set of every game that is obtained after deleting a strategy that is at minimum probability in any ε-perfect equilibrium close to the stable set.

In light of Theorem 1 we can see that (β) implies that for generic negative plurality games, every stable set generates a unique probability distribution on candidates. A similar conclusion can be reached about generic approval voting games with three candidates.

It is well-known that the reduced game that is obtained after applying iterated deletion of dominated strategies depends on the order of elimination. Farquharson (1969) gets rid of this problem when defining sophisticated voting by imposing stability only against iterated elimination of all dominated strategies at each round. Following De Sinopoli (2000), we drop that constraint and define the *sophisticated outcome* as the outcome that is isolated by at least one order of elimination. The following proposition is a generalization of a result contained in De Sinopoli (2000) about plurality rule.

Proposition 2. *If a voting game has finitely many Nash equilibrium outcomes and a sophisticated outcome exists then it is unique and it coincides with the unique stable outcome of the game.*

Proof. Consider a voting game with finitely many Nash equilibrium outcomes. Property (β) ensures that every point of each stable set generates the same outcome. In turn, property (γ) implies that every stable set contains a stable set of any game that is obtained after deleting a dominated strategy. Therefore, if the sophisticated outcome exists it is contained in every stable set, making it the unique stable outcome.[9] □

The counterexample offered by Govindan and McLennan (2001) shows that once we fix the function mapping pure strategy profiles into outcomes we may run into continua of equilibrium outcomes even if preferences are generic. The generic finiteness results proved here and in De Sinopoli (2001) imply that we can apply Proposition 2 to generic voting environments when the voting system is plurality or negative plurality. Additionally, it can also be applied to approval as long as the number of candidates is equal to three.

6. Comparing Voting Systems

In this section, we use the results obtained above to compute the stable sets of equilibria in some simple examples and to show some of the fundamental ways in which the electoral system determines the political outcome. In this series of examples, we will take the viewpoint that the Condorcet winner, whenever it exists, is the most desirable alternative from a social perspective and that the Condorcet loser is the least desirable alternative. We begin with the formal definitions:

Definition 2 (Condorcet Winner)**.** A candidate $c \in K$ is the *Condorcet winner* if

$$\#\Big\{i \in N : u_i(c) > u_i(d)\Big\} > \#\Big\{i \in N : u_i(c) < u_i(d)\Big\} \text{ for all } d \neq c.$$

Furthermore, we say that a candidate $c \in K$ is a *weak-Condorcet winner* if

$$\#\Big\{i \in N : u_i(c) > u_i(d)\Big\} \geq \#\Big\{i \in N : u_i(c) < u_i(d)\Big\} \text{ for all } d \in K, \text{ and}$$
$$\#\Big\{i \in N : u_i(c) > u_i(d')\Big\} > \#\Big\{i \in N : u_i(c) < u_i(d')\Big\} \text{ for some } d' \neq c.$$

If the Condorcet winner exists then it is unique. On the other hand, if there is a weak-Condorcet winner then it may not be the only one. The definitions of Condorcet loser and weak-Condorcet loser are the obvious ones.

In what follows, we consider voting environments with three candidates that are always labelled a, b and c. Note that, with three candidates, each voter has only two undominated strategies: under approval, (1) approving their best candidate and (2) approving their two best candidates; under plurality, (1) voting for their best candidate and (2) voting for their second-best candidate; under negative plurality, (1) voting against their worst candidate and (2) voting against their second-best candidate. To economize on notation, we may say that a voter has preferences $u = (u_a, u_b, u_c)$ if the utility value that she derives when candidate a, b or c wins the election is, respectively, u_a, u_b and u_c.

[9] Note that the proposition also holds if we defined the sophisticated outcome as Nash equilibrium outcome that is isolated after some order of elimination of dominated strategies.

Before continuing we remark that every voting environment that we consider has a neighborhood that is contained in the generic set of utilities where the plurality, negative plurality and approval voting games generate a finite set of equilibrium outcomes.

