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Abstract

The time evolution of prices and savings in a stock market is modeled by a discrete-delay nonlinear dynamic system. The proposed model has a unique and unstable steady-state, so its time evolution is determined by the nonlinear effects acting out of the equilibrium. We perform the analysis of the linear approximation through the study of the eigenvalues of the Jacobian matrix in order to characterize the local stability properties and the local bifurcations in the parameter space. If the delay is equal to zero, Lyapunov exponents are calculated. For certain values of the parameters, we prove that the system has a chaotic behaviour. The discrete nonlinear model is associated with a discrete stochastic model. For the liniarization of this model, we establish the conditions for which the mean and quadratic mean values of the state variables are asymptotically stable. Some numerical examples are finally given to justify the theoretical results.

Keywords: price index, mutual fund, stock market, nonlinear dynamic model, Lyapunov exponents.

1 Introduction

Describing the stock market behavior has been one of the main cornerstones in the modern theory of finance. Indeed, identifying models able to explain the workings of the financial markets significantly improved our general understanding and informed efficient policies for the stock market participants. In this context, asset prices were shown to exhibit a volatile or erratic behavior. As a result, a rich part of the literature on this topic (Kyle, 1985; Lo

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and MacKinlay, 1988; Bouchaud, 2005; Chiarella and Wang, 2006; Dieci et al., 2006) has focused on different market structures or modeling approaches (continuous vs. discrete), all meant to capture this behavior.

This paper extends the literature by considering a discrete-delay time deterministic model that captures the interactions between the stock market price index and the net stock of savings collected by a mutual fund. We focus on this framework as it accurately reflects the current structure of financial markets, characterized to some extent by the removal of intermediaries. Currently, a portion of the funds involved in saving and financing flows directly in the markets instead of being passed through the standard intermediary operations (i.e., banks lending, deposit operations, etc.). This is due to the public's interest in directly or indirectly investing in the stock market. Those who invest directly are considered 'dealers' and are directly admitted to securities negotiation. Alternatively, the agents can invest in professionally managed mutual funds. These funds are collective investment schemes that pool investors and buy shares in a portfolio of stocks, bonds, and other securities. These agents represent the 'savers',who intend to invest in the stock market but, being scarcely informed, prefer to underwrite shares of a mutual fund. The link between the two categories is the fact that, in the model, the dealers choose the securities on the market and sell fractions of the whole portfolio (shares of the mutual fund).

The paper proceeds as following: Section 2 describes the general economic motivation and structure of the model that considers savings at time $n-m$ (m being the delay). We show that, subject to an economically feasible set of parameters, the model has a unique steady state. In Section 3 we analyze the characteristic equation and the value of the bifurcation for the parameter a (that determines the agent's attitude to realize capital gains), when there is a delay and when there is not. Section 4 derives the normal form, and Section 5 offers a numerical example for fixed parameters. For the 'no delay' case ($m=0$), in Section 6 we present the method for assigning the Lyapunov exponents. For certain parameter values, it follows that the first Lyapunov exponent is positive and the system behaves chaotically. Using a program in Maple 12, we visualize the orbits of the state variables. Section 7 analyses the stochastic perturbation of the deterministic system and establishes the conditions for which the mean and quadratic mean values of the state variables are asymptotically stable. In Section 8 we perform the numerical simulations for the stochastic case. Section 9 concludes.

2 The deterministic discrete-delay model

We assume one day as unit of time. This allows us to give the following simple description of the rules that regulate the time evolution of the stock market price index, p, and of the net stock of savings collected by the mutual fund, s. The model structure is based on the assumption that there are two different kinds of economic agents interacting in the market (dealers and savers), and they belong to two distinct markets. As a result, the dynamic process denoting the evolution of the state variables (s, p) will reflect this 'segmentation'.[1]

If the price level is p_n and the savings collected by the funds is s_{n-m} at times $n-m$, $m \geq 0$, then at time $n+1$, the stock market, where only the dealer participates to the negotiations, will open with a new value of the index p_{n+1}, determined by a law of the kind:

$$p_{n+1} - p_n = g\left(s_{n-m}, p_n\right). \tag{1}$$

This equation captures the direct dependency of the market price variations on the volume of capital detained in the mutual funds and on the level of price currently reached. Afterwards, the stock market being closed, the savers, who act by underwriting shares of the mutual fund or asking for the repayment of the ones already held, will buy or sell. Such choices give rise to the new value of savings, through a law of the kind:

$$s_{n+1} - s_n = f\left(s_{n-m}, p_{n+1}, p_{n+1} - p_n\right). \tag{2}$$

Equation (2) captures the savers' responsiveness to the stock market: the savers will choose the funds based not only on the actual price level for the past two periods, but also on the price trend.

