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Australian School of Business Research Paper No. 2013 ECON 12

Likelihood-Based Confidence Sets for the Timing of Structural Breaks

Yunjong Eo
James Morley

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Yunjong Eo†
University of Sydney

James Morley‡
University of New South Wales

March 30, 2013

Abstract

We propose the use of likelihood-based confidence sets for the timing of structural breaks in parameters from time series regression models. The confidence sets are valid for the broad setting of a system of multivariate linear regression equations under fairly general assumptions about the error and regressors and allowing for multiple breaks in mean and variance parameters. In our asymptotic analysis, we determine the critical values for a likelihood ratio test of a break date and the expected length of a likelihood-based confidence set constructed by inverting the likelihood ratio test. Notably, the likelihood-based confidence set is considerably shorter than for other methods employed in the literature. Monte Carlo analysis confirms better performance than other methods in terms of length and coverage accuracy in finite samples, including when the magnitude of breaks is small. An application to postwar U.S. real GDP and consumption leads to a much tighter 95% confidence set for the timing of the "Great Moderation" in the mid-1980s than previously found. Furthermore, when taking cointegration between output and consumption into account, confidence sets for structural break dates are even more precise and suggest a sudden "productivity growth slowdown" in the early 1970s and an additional large, abrupt decline in long-run growth in the mid-1990s.

Keywords: Inverted Likelihood Ratio Confidence Sets; Multiple Breaks; Great Moderation; Productivity Growth Slowdown

JEL classification: C22; C32; E20

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*We thank the editor, Elie Tamer, and two anonymous referees for suggestions that vastly improved the paper. We acknowledge the use of GAUSS code made available by Zhongjun Qu and Pierre Perron and can provide compatible code upon request that calculates likelihood-based confidence sets for structural breaks within their modeling framework. Junsoo Lee, Irina Panovska, Jeremy Piger, Werner Ploberger, Barbara Rossi, Farshid Vahid, Tao Zha, and seminar and conference participants at Monash University, University of Alabama, Universitat Pompeu Fabra, 2008 North American Summer Meeting of the Econometric Society, 2008 Missouri Economics Conference, 2008 NBER-NSF Time Series Conference, 2008 Midwest Econometrics Group, and 2009 Society for Nonlinear Dynamics and Econometrics Symposium provided helpful comments. All errors are our own.

†Yunjong Eo, School of Economics, University of Sydney, Sydney 2006, Australia; Tel: +61 2 9351 3075; Fax: + 61 2 9351 4341; Email: yunjong.eo@sydney.edu.au

‡*Corresponding Author:* James Morley, School of Economics, University of New South Wales, Sydney 2052, Australia; Tel: +612 9385 3366; Fax: +612 9313 6337; Email: james.morley@unsw.edu.au

1 Introduction

The exact timing of structural breaks in parameters from time series models is generally unknown *a priori*. Much of the literature on structural breaks has focused on accounting for uncertainty about this timing when testing for the existence of structural breaks (e.g., Andrews (1993)). However, there has also been considerable interest in how to make inference about the timing itself, with an important contribution made by Bai (1997). Employing the asymptotic thought-experiment of a slowly shrinking magnitude of a break, Bai derives the distribution of a break date estimator in a linear time series regression model and uses a related statistic to construct confidence intervals for the timing of breaks. One problem for Bai's approach is that the confidence intervals tend to undercover in finite samples, even given moderately-sized breaks. Elliott and Müller (2007) propose a different approach based on the inversion of a test for an additional break under the null hypothesis of a given break date and employing the asymptotic thought-experiment of a quickly shrinking magnitude of break. Their approach produces a confidence set (not necessarily an interval) for the timing of a break that has very accurate coverage rates in finite samples, even given small breaks. However, it is only applicable for a single break and tends to produce wide confidence sets, including when breaks are large.