6.1. *Sophisticated Voting*

Our discussion in the previous section allows us to easily compute the unique stable outcome of those voting games that are dominance solvable.[10] To start getting a feel for the different strategic incentives generated by plurality, negative plurality and approval, we can do a simple exercise of computing and reporting those stable outcomes. Consider three candidates, a, b and c, and let the number of voters vary from three to fifteen.[11] Take the set of all possible preference orderings such that the Condorcet winner is candidate a and candidate b wins or ties with candidate c in a pairwise contest. Then eliminate "nongeneric" preference orderings where some voter is indifferent between two candidates or between her middle-ranked candidate and the lottery $1/3a + 1/3b + 1/3c$. This leaves us with 2,590,345 ordering profiles, each of which can be represented by a generic utility vector. Using those generic utility vectors and for each voting system, we report in Table 1 the number of dominance solvable games that, consequently, have a unique stable outcome. We also report how many times the unique stable outcome assumes each one of its possible values. Outcome bc, for instance, means that candidates b and c both win the election and the final candidate is elected by a fair lottery (the outcomes ab and ac never occur).

6.2. *Above the Fray*

Using Poisson games, Myerson (2002) studies voting rules in terms of their tendency to admit discriminatory equilibria in which voters disregard a candidate as not a serious contender. He finds that plurality rule tends to generate too many discriminatory equilibria.

In the same vein, we show how a universally liked candidate can win with probability zero in a stable set of a plurality game. We borrow Myerson's terminology "above the fray" to indicate candidate a's privileged position in the election.

[10] Dhillon and Lockwood (2004) provide sufficient conditions for a plurality voting game to be dominance solvable. Sufficient conditions for dominance solvability of other scoring rules, including approval, are offered in Buenrostro et al. (in print). Given the results in the current paper, whenever applicable, those conditions can be seen as sufficient conditions so that the relevant voting game has a unique stable outcome.

[11] We wrote a MATLAB program (available upon request) to apply iterated deletion of dominated strategies in the current setting. The computation slows down considerably when the number of voters is above fifteen.

TABLE 1. Dominance solvable voting games with generic utilities, with a unique Condorcet winner and where the number of voters varies from 3 to 15. The total number of preference profiles considered (modulo renaming of the candidates) is 2,590,345.

System	Dom. Solvable	a	b	c	bc	abc
Approval	1,745,797	1,745,479	0	0	0	318
Plurality	513,650	513,644	1*	0	0	5
Negative	25,720	8*	17,837	3,482	1,142	3,251

* These outcomes are obtained when the number of voters is equal to three.

Example 2. There are three candidates a, b and c and, for some integer m, $2m$ voters grouped in two subsets. The first subset has m voters with preferences $a \succ b \succ c$. Voters in the second subset have preferences $a \succ c \succ b$. From a social perspective it is clear that candidate a is the most preferred alternative. However, the plurality game has an stable set where voters in the first subset vote for b and voters in the second subset vote for c. It is easy to see that this strategy combination is a strict Nash equilibrium and, therefore, a singleton stable set.

It should be noted that with the same preferences as in the previous example, the approval game is dominance solvable. Therefore, by Proposition 2, it has a unique stable set. This stable set leads to the election of candidate a with probability one. It can also be proven that, once we eliminate dominated strategies, every Nash equilibrium of the negative plurality game leads to the election of candidate a too.[12] By properties (α), (β) and (γ), candidate a wins with probability one in the unique stable outcome of the negative plurality game.

6.3. *One Bad Apple*

We have just illustrated how negative plurality may eliminate discriminatory equilibria. Paralleling Myerson (2002) we now show that negative plurality may, in fact, generate too few discriminatory equilibria, which can also be harmful.

We show how a universally disliked candidate can win with positive probability in a stable set of the negative plurality game. Again, we borrow the phrase "one bad apple" from Myerson to express the idea that the existence of one bad candidate can spoil the whole election.

Example 3. There are three candidates a, b and c and, for some integer m, $3m$ voters that are grouped into three equally sized subsets. Voters in the first subset have preferences $(3, 1, 0)$, voters in the second subset have preferences $(1, 3, 0)$ and

[12] That is, a is the sophisticated outcome even though the negative plurality game is not dominance solvable. Thus, the figures given in Table 1 should be taken with appropriate care.

voters in the third subset have preferences that are either $a \succ b \succ c$ or $b \succ a \succ c$. It should be clear that from a social perspective, candidate c should not win the election. The negative plurality game has a stable set such that voters in the first subset vote against b, voters in the second subset vote against a and voters in the third subset vote against c. The strategy profile is a strict Nash equilibrium and, therefore, a singleton stable set.