The two functions, g and f, are supposed to be at least $\mathbb{C}^1$ and to satisfy the following assumptions:

[1] See Bischi and Valori, 2000.

$$1.)\frac{\partial g}{\partial s_{n-m}}>0;\ 2.)\frac{\partial g}{\partial p_n}<0;\ 3.)\frac{\partial f}{\partial s_{n-m}}<0;\ 4.)\frac{\partial f}{\partial p_{n+1}}<0;\ 5.)\frac{\partial f}{\partial\left(p_{n+1}-p_n\right)}>0.$$

It is easy to verify these assumptions, since the previous five hypotheses are sufficient to ensure the uniqueness of the equilibrium, if it exists. Such equilibrium values, say $\overline{s}$ and $\overline{p}$, can be considered 'natural levels', and deduced from general macroeconomic considerations on what agents perceive as the reference value to which they compare the current situation in order to make the investment decision. Under these assumptions the model can be rewritten as:

$$\begin{aligned} s_{n+1}-s_n &= F\left(s_{n-m}-\overline{s}, p_{n+1}-\overline{p}, p_{n+1}-p_n\right), \\ p_{n+1}-p_n &= G\left(s_{n-m}-\overline{s}, p_n-\overline{p}\right). \end{aligned} \tag{3}$$

The functions F and G satisfy the same first-order conditions defined for f and g, and in this context it is natural to claim that $G(0,0)=F(0,0,0)=0$.

After the following change of variables:

$$S_n = s_n-\overline{s}, \qquad S_{n-m}=s_{n-m}-\overline{s}, \qquad P_n=p_n-\overline{p}, \tag{4}$$

we obtain the system:

$$\begin{aligned} S_{n+1}-S_n &= F\left(S_{n-m}, P_{n+1}, P_{n+1}-P_n\right) \\ P_{n+1}-P_n &= G\left(S_{n-m}, P_n\right). \end{aligned} \tag{5}$$

for which the unique equilibrium is the origin. However, these functions are too general to attempt an analytic analysis. Following Antoci (1989), and considering the assumptions 1.)-5.) about the prevailing behavior of the agents, we specify the map for F and G in a polynomial form as

$$F(x,y,z) = -Ay - Bx^3 + Hz,$$

$$G(x,y) = Cx - Dy^3.$$

The system is therefore given by:

$$\begin{aligned} S_{n+1} - S_n &= -AP_{n+1} - BS_{n-m}^3 + H(P_{n+1} - P_n) \\ P_{n+1} - P_n &= CS_{n-m} - DP_n^3, \end{aligned} \tag{6}$$

where all the coefficients A, B, C, D, H, whose meaning can be easily deduced from the previous discussion about the assumptions 1.)...5.), are real and positive.[2] The first equation shows that close to the equilibrium, the dynamics is influenced positively by the price trend and negatively by the deviation from the natural level $\overline{p}$. This dynamics depends only marginally on the difference $(s_n - \overline{s})$. However, if $|s_n - \overline{s}|$ is sufficiently large, it will determine the system to converge towards the equilibrium. Similarly, for the second equation, the dynamics of the prices in a neighborhood of the equilibrium is particularly sensitive to deviations of s_n from $\overline{s}$. In this case, the difference $(p_n - \overline{p})$ is important to redirect p_n towards $\overline{p}$ for values of p_n far from the equilibrium.

Close to the equilibrium, each variable's dynamics is endogenously determined with respect to the other variables in the model. Also, the supply of securities is rather inelastic with respect to the price deviation from the natural level. However, when the system is far from the equilibrium, only a high deviation of the savings from their natural level will convince the savers to rebalance their portfolios.

Denoting $S_{n-m} = x^1, ...S_n = x^{m+1}, P_n = x^{m+2}$, the time evolution of the system (6) is obtained by the iteration of the $(m+2)$-dimensional map, defined by:

[2] From the qualitative analysis perspective, this system is part of the autonomous polynomial systems (Andronov and Chaikin, 1949; Andronov et al., 1973; Ye Yian Qian, 1986).

$$\begin{pmatrix} x^1 \\ \dots \\ x^m \\ x^{m+1} \\ x^{m+2} \end{pmatrix} \rightarrow \begin{pmatrix} x^2 \\ \dots \\ x^{m+1} \\ cex^1 - b(x^1)^3 + x^{m+1} - ax^{m+2} - de(x^{m+2})^3 \\ cx^1 + x^{m+2} - d(x^{m+2})^3 \end{pmatrix}, \tag{7}$$

and

$$a = A,\ b = B,\ c = C,\ d = D,\ e = H - A. \tag{8}$$

According to the above discussion, the parameters a, b, c and d are positive, whereas the coefficient e can take negative values conditional on $e + a > 0$ being satisfied. Specifically, b and d represent the 'stabilizing' coefficients, and measure the force with which the system tends to converge to the equilibrium once is sufficiently far from it. The coefficient e can be interpreted as an index of responsiveness of the savers, and thus can be positively correlated with the speed of adjustment of their expectancies, becoming an index of the 'speculative nature' of the market.