In this paper, we propose the use of likelihood-based confidence sets for the timing of structural breaks in parameters from time series regression models. Under the asymptotic thought-experiment of a judiciously-chosen slowly shrinking magnitude of break, as in Bai (1997) and originally proposed by Picard (1985), we show that likelihood-based confidence sets are valid in Qu and Perron's (2007) broad setting of quasi-maximum likelihood estimation based on Normal errors for a system of multivariate linear regression equations under fairly general assumptions about regressors and errors. This setting allows for multiple breaks in mean and variance parameters and, as shown by Bai, Lumsdaine, and Stock (1998) and Qu and Perron (2007), potentially produces more precise inferences as additional equations are added to the system. Our asymptotic analysis provides critical values for a likelihood ratio test of a hypothesized break date and an analytical expression for the expected length of a likelihood-based confidence set based on inverting the likelihood ratio test.

Notably, we find that the asymptotic expected length of a 95% likelihood-based confidence set is approximately half of that for Bai's (1997) approach.[1]

Our proposed approach is motivated by Siegmund (1988), who considers likelihood-based confidence sets in the simpler context of a changepoint model of independent Normal observations with a one-time break in mean. We follow Siegmund's suggestion of constructing an inverted likelihood ratio (ILR) confidence set for the break date.[2] In addition, our calculation of the asymptotic expected length of an ILR confidence set follows from his analysis in the simpler setting. Another related study is by Dümbgen (1991), who derives the asymptotic distribution of a break date estimator given independent, but not necessarily Normal observations and proposes inverting a bootstrap version of a likelihood ratio test to construct a confidence set for the break date. More recently, Hansen (2000) proposes the use of an ILR confidence set in the related context of a threshold regression model. However, he maintains the assumption of a stationary threshold variable, thus precluding the use of a deterministic time trend as a threshold variable in order to capture a structural break. Despite a somewhat different setup, our asymptotic analysis builds closely off of Hansen's.

We consider a range of Monte Carlo experiments in order to evaluate the finite-sample performance of the competing methods for constructing confidence sets of structural break dates. We allow both large and small breaks in mean and variance, including in the presence of serial correlation, multiple breaks, and a multivariate setting.[3] The Monte Carlo analysis supports the asymptotic results in the sense that the ILR confidence sets always

[1]Expected length is more difficult to determine for Elliott and Müller's (2007) approach. However, if the asymptotic power for the test of an additional break is strictly less than one when the true break date is within some fixed fraction of the sample period away from the hypothesized break, the expected length of their confidence set will increase with the sample size. This pattern is confirmed in our Monte Carlo analysis, even for an extremely large magnitude of break for which the power of a test for the existence of a break will be high regardless of its timing.

[2]Siegmund (1988) also suggests constructing a likelihood-based confidence set using what can be thought of as the marginal "fiducial distribution" of a break date. In particular, a marginal fiducial distribution of a break date is equivalent to a Bayesian marginal posterior distribution for the break date given a flat prior and integrating out other parameters over the likelihood. The motivation for using a fiducial distribution to construct a frequentist confidence set for a break date, which Siegmund (1988) attributes to Cobb (1978), ultimately comes from Fisher's (1930) idea of using fiducial inference to construct a confidence set for a location parameter. In practice, we find that both methods of constructing likelihood-based confidence sets perform very similarly, but inverting a likelihood ratio test is far more computationally efficient. Thus, we focus on ILR confidence sets in this paper.

[3]Following Elliott and Müller (2007), we refer to 'large' breaks as those that can be detected with near certainty using a test for structural instability and 'small' breaks as those that cannot.

have the shortest length, at the same time maintaining accurate, if somewhat conservative, coverage. Bai's (1997) approach produces confidence intervals that are much longer than for the likelihood-based approach, consistent with the asymptotic results, and they tend to undercover for even moderately-sized breaks, including when considering a bootstrap version. Meanwhile, as emphasized by Elliott and Müller (2007), their approach has accurate coverage in finite samples. However, their confidence sets are always longer than for the likelihood-based approach, including for small breaks and especially for larger sample sizes.