With the same preferences as before, both the approval voting game and the plurality game are dominance solvable. The unique sophisticated equilibrium leads to the election of the Condorcet winner in both games (in this example this could be candidate a, b, or both could be weak-Condorcet winners). By Proposition 2 this is the unique stable outcome under both plurality and approval.

6.4. *Electing Condorcet Losers*

We now show a striking property of negative plurality. In an election where the Condorcet loser exists, negative plurality may select it with probability one.

Example 4 (Negative plurality selects the Condorcet loser with probability one)**.** Take five voters with preferences

$$u_1 = u_2 = (3, 0, 1)$$
$$u_3 = u_4 = (0, 3, 1)$$
$$u_5 = (3, 2, 0).$$

Candidate a is the Condorcet winner and candidate c is the Condorcet loser. The negative plurality game is dominance solvable. First, eliminate every dominated strategy in the game. In the reduced game, voter 3 and 4's dominant strategy is to vote against a. Given that, voter 5's dominant strategy is to vote against c. In the last round of elimination we find that voters 1 and 2 vote against candidate b. Therefore, in the unique stable outcome of the negative plurality game, the Condorcet loser wins the election.

With the same preferences, consider approval voting. In the first round of elimination, keep only the strategies where every voter approves her most preferred candidate and does not approve her least preferred one. In the next round, we find that voter 1 and 2's dominant strategy is to approve only candidate a. Given that, voters 2 and 3 approve only candidate b. Finally, voter 5 approves candidate a, which makes a the winner of the election.

In Example 2, we have seen how plurality rule generates stable outcomes where the set of weak Condorcet losers wins with probability one. Again, in that example approval voting keeps inducing the "right" outcome. However, we can now see that approval voting suffers from a similar flaw too.

Example 5. Take six voters with preferences

$$u_1 = u_2 = (1,3,0)$$
$$u_3 = u_4 = (1,0,3)$$
$$u_5 = (3,2,0)$$
$$u_6 = (3,0,2).$$

Candidate a is the Condorcet winner and candidates b and c are the weak Condorcet losers. Consider the strategy profile $\sigma = (b,b,c,c,ab,ac)$ which yields the probability distribution $1/2b+1/2c$. We prove that σ is an *absorbing retract* (Kalai and Samet, 1984) and, therefore, contains a stable set (Mertens, 1992, p. 562).[13]

Voters 1 through 4 are playing a strict best response. Therefore, they are also playing a best response against every sufficiently close strategy profile. A similar argument applies for voters 5 and 6. If they do not approve their second-best candidate their utility strictly decreases. Thus, this also holds for every sufficiently close strategy profile. On the other hand, if they play an undominated strategy then they must approve their best candidate. We conclude that the indicated strategy profile is a singleton stable set of the approval game.

6.5. *More on Approval Voting*

Approval voting has received a lot attention by the literature on political economy, see for instance Brams and Fishburn (1978), Fishburn and Brams (1981), or more recently, Brams and Sanver (2006). The computations given in Table 1 suggest that approval voting could potentially improve upon other voting systems (cf. Buenrostro et al. (in print)). However, there are generic examples where approval voting does not behave as one would like (e.g. see De Sinopoli et al. (2006)). As we show below there are also generic voting environments where approval seems to be outperformed by plurality or even negative plurality. Even if one decides to advocate for approval voting over other voting systems, it is important to understand its limitations and the kind of situations where it is not the ideal voting system. In any case, we think that these examples prove that none of the voting systems considered here is unambiguously superior to the rest.

In this section we provide three new examples. In the first two we compare approval with plurality and in the third we compare approval with negative plurality. We begin with an example where approval selects the Condorcet winner and the Condorcet loser with the same probability in the unique stable set. Meanwhile, plurality selects the Condorcet winner with probability one.

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[13] A strategy combination is an absorbing retract if it is a best reply to all sufficiently close strategy combinations.