We now turn to studying the necessary and sufficient conditions for the parameters a, b, c, and d such that the system (6) may accept a closed and stable curve in the neighborhood of the fixed point $(0, 0, .., 0) \in \mathbb{R}^{m+2}$. For $m = 0$, the system (6) is analyzed in Bischi and Valori (2000), but to fix notation we will present both cases (i.e., $m = 0$ and $m \geq 1$).

3 The analysis of the characteristic equation for (7)

The first step in the qualitative analysis of the dynamic model given by (6) is the localization of the steady states of the $(m + 2)$-dimensional map given by (7). In our case, using Ford and Wulf (1998) and Kuznetsov (1995), the following results hold:

Proposition 1 *(i). For each economically feasible set of parameters, the* $(m + 2)$ *-dimensional map given by (7) has the unique steady state* $O = (0, 0, .., 0) \in \mathbb{R}^{m+2}$.

(ii). If $m = 0$, the Jacobi matrix of (7) computed at O is:

$$A_1 = \begin{pmatrix} 1+ec & -a \\ c & 1 \end{pmatrix}, \tag{9}$$

and if $m \geq 1$, the Jacobi matrix of (7) computed at O is:

$$A_1 = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \\ ce & 0 & \ldots & 1 & -a \\ c & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{10}$$

(iii). If $m = 0$, the characteristic equation is given by:

$$\lambda^2 - (2 + ce)\lambda + 1 + ce + ac = 0, \tag{11}$$

and if $m \geq 1$, the characteristic equation is given by:

$$\lambda^m (\lambda - 1)^2 - ce(\lambda - 1) + ac = 0. \tag{12}$$

Given the respective eigenvalues of equations (11) and (12), the instability of O follows for any feasible set of the parameters. Consequently, the time evolution of the variables S_n and P_n never converges to the equilibrium values.

However, we are interested to understand the types of non-stationary asymptotic dynamics of the model. Thus, we will focus our analysis on the sets of parameters which are out of the economically feasible region. Also, the local bifurcations of A_1 for economically non-feasible sets of parameters may provide useful insights on the behavior of the model in the feasible parameter region.

In the following analysis, we shall consider fix positive values of the parameters b, c, d and we will investigate the effect of changes in the parameters $e \in \mathbb{R}$ and $a \in \mathbb{R}$. The choice of the parameters a and e as bifurcation parameters is related to the fact that they give a measure of the two opposite forces that determine the relative weight of the agent's attitude to realize the capital gains, measured by the parameter a, and their speculative attitude, measured by the parameter e.

If $m = 0$, using Kuznetsov (1995) and Mircea et al. (2003), the following results hold:

Proposition 2 *(i). If $c > 0$, $e \in \left(-\frac{4}{c}, 0\right) - \left\{-\frac{3}{c}, -\frac{2}{c}\right\}$ and $a = a_0 = -e$, then the characteristic equation has two eigenvalues and there exists a unit circle, given by $\lambda_1(a_0) = exp(i\theta(a_0)), \lambda_2(a_0) = \overline{\lambda_1(a_0)}$, where $\theta(a_0) = \arccos\left(\frac{2+ce}{2}\right)$.*

(ii). Consider the variable transformation:

$$a\left(\beta\right) = a_0 + \frac{(1+\beta)^2 - 1}{c}, \tag{13}$$

where $|\beta|$ is sufficiently small.

With (13), the characteristic equation (11) becomes

$$\mu^2 - (2 + ce)\mu + (1 + \beta)^2 = 0. \tag{14}$$

The eigenvalues of equation (14) are given by

$$\mu_1(\beta) = (1 + \beta) \exp(i\omega(\beta)), \quad \mu_2(\beta) = \overline{\mu_1(\beta)}, \tag{15}$$

where

$$\omega(\beta) = \arccos\left(\frac{2 + ce}{2(1 + \beta)}\right). \tag{16}$$

(iii). If $\mu = \mu_1(\beta)$ is the eigenvalue of (14), the eigenvectors $q \in \mathbb{R}^2$, $p \in \mathbb{R}^2$ corresponding to A_1 and A_1^T respectively, have the following components:

$$q_1 = 1, \quad q_2 = \frac{1 + ec - \mu}{a}, \quad p_1 = \frac{1 - \overline{\mu}}{2(1 - \overline{\mu}) + ec}, \quad p_2 = \frac{a}{2(1 - \overline{\mu}) + ec} p_1, \tag{17}$$

where $a = a(\beta)$.

From Proposition 2, it follows that $\mu_1(0) = \mu_1(a_0)$. Also, all the assumption for the occurrence of a Neimark-Sacker hold for the parameter β and, so, also for a_0.