To demonstrate the empirical relevance of the shorter expected length of the ILR confidence sets, we apply the various methods to make inference about the timing of structural breaks in postwar U.S. real GDP and consumption. Consistent with the asymptotic and Monte Carlo results, we find the ILR confidence set for the timing of the so-called "Great Moderation" in quarterly output growth is less than half of what it is for standard approaches. Indeed, the 95% ILR confidence set is similar to the 67% confidence interval reported in Stock and Watson (2002) based on Bai's (1997) approach.[4] The short length of the ILR confidence set supports the idea that the Great Moderation was an abrupt change in the mid-1980s, rather than a gradual reduction in volatility, potentially providing insight into its possible sources (see, Morley (2009)). Meanwhile, when taking cointegration between output and consumption into account, confidence sets for structural break dates are even more precise, which is consistent with the findings in Bai, Lumsdaine, and Stock (1998). In addition to the Great Moderation, we find evidence of large, abrupt declines in the long-run growth rate of the U.S. economy in the early 1970s, corresponding to the "productivity growth slowdown", and again in the mid-1990s.

The rest of this paper is organized as follows. Section 2 establishes the asymptotic properties of the likelihood-based confidence sets for the timing of structural breaks in parameters

[4] Stock and Watson (2002) consider the four-quarter growth rate for U.S. real GDP, rather than the annualized quarterly growth rate, as considered here. They discuss that because they use Bai's (1997) approach by regressing the absolute value of residuals from an autoregression of real GDP growth on a constant and allowing a break in the constant from the auxiliary regression, the break estimator has a non-Normal and heavy-tailed distribution, and the 95% confidence interval would be very wide, hence their reporting of the 67% interval. Meanwhile, our ILR confidence sets are much more similar to the 95% credibility set for the timing of the Great Moderation found in Kim, Morley, and Piger (2008) based on the marginal posterior distribution of the break date given a flat/improper prior for the parameters of a linear time series regression model, which is computationally (but not conceptually) equivalent to the approach based on a fiducial distribution suggested by Siegmund (1988).

from time series regression models. Section 3 presents Monte Carlo analysis comparing the finite-sample performance of the likelihood-based approach to the widely-used methods developed by Bai (1997) and Elliott and Müller (2007). Section 4 provides an application to the timing of structural breaks in postwar U.S. real GDP and consumption. Section 5 concludes.

2 Asymptotics

In this section, we make explicit some assumptions for which the likelihood-based confidence set of a structural break date is asymptotically valid. In particular, we consider Qu and Perron's (2007) broad setting of a system of multivariate linear regression equations with possible multiple breaks in mean and variance parameters. However, it should be emphasized that this setting encompasses the simpler univariate and single-equation models that are often considered in structural break analysis (see, for example, Bai (1997) and Bai and Perron (1998, 2003)).

Our asymptotic analysis proceeds as follows: First, we present the general model and assumptions. Second, we discuss quasi-maximum likelihood estimation of the model and establish results for the asymptotic distribution of the likelihood ratio test of a structural break date and a confidence set for the break date based on inverting the likelihood ratio test.

2.1 Model and Assumptions

We consider a multivariate regression model with multiple structural changes in the regression coefficients and/or the covariance matrix of the errors. The model is assumed to have n equations with m structural breaks (i.e. $m+1$ regimes) with break dates $\Upsilon = (\tau_1, \ldots, \tau_m)$.