Example 6. For any integer $m > 0$, define a voting environment with $3m+2$ voters and preferences:

number of voters	preferences
$m+1$	$(3, 1, 0)$
m	$(1, 3, 0)$
m	$(1, 0, 3)$
1	$(0, 3, 2)$

As usual, candidate a is the Condorcet winner and candidate c is the Condorcet loser. The approval game is dominance solvable and gives, as unique outcome, a three way lottery where the Condorcet winner and the Condorcet loser are elected with the same probability. The plurality game is also dominance solvable and the Condorcet winner is elected with probability one.

In the next example, the Condorcet winner exists and the unique stable outcome of the approval game varies continuously with the utility values of the voters—but their preference orderings over candidates remain constant. This example shows that for any $\varepsilon > 0$, there exists an open set of utilities such that approval voting selects the Condorcet winner with probability smaller than ε. On the other hand, plurality selects the Condorcet winner with probability one.

Example 7. Consider five voters with preferences:

$$\begin{aligned} u_1 = u_2 &= (1, 3, 0) \\ u_3 &= (3\alpha, 0, 3) \\ u_4 &= (3, 3\beta, 0) \\ u_5 &= (3, 0, 2) \end{aligned}$$

where $0 < \alpha < 1/2$ and $1/2 < \beta < 1$. Candidate a is the Condorcet winner. Apply iterated deletion of dominated strategies until the game is reduced to a two-player game between voter 4 and voter 5. This reduced game has a unique equilibrium that determines the unique stable set of the approval game. Such a stable set consists of the strategy profile

$$\sigma^* = \left(b, b, \frac{3-3\beta}{1+\beta}c + \frac{4\beta-2}{1+\beta}ac, \frac{3\alpha}{2-\alpha}a + \frac{4\beta-2}{1+b}ab, ac\right).$$

The probability that the Condorcet winner wins the election is $p(a \mid \sigma^*) = (\alpha\beta + \alpha + 4\beta - 2)/(2 + 2\beta - \alpha - \alpha\beta)$. This probability is arbitrarily close to zero when α is close enough to 0 and β is close enough to $1/2$. On the other hand, this probability is arbitrarily close to one when α is close enough to $1/2$ and β is close enough to 1.

The plurality game is dominance solvable and leads to a unique stable outcome where candidate a wins with probability one no matter what the values of $0 < \alpha < 1/2$ and $1/2 < \beta < 1$ are.

In our last example, we compare approval voting with negative plurality. In this example, there is a unique weak-Condorcet winner and a unique weak-Condorcet loser. Moreover, every voter prefers the weak-Condorcet winner to the weak-Condorcet loser. In the unique stable set of the approval game both candidates are elected with the same probability. In the negative plurality game, the unique stable set has the weak-Condorcet winner being elected with probability one.

Example 8. Take six voters with preferences:

$$u_1 = u_2 = u_3 = (1, 3, 0)$$
$$u_4 = u_5 = u_6 = (3, 0, 2)$$

Candidate a is the unique weak-Condorcet winner: it ties with b and wins c in pairwise contests. Meanwhile candidate b ties with both a and c in pairwise contests. It follows that c is the unique weak Condorcet loser.

The approval game is dominance solvable. First eliminate dominated strategies so that voters 1, 2 and 3 are left with b and ab and voters 4, 5 and 6 are left with a and ac. Take voter 1, she will approve candidate b for sure. But when considering whether or not approving her second-ranked candidate a she knows that a will receive at least 3 approval votes, that b will receive exactly 3 approval votes (counting hers) and that c will receive at most 3 approval votes. Since the only case in which approving a pays off for voter 1 is when a and c are the only two candidates tied at the top and this case is impossible, approving both a and b is dominated by approving only b. The same is true for voters 2 and 3.

Voter 4 approves candidate a anyway. It follows that she knows that both a and b will receive exactly three approval votes and c at most 2. So the only case where approving c matters is when it takes exactly 2 votes. In such a case, voter 4 prefers a three-way lottery among the three candidates to a two-way lottery between candidates a and b. The analysis is symmetric for voters 5 and 6, hence, they all will approve both a and c.

This yields a unique stable outcome where the three candidates are elected with probability $1/3$. That is, both the weak Condorcet winner and the weak Condorcet loser are elected with the same probability *even though every voter strictly prefers a to c*. On the other hand, the negative plurality game is also dominance solvable (we leave the analysis to the reader) and leads to a unique stable outcome where candidate a is elected with probability one.

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