If $m = 1$, using Ford and Wulf (1998) and equation (12), the following result holds:

Proposition 3 *(i). If $c > 0$, $e \in \left(-\frac{1}{c}, 0\right)$ and $a = a_0$, where*

$$a_0 = \frac{1 - ec - \sqrt{1 + ec}}{c}, \tag{18}$$

then the characteristic equation has two eigenvalues and there exists a unit circle, given by $\lambda_1(a_0)$ and $\lambda_2(a_0) = \lambda_1(a_0)$, and an eigenvalue $\lambda_3(a_0)$ with absolute value less than one:

$$\lambda_1(a_0) = \lambda_2(a_0) = \exp\left(\frac{2 - c(a_0 + e)}{2} i\right), \quad \lambda_3(a_0) = -c(a_0 + e). \tag{19}$$

(ii). For $|\beta|$ sufficiently small, consider the variable transformation:

$$a(\beta) = a_0 + f(\beta), \tag{20}$$

where

$$f(\beta) = \frac{(1 + \beta)^2(1 - \sqrt{(1 + \beta)^2 + ec}) - (1 - \sqrt{1 + ec})}{c}. \tag{21}$$

With (20), the characteristic equation (12) becomes

$$\mu^3 - 2\mu^2 + (1 - ec)\mu + c(a_0 + f(\beta) + e) = 0. \tag{22}$$

The eigenvalues of equation (22) are given by

$$\begin{aligned} \mu_1(\beta) &= (1 + \beta) \exp(i\omega(\beta)), \\ \mu_2(\beta) &= \overline{\mu_1(\beta)}, \\ \mu_3(\beta) &= -\frac{c(a_0 + f(\beta) + e)}{(1 + \beta)^2}, \end{aligned} \tag{23}$$

where

$$\omega(\beta) = \arccos\left(\frac{2(1+\beta)^2 - c(a_0 + f(\beta) + e)}{2(1+\beta)^2}\right). \tag{24}$$

(iii). If $\mu = \mu_1(\beta)$ is the eigenvalue of (22), the eigenvectors $q \in \mathbb{R}^3$, $p \in \mathbb{R}^3$ corresponding to A_1 and A_1^T respectively, have the following components:

$$q_1 = 1, \quad q_2 = \mu, \quad q_3 = \frac{ec + \mu(1-\mu)}{a}, \tag{25}$$

$$p_1 = \frac{(\overline{\mu}-1)(\mu-1)^2}{\mu(\overline{\mu}-1)(\mu-1)^2 - (ec+1-\overline{\mu})(\overline{\mu}-1)}, \quad p_2 = \frac{p_1}{\overline{\mu}}, \quad p_3 = \frac{p_1}{(\overline{\mu}-1)\overline{\mu}^2}. \tag{26}$$

From Proposition 3, it follows that $\mu_1(0) = \lambda_1(a_0)$, $\mu_3(0) = \lambda_3(a_0)$ and all the assumptions for the occurrence of a Neimark-Sacker hold for the parameter β and, so, also for a_0.

4 The normal form of the map (7)

In order to write the normal form of the map (7), we take into account that the RHS of the system (7) contains only first and third order terms and apply the method from Kuznetsov (1995).

If $m = 0$, the following result holds:

Proposition 4 *(i). The normal form for equation (7) is given by*

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{6}g_{30}z_n^3 + \frac{1}{2}g_{21}z_n^2\overline{z_n} + \frac{1}{2}g_{12}z_n\overline{z_n}^2 + \frac{1}{6}g_{03}\overline{z_n}^3, \tag{27}$$

where $\mu(\beta)$ is given by equation (15), $z_n \in \mathbb{C}^2$ and

$$\begin{aligned}
g_{30} &= g_{30}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2^3,\\
g_{21} &= g_{21}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2\overline{q_2}^2,\\
g_{12} &= g_{12}(\beta) = -6bp_1 - 6d(p_1e + p_2)q_2^2\overline{q_2},\\
g_{03} &= g_{03}(\beta) = -6bp_1 - 6d(p_1e + p_2)\overline{q_2}^3,
\end{aligned} \tag{28}$$

with p_1, p_2, q_2 given by (17).

(ii). Let us consider $l_1(0) = \mathrm{Re}(\exp(-i\omega(0)g_{21}(0)))$, where $\omega(0)$ is given by (16). The condition for a supercritical bifurcation is $l_1(0) < 0$.

(iii). The orbits of system (6), are given by

$$S_n = z_n + \overline{z_n}, \quad P_n = q_2 z_n + \overline{q_2 z_n}, \tag{29}$$

where z_n is a solution of equation (26).

If $m = 1$, the following result holds:

Proposition 5 *(i). The normal form associated to equation (7) yields*

$$z_{n+1} = \mu(\beta)z_n + \frac{1}{2}g_{21}(\beta)z_n^2\overline{z_n}, \tag{30}$$

where $\mu(\beta)$ is given by equation (23), $z_n \in \mathbb{C}^2$ and

$$g_{21} = g_{21}(\beta) = -6bp_1 - 6d(p_1e + p_3)q_3\overline{q_3}^2,$$

with p_1, p_2, q_3 given by (25)-(26).