Following the notation of Qu and Perron (2007), the model in the jth regime is given as

$$y_t = (I_n \otimes z_t')S\beta_j + u_t, \text{ for } \tau_{j-1} < t \leq \tau_j \tag{1}$$

where y_t is a $n \times 1$ vector, $z_t = (z_{1t}, \ldots, z_{qt})'$ is a $q \times 1$ vector of regressors, β_j is a $p \times 1$ vector of regression coefficients, u_t is a $n \times 1$ vector of errors with mean 0 and covariance matrix Σ_j, and $j = 1, ..., m+1$. The matrix S is a selection matrix for regressors z_t. It consists of 0 or 1 elements and has the dimension $nq \times p$ with full column rank.[5] Note that, if all the regressors are included in each equation, $nq = p$. Also, it is possible to impose a set of r cross- and within-equation restrictions across or within structural regimes in the general form of

$$g(\beta) = 0$$

where $\tilde{\beta} = (\beta_1', \ldots, \beta_{m+1}')$ and $g(\cdot)$ is an r-dimensional vector. Then, for notational simplicity, we can rewrite (1) as

$$y_t = x_t'\beta_j + u_t \tag{2}$$

where the $p \times n$ matrix x_t is defined by $x_t' = (I_n \otimes z_t')S$.

In developing our asymptotic results, we closely follow the assumptions in Bai (1997, 2000) and Qu and Perron (2007). Let $||X||_r = \left(\sum_i \sum_j E|X_{ij}|^r\right)^{1/r}$ for $r \geq 1$ denote the L_r norm of a random matrix X, $< \cdot >$ denote the usual inner product, λ_{min} and λ_{max} denote the smallest and largest eigenvalues respectively, and $[\cdot]$ denote the greatest integer function. Also, let the true values of the parameters be denoted with a superscript 0. Then, the assumptions are given as follows:

Assumption 1 *$\tau_j^0 = [T\lambda_j^0]$ for $j = 1, \ldots, m+1$ with $0 < \lambda_1^0 < \cdots < \lambda_m^0 < 1$.*

Assumption 2 *For each $j = 1, \ldots, m+1$ and $l_j \leq \tau_j^0 - \tau_{j-1}^0$, $(1/l_j) \times \sum_{t=\tau_{j-1}^0+1}^{\tau_{j-1}^0+l_j} x_t x_t' \xrightarrow{a.s.} H_j^0$ as $l_j \to \infty$ with H_j^0 a nonrandom positive definite matrix not necessarily the same for all j. In addition, for $\Delta\tau_j^0 = \tau_j^0 - \tau_{j-1}^0$, as $\Delta\tau_j^0 \to \infty$, uniformly in $s \in [0,1]$, $(1/\Delta\tau_j^0) \times \sum_{t=\tau_{j-1}^0+1}^{\tau_{j-1}^0+[s\Delta\tau_j^0]} x_t x_t' \xrightarrow[p]{} sH_j^0$.*

[5] Suppose there are two equations ($n = 2$) and three regressors ($q = 3$). If the first and second regressors are used in the first equation and the first and third regressors are used in the second equation, the selection matrix S would be specified as follows.

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Assumption 3 *There exists* $l_0 > 0$ *such that for all* $l > l_0$*, the matrices* $(1/l) \times \sum_{t=\tau_j^0+1}^{\tau_j^0+l} x_t x_t'$ *and* $(1/l) \times \sum_{t=\tau_j^0-l}^{\tau_j^0} x_t x_t'$ *have the minimum eigenvalues bounded away from zero for all* $j = 1, \ldots, j$*.*

Assumption 4 *The matrix* $\sum_{t=k}^{l} x_t x_t'$ *is invertible for* $l - k \geq k_0$ *for some* $0 < k_0 < \infty$*.*

Assumption 5 *If* $x_t u_t$ *is weakly stationary within each segment, then*

(a) $\{x_t u_t, \mathcal{F}_t\}$ *form a strongly mixing* ($\alpha - mixing$) *sequence with size* $-4r/(r-2)$ *for some* $r > 2$ *for* $\mathcal{F}_t = \sigma - fields\,\{\ldots, x_{t-1}, x_t, , x_{t+1}, \ldots, , u_{t-2}, u_{t-1}\}$*,*

(b) $E(x_t u_t) = 0$ *and* $||x_t u_t||_{2r+\delta} < M < \infty$ *for some* $\delta > 0$ *and,*

(c) letting $S_{k,j}(l) = \sum_{t=\tau_{j-1}^0+l+1}^{\tau_{j-1}^0+l+k} x_t u_t, j = 1, \ldots, m+1$*, for each* $e \in R^n$ *of length 1, var*$(< e, S_{k,l}(0) >) \geq v(k)$ *for some function* $v(k) \to \infty$ *as* $k \to \infty$*.*

Or, if $x_t u_t$ *is not weakly stationary within each segment, assume (a)-(c) holds and, in addition, there exists a positive definite matrix* $\Omega = [w_{i,s}]$ *such that, for any* $i, s = 1, \ldots, p$*, we have, uniformly in* l*, that* $|k^{-1} E((S_{k,j}(l)_i S_{k,j}(l)_s) - w_{i,s}| \leq C_2 k^{-\psi}$ *for some* C_2 *and* $\psi > 0$*.*

Assumption 6 *Assumption 5 holds with* $x_t u_t$ *replaced by* u_t *or* $u_t u_t' - \Sigma_j^0$ *for* $\tau_{j-1} < t \leq \tau_j \quad (j = 1, \ldots, m+1)$*.*

Assumption 7 *The magnitudes of the shifts satisfy* $\beta_{T,j+1}^0 - \beta_{T,j}^0 = v_T \delta_j$*, and* $\Sigma_{T,j+1}^0 - \Sigma_{T,j}^0 = v_T \Phi_j$ *where* $(\delta_j, \Phi_j) \neq 0$ *and they are independent of* T*. Moreover,* v_T *is a sequence of positive numbers that satisfy* $v_T \to 0$ *and* $T^{1/2} v_T / (logT)^2 \to \infty$*. (Note that, for simplicity, we use* β_j^0 *and* Σ_j^0 *from now on, suppressing the subscript* T*.)*

Assumption 8 $(\beta^0, \Sigma^0) \in \overline{\Theta}$ *with* $\overline{\Theta} = \{(\beta, \Sigma) : ||\beta|| \leq c_1, \lambda_{min}(\Sigma) \geq c_2, \lambda_{max}(\Sigma) \leq c_3\}$ *for some* $c_1 \leq \infty$*,* $0 < c_2 \leq c_3 < \infty$*.*

Again, these assumptions are based on Qu and Perron (2007) and are discussed in detail in their paper. However, we provide a brief explanation here. Assumption 1 restricts the break dates to be asymptotically distinct and implies that we should allow for a reasonable number of observations between the break dates. Assumption 2 is used for the central limit theorem and allows the regressors to have different distributions, although it excludes unit root regressors and trending regressors. Assumption 3 requires that there is no local collinearity in the regressors near the break dates. Assumption 4 is a standard invertibility condition to ensure well-defined estimates. Assumptions 5 and 6 determine the structure of the $x_t u_t$ and u_t processes and imply short memory for $x_t u_t$ and $u_t u_t'$ with bounded fourth moments. These assumptions guarantee strongly consistent estimates and a well-behaved likelihood function while, at the same time, being mild in the sense of allowing for substantial conditional heteroskedasticity and autocorrelation and encompassing a wide range of econometric models, as discussed in Qu and Perron (2007). Assumption 7 implies that, although the magnitude of structural change shrinks as the sample size increases, it is large enough so that we can derive the limiting distributions for the estimates of the break dates which are independent of the exact distributions of regressors and errors. This assumption follows from Picard (1985) and Bai (1997), among many others. Note that Elliott and Müller (2007) make the assumption that v_T shrinks at a faster rate to consider a *smaller* magnitude of break. We will also consider the implications of this different assumption in our Monte Carlo analysis in the next section. Finally, Assumption 8 implies that the data are generated by innovations with a nondegenerate covariance matrix and a finite conditional mean.