(ii). Let us consider $l_1(0) = \mathrm{Re}(\exp(-i\omega(0)g_{21}(0)))$, where $\omega(0)$ is given by (24). The condition for a supercritical bifurcation is $l_1(0) < 0$.

(iii). The orbits of system (6), are given by

$$\begin{aligned}
S_n &= q_2 z_n + \overline{q_2 z_n} + k_1\mu_3(\beta)^n, \\
P_n &= q_3 z_n + \overline{q_3 z_n} + k_1\left(\frac{ec + \mu_3(\beta) - \mu_3(\beta)^2}{a}\right)^n, \\
S_{n-m} &= U_n = z_n + \overline{z_n} + k_1,
\end{aligned} \tag{31}$$

where z_n is a solution of equation (29), $\mu_3(\beta)$ is given by equation (23) and $k_1 \in \mathbb{R}$.

5 Numerical simulation for the deterministic model

Using a Maple 12 program, for $m = 0$ and $b = 0.5, c = 0.4$, $d = 0.1$, $e = -2$, $\beta = -0.001$, $l_1(0) = 0.2449$, $n = 1300$, just after the Neimark-Hopf bifurcation, a trajectory is numerically generated starting from an initial condition close to the fixed point O. The bifurcation value is $a_0 = -e$ and the trajectory is presented in the phase plane (Sn, Pn) in Figure 1.

For $m = 1$ and $b = 0.5$, $c = 0.4$, $d = 0.1$, $e = -2$, $\beta = -0.001$, $l_1(0) = 0.2449$, $n = 1300$, just after the Neimark-Hopf bifurcation, we numerically generate a similar trajectory, starting from an initial condition close to the fixed point O. The bifurcation value in this case is different from the $m = 0$ case, i.e. $a_0 = 3.3819$. As before, we present the trajectory in the (Sn, Pn) phase plane in Figure 2 and in the (Un, Sn) phase plane in Figure 3.

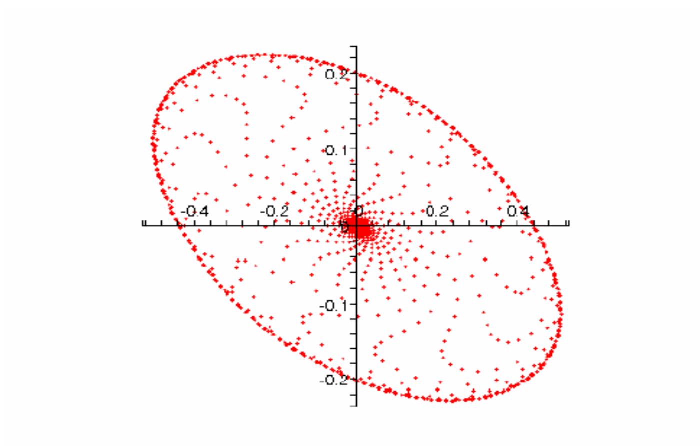

Fig 1. The trajectory in the phase plane (S_n, P_n)

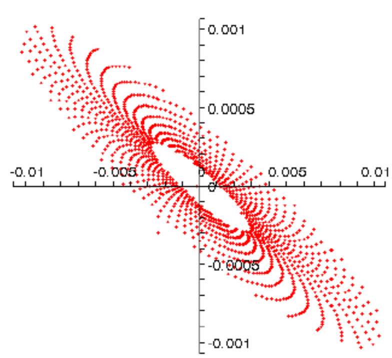

Fig 2. The trajectory in the phase plane (S_n, P_n)

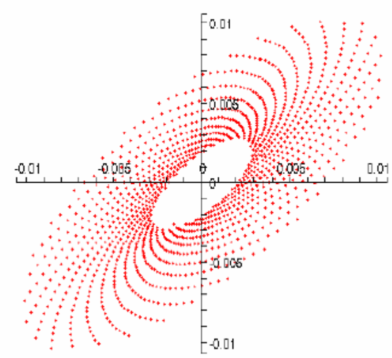

Fig 3. The trajectory in the phase plane (U_n, S_n)

6 The Lyapunov exponent for the system (6) with no delay

In this section we analyze the behavior of system (6) solutions for $m = 0$ and $A > 0, B < 0, C > 0, D < 0, H > 0$ and calculate the Lyapunov exponents.

The system is given by:

$$\begin{aligned} x_{n+1} &= (1 - C(A-H))x_n - Ay_n - Bx_n^3 + DAy_n^3 \\ y_{n+1} &= Cx_n + y_n - Dy_n^3, \end{aligned} \tag{32}$$

where $x_n = S_n$ and $y_n = P_n$.

The Lyapunov exponents can be obtained[3] by solving the system (32) and the system:

$$\begin{aligned} z_{n+1} &= \arctan(\frac{F_1(x)}{F_2(x)}) \\ \lambda_{n+1} &= \lambda_n + \ln\left(|\cos z_n \cos z_{n+1}(f_{11} - f_{21}\tan z_{n+1}) - \sin z_n \cos z_{n+1}(f_{12} - f_{22}\tan z_{n+1})|\right) \\ \mu_{n+1} &= \mu_n + \ln\left(|\sin z_n \cos z_{n+1}(f_{11}\tan z_{n+1} + f_{21}) + \cos z_n \cos z_{n+1}(f_{12}\tan z_{n+1} + f_{22})|\right), \end{aligned} \tag{33}$$

where:

$$\begin{aligned} & f_{11} = \frac{\partial f_1}{\partial y_1}, \quad f_{12} = \frac{\partial f_1}{\partial y_2}, \quad f_{21} = \frac{\partial f_2}{\partial y_1}, \quad f_{22} = \frac{\partial f_2}{\partial y_2}, \\ & f_1 = (1 - C(A-H))y_1 - Ay_2 - By_1^3 + DAy_2^3, \\ & f_2 = Cy_1 + y_2 - Dy_2^3, \\ & F_1(x) = f_{22}\sin z_n - f_{21}\cos z_n, \\ & F_2(x) = f_{11}\cos z_n - f_{12}\sin z_n. \end{aligned}$$

The Lyapunov exponents are:

$$\lambda = \lim_{n\to\infty} \frac{\lambda_n}{n}, \quad \mu = \lim_{n\to\infty} \frac{\mu_n}{n}. \tag{34}$$

For parameter values $A = 0.4$, $B = -0.3$, $C = 0.2$, $D = -0.1$, $H = 0.2$, the Lyapunov exponents are $\lambda \cong 0.047, \mu = -0.178$. Since $\lambda > 0$, it follows that the system (32) is chaotic.

A similar analysis can be conducted for $m = 1$.

[3] See Janaki and Rangarajan (1999).

7 The stochastic discrete-delay model

Let (Ω, F, P) be a probability space, and let $\xi_n = \xi(n, \varpi)$, $n \in \mathbb{N}, \varpi \in \Omega$ be a random variable with null mean value, $E(\xi_n) = 0$, and quadratic mean value $E(\xi_n^2) = \sigma^2$, with $\sigma > 0$ (see Kloeden and Platen, 1995).

For $m = 0$, the stochastic perturbation of the deterministic system (6) is

$$\begin{aligned} S_{n+1} &= S_n - AP_{n+1} - BS_{n-1}^3 + H(P_{n+1} - P_n) + b_1 S_n \xi_n \\ P_{n+1} &= P_n + CS_{n-1} - DP_n^3 + b_2 P_n \xi_n, \end{aligned} \tag{35}$$

with $b_1, b_2 \in \mathbb{R}$.

The linear discrete dynamic stochastic model associated to the system (35) in the fix point $(0,0) \in \mathbb{R}^2$ is[4]

$$u_{n+1} = A_1 u_n + B u_n \xi_n, \tag{36}$$

where $u_n = (u_n^1, u_n^2)^T \in \mathbb{R}^2$, $A_1 = \begin{pmatrix} 1 + ec & -a \\ c & 1 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$.

We note

$$\begin{aligned} E(u_n) &= \left(E(u_n^1), E(u_n^2)\right)^T = E_n, \\ E(u_n u_n^T) &= \begin{pmatrix} (v_n^1)^2 & v_n^{12} \\ v_n^{12} & (v_n^2)^2 \end{pmatrix} = V_n, \end{aligned} \tag{37}$$

where $(v_n^1)^2 = E\left((u_n^1)^2\right)$, $v_n^{12} = E\left(u_n^1 u_n^2\right)$ and $(v_n^2)^2 = E\left((u_n^2)^2\right)$.

Proposition 6 *(i). The mean values E_n satisfy the system of equations*

$$E_{n+1} = A_1 E_n, \ n \in \mathbb{N}. \tag{38}$$

Proof. *Since $E(\xi_n) = 0$, from (36) it directly results (38).* ■

[4] See Babus and de Vries (2010).