2.2 Estimation, Likelihood Ratio, and Likelihood-Based Confidence Set

Following Qu and Perron (2007), the model in (2) can be consistently estimated by restricted quasi-maximum likelihood based on the (potentially false) assumption of serially-uncorrelated Normal errors in order to construct the likelihood. Let $\tilde{\beta} = (\beta_1, \ldots, \beta_{m+1})$ and $\tilde{\Sigma} = (\Sigma_1, \ldots, \Sigma_{m+1})$. Then, conditional on a set of break dates $\Upsilon = (\tau_1, \ldots, \tau_m) = (T\lambda_1, \ldots, T\lambda_m)$ where $(\lambda_1, \ldots, \lambda_m) \in \bigwedge_\epsilon = \{(\lambda_1, \ldots, \lambda_m); |\lambda_{j+1} - \lambda_j| \geq \epsilon, \lambda_1 \geq \epsilon, \lambda_m \geq \epsilon\}$, the quasi-likelihood function is

$$L_T(\Upsilon, \tilde{\beta}, \tilde{\Sigma}) = \prod_{j=1}^{m+1} \prod_{t=\tau_{j-1}+1}^{\tau_j} f\left(y_t | x_t; \beta_j, \Sigma_j\right)$$

where

$$f\left(y_t|x_t;\beta_j,\Sigma_j\right)=\frac{1}{(2\pi)^{n/2}|\Sigma_j|^{1/2}}\exp\left\{-\frac{1}{2}(y_t-x_t'\beta_j)\Sigma_j^{-1}(y_t-x_t'\beta_j)\right\}.$$

Then, the quasi log-likelihood ratio for the timing of break dates (not the existence of structural breaks) evaluated at the true parameter values $(\tilde{\beta}^0,\tilde{\Sigma}^0)$ is given by

$$\begin{aligned} lnLR_T(\Upsilon,\tilde{\beta}^0,\tilde{\Sigma}^0) &= \sum_{j=1}^{m}\sum_{t=\tau_{j-1}+1}^{\tau_j} lnf\left(y_t|x_t;\beta_j^0,\Sigma_j^0\right)-\sum_{j=1}^{m}\sum_{t=\tau_{j-1}^0+1}^{\tau_j^0} lnf\left(y_t|x_t;\beta_j^0,\Sigma_j^0\right) \\ &= \sum_{j=1}^{m} lr_j(\tau_j-\tau_j^0) \end{aligned}$$

where, letting $r=\tau_j-\tau_j^0$,

$$\begin{aligned} lr_j(r) &= 0 \text{ for } r=0 \\ lr_j(r) &= -\frac{r}{2}(ln|\Sigma_j^0|-ln|\Sigma_{j+1}^0|) \\ &\quad -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0}(y_t-x_t'\beta_{j+1}^0)(\Sigma_{j+1}^0)^{-1}(y_t-x_t'\beta_{j+1}^0)-(y_t-x_t'\beta_j^0)(\Sigma_j^0)^{-1}(y_t-x_t'\beta_j^0) \text{ for } r<0, \\ lr_j(r) &= -\frac{r}{2}(ln|\Sigma_j^0|-ln|\Sigma_{j+1}^0|) \\ &\quad -\frac{1}{2}\sum_{t=\tau_j^0+1}^{\tau_j^0+r}(y_t-x_t'\beta_j^0)(\Sigma_j^0)^{-1}(y_t-x_t'\beta_j^0)-(y_t-x_t'\beta_{j+1}^0)(\Sigma_{j+1}^0)^{-1}(y_t-x_t'\beta_{j+1}^0) \text{ for } r>0. \end{aligned}$$