(ii). The quadratic mean values V_n satisfy the system of equations

$$V_{n+1} = A_1 V_n A_1^T + \sigma^2 B V_n B^T,\ n \in \mathbb{N}. \tag{39}$$

Proof. *The system (36) yields*

$$u_{n+1}u_{n+1}^T = A_1 u_n u_n^T A_1^T + \xi_n (A_1 u_n u_n^T B^T + B u_n u_n^T A_1^T) + \xi_n^2 B u_n u_n^T B^T \tag{40}$$

and since $E(\xi_n) = 0$ and $E(\xi_n^2) = \sigma^2$, from (40) it follows (39). ■

From (38) and (39) it follows that:

Proposition 7 *(i). The characteristic polynomial for the system (38) is*

$$P_1(\lambda) = det(\lambda I - A_1). \tag{41}$$

(ii). The characteristic polynomial for the system (39) is

$$P_2(\lambda) = det(\lambda I - A_2), \tag{42}$$

where

$$A_2 = \begin{pmatrix} (1+ec)^2 + \sigma^2 b_1^2 & a^2 & a\,(1+ec) \\ c^2 & 1+\sigma^2 b_1^2 & 2c \\ c\,(1+ec) & -a & c\,(1+ec) - a + \sigma^2 b_1 b_2 \end{pmatrix}. \tag{43}$$

(iii). If the characteristic roots of equation $P_1(\lambda) = 0$ are in absolute value less than one, then the mean values of the variables in (38) are asymptotically stable.

(iv). If the characteristic roots of equation $P_2(\lambda) = 0$ are in absolute value less than one, then the quadratic mean values of the variables in (39) are asymptotically stable.

A similar analysis can be performed for the stochastic system for $m = 1$.

8 Numerical simulation for the stochastic system

Consider $b = 0.5$, $c = 0.4$, $d = 0.1$, $e = -2$, $b_1 = 0.03$, $b_2 = 0.02$ and $\sigma = 0.2$. For $a = -e - 0.01$, the roots of the equation $P_1(\lambda) = 0$ are $\lambda_1 = 0.6 - 0.79i$ and $\lambda_2 = 0.6 + 0.79i$, and so, the mean values of the state variables in expression (36) are asymptotically stable. The roots of the equation $P_2(\lambda) = 0$ are $\lambda_1 = -0.27 - 0.95i$, $\lambda_2 = -0.27 + 0.95i$ and $\lambda_3 = 0.99$, and so, the quadratic mean values of the state variables in (36) are asymptotically stable. The trajectories in the phase plane $(S_n(\varpi), P_n(\varpi))$, $(n, S_n(\varpi))$ and $(n, P_n(\varpi))$ are presented in Figures 4 - 6, respectively.