From Assumption 7, let $\Delta\beta_j=\delta_j v_T$ and $\Delta\Sigma_j=\Phi_j v_T$ where $v_T\to 0$ such that $T^{1/2}v_T/(lnT)^2\to\infty$. Also, letting $\eta_t=(\Sigma_j^0)^{-1/2}u_t$ denote standardized errors, define some parameters as follows:

$$
\begin{aligned}
B_{1,j} &= (\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Delta\Sigma_j(\Sigma_j^0)^{-1/2}, \\
B_{2,j} &= (\Sigma_{j+1}^0)^{1/2}(\Sigma_j^0)^{-1}\Delta\Sigma_j(\Sigma_{j+1}^0)^{-1/2}, \\
Q_{1,j} &= \underset{T\to\infty}{plim}(\tau_j^0-\tau_{j-1}^0)^{-1}\sum_{t=\tau_{j-1}^0+1}^{\tau_j^0} x_t(\Sigma_{j+1}^0)^{-1}x_t', \\
Q_{2,j} &= \underset{T\to\infty}{plim}(\tau_{j+1}^0-\tau_j^0)^{-1}\sum_{t=\tau_j^0+1}^{\tau_{j+1}^0} x_t(\Sigma_j^0)^{-1}x_t', \\
\Pi_{1,j} &= \lim_{T\to\infty} var\left\{(\tau_j^0-\tau_{j-1}^0)^{-1/2}\left[\sum_{t=\tau_{j-1}^0+1}^{\tau_j^0} x_t(\Sigma_{j+1}^0)^{-1}(\Sigma_j^0)^{1/2}\eta_t\right]\right\}, \\
\Pi_{2,j} &= \lim_{T\to\infty} var\left\{(\tau_{j+1}^0-\tau_j^0)^{-1/2}\left[\sum_{t=\tau_j^0+1}^{\tau_{j+1}^0} x_t(\Sigma_j^0)^{-1}(\Sigma_{j+1}^0)^{1/2}\eta_t\right]\right\}, \\
\Omega_{1,j} &= \lim_{T\to\infty} var\left\{vec\left[(\tau_j^0-\tau_{j-1}^0)^{-1/2}\sum_{t=\tau_{j-1}^0+1}^{\tau_j^0}(\eta_t\eta_t'-I)\right]\right\}, \\
\Omega_{2,j} &= \lim_{T\to\infty} var\left\{vec\left[(\tau_{j+1}^0-\tau_j^0)^{-1/2}\sum_{t=\tau_j^0+1}^{\tau_{j+1}^0}(\eta_t\eta_t'-I)\right]\right\}, \\
\Gamma_{1,j} &= \left(\frac{1}{4}vec(B_{1,j})'\Omega_{1,j}^0 vec(B_{1,j})+\Delta\beta_j'\Pi_{1,j}\Delta\beta_j\right)^{1/2} \\
\Gamma_{2,j} &= \left(\frac{1}{4}vec(B_{2,j})'\Omega_{2,j}^0 vec(B_{2,j})+\Delta\beta_j'\Pi_{2,j}\Delta\beta_j\right)^{1/2} \\
\Psi_{1,j} &= \left(\frac{1}{2}tr(B_{1,j}^2)+\Delta\beta_j'Q_{1,j}\Delta\beta_j\right) \\
\Psi_{2,j} &= \left(\frac{1}{2}tr(B_{2,j}^2)+\Delta\beta_j'Q_{2,j}\Delta\beta_j\right)
\end{aligned}
$$

Given this setup, we can establish the following results for the asymptotic distribution of a likelihood ratio test for a structural break date and the confidence set for the break date based on inverting the likelihood ratio test:

Proposition 1 *Under Assumptions 1-8, for the* j*th break date*

$$lr_j(\hat{\tau}_j - \tau_j^0) \Rightarrow \max_v \begin{cases} \omega_{1,j}\left(-\frac{1}{2}|v| + W_j(v)\right) & \textit{for} \quad v \in (-\infty, 0] \\ \omega_{2,j}\left(-\frac{1}{2}|v| + W_j(v)\right) & \textit{for} \quad v \in (0, \infty) \end{cases} \tag{3}$$

where $W_j(v)$ *is a standard Wiener processes defined on the real line,*

$$\omega_{1,j} = \frac{\Gamma_{1,j}^2}{\Psi_{1,j}}, \text{ and } \omega_{2,j} = \frac{\Gamma_{2,j}^2}{\Psi_{2,j}}.$$