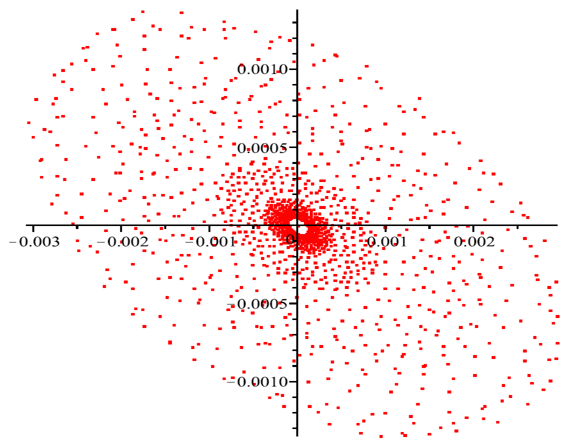

Fig 4. The trajectory in the phase plane $(S_n(\varpi), P_n(\varpi))$

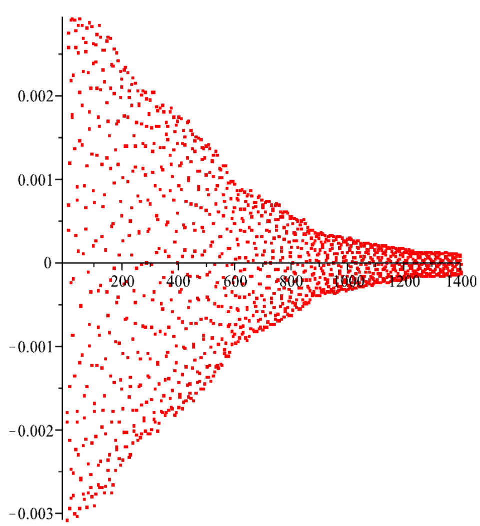

Fig 5. The trajectory in the phase plane $(n, S_n(\varpi))$

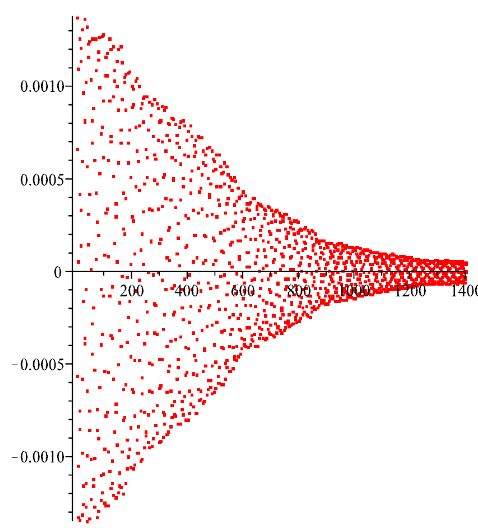

Fig 6. The trajectory in the phase plane $(n, P_n(\varpi))$

Similarly, for $a = -e + 0.1$, it follows that the roots of the equation $P_1(\lambda) = 0$ are $\lambda_1 = 0.6 - 0.8i$ and $\lambda_2 = 0.6 + 0.8i$, and so, the mean values of the state variables in expression (36) are cyclical. The roots of the equation $P_2(\lambda) = 0$ are $\lambda_1 = -0.28 - 0.96i$, $\lambda_2 = -0.28 + 0.96i$ and $\lambda_3 = 1.004$, and so, the quadratic mean values of the state variables in (36) are unstable. The trajectory in $(S_n(\varpi), P_n(\varpi))$ plane is presented in Figure 7, while the trajectories in the $(n, S_n(\varpi))$ and $(n, P_n(\varpi))$ are given in Figure 8 and 9, respectively.

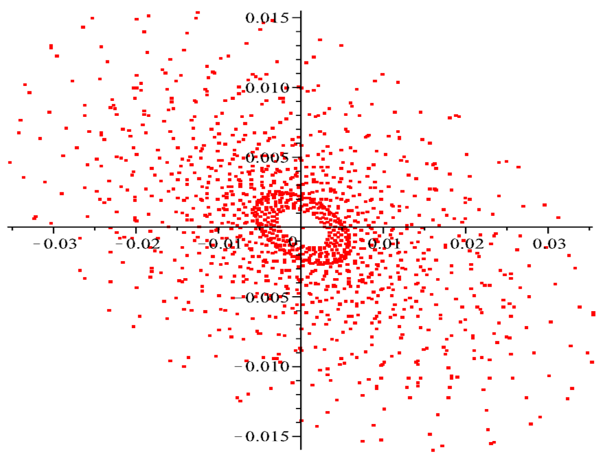

Fig 7. The trajectory in the phase plane

$(S_n(\varpi), P_n(\varpi))$

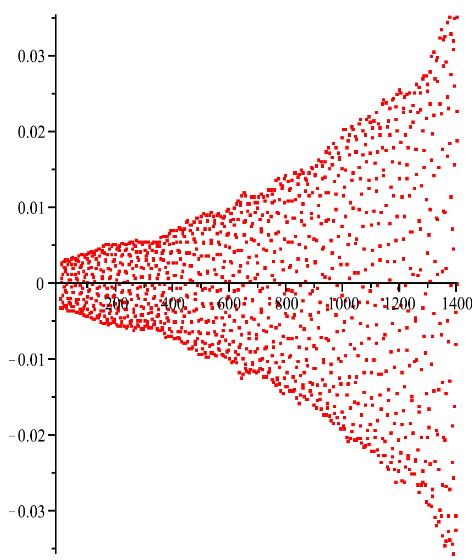

Fig 8. The trajectory in the phase plane $(n, S_n(\varpi))$

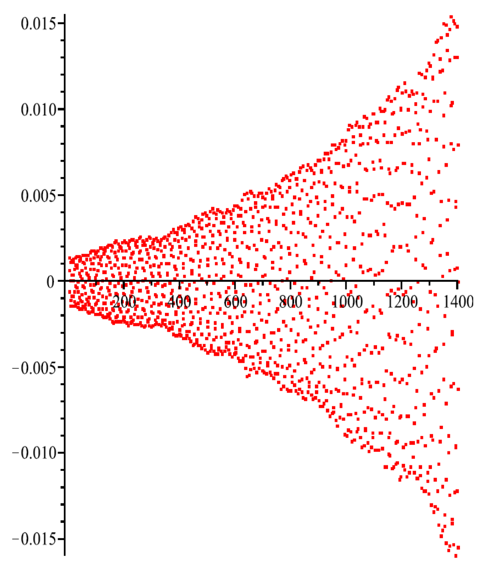

Fig 9. The trajectory in the phase plane $(n, P_n(\varpi))$

9 Conclusions

This paper developed a deterministic model on the time evolution and interactions between savings and the price level in a stock market. Although not based on the market microstructure or the optimizing behavior of the agents, it gives a fairly general description of the main (nonlinear) interactions between the two kinds of agents that we assume are acting in two different sections of the market: the dealers, administrators of mutual funds, directly admitted to the securities negotiation, and the savers, who, after taking their investment decision, buy or sell shares of the mutual funds.

The model takes into account the savings s_{n-m} collected by funds at the time $(n-m)$, for $m=0$ and $m=1$. Considering parameter a as the bifurcation parameter, we obtained the normal form of the model. Knowing its solution, we described the dynamics of the model and show that for certain parameter values, the system display a chaotic behavior due to the fact that the first Lyapunov exponent is positive.

Furthermore, we describe the stochastic perturbation model associated with the deterministic model and establish the conditions for which the mean and quadratic mean values of the state variables are asymptotically stable.

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