Proposition 1 establishes the asymptotic distribution of the likelihood ratio test. Note that the distribution is asymmetric unless $\omega_{1,j} = \omega_{2,j}$, in which case it is just a rescaled version of the distribution $\max_v -\frac{1}{2}|v| + W(v)$ studied in Bhattacharya and Brockwell (1976). In practice, we can replace the true values of $\omega_{1,j}$ and $\omega_{2,j}$ with consistent estimates.

Proposition 2 *Under Assumptions 1-8, the confidence set for the* j*th break date is given by* $C_j(y) = \{t | \max_{\tau_j} \ln L(\tau_j|y) - \ln L(t|y) \leq \kappa_{\alpha,j}\}$ *where the asymptotic critical value used to construct the* $1-\alpha$ *likelihood-based confidence set is* $\kappa_{\alpha,j}$ *such that*

$$\left(1 - \exp(-\frac{\kappa_{\alpha,j}}{\omega_{1,j}})\right)\left(1 - \exp(-\frac{\kappa_{\alpha,j}}{\omega_{2,j}})\right) = 1 - \alpha.$$

Proposition 2 shows how to construct the likelihood-based confidence set for the jth break date from the likelihood ratio test in Proposition 1, including how to determine the asymptotic critical value at level α. By inverting the likelihood ratio test, we can construct a confidence set given the critical value. Note that the likelihood-based confidence sets in Proposition 2 are constructed under the assumption that the magnitude of the break $\Delta\beta_{T,j} \to 0$ and $\Delta\Sigma_{T,j} \to 0$ as $T \to \infty$, so that the actual coverage should exceed the desired level $(1-\alpha)$ for a given fixed magnitude of break, at least for Normal errors (see Hansen (2000)).

Proposition 3 *Under Assumptions 1-8 and as* $\alpha \to 0$*, the expected length of a* $1-\alpha$

likelihood-based confidence set is

$$
\begin{aligned}
& 2\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)\left(1-exp(-\kappa_{\alpha,j}/\omega_{1,j})\right)\left\{\kappa_{\alpha,j}/\omega_{1,j}-\frac{1}{2}(1-exp(-\kappa_{\alpha,j}/\omega_{1,j}))\right\} \\
+\ & 2\left(\Gamma_{2,j}^2/\Psi_{2,j}^2\right)\left(1-exp(-\kappa_{\alpha,j}/\omega_{2,j})\right)\left\{\kappa_{\alpha,j}/\omega_{2,j}-\frac{1}{2}(1-exp(-\kappa_{\alpha,j}/\omega_{2,j}))\right\}.
\end{aligned}
$$

Proposition 3 establishes the expected length of $1-\alpha$ likelihood-based confidence set. The length is calculated by measuring the expected size of the break dates τ_j such that $lr_j(\hat{\tau}_j-\tau_j)\leq\kappa_{\alpha,j}$. Siegmund (1986, 1988) considers the simple case of a Normal random variable with a structural break in mean and variance of unity. When the magnitude of mean shift is fixed and known, he shows that the likelihood ratio $lr_j(\hat{\tau}-\tau^0)$ can be approximated by the distribution of $\max_r -\frac{1}{2}|r|+W(r)$ where $r=\hat{\tau}-\tau^0$ and calculates the expected length as $\alpha\to 0$. Here, we consider a more general setting for which the limiting theory requires a shrinking magnitude of break to allow for analytical results. Therefore, the distance between the break dates under the null and alternative hypotheses is scaled using a change in variables to attain the distribution in (3). As a result, v in $\max_v -\frac{1}{2}|v|+W(v)$ is not the distance between two break dates, unlike $r=\hat{\tau}-\tau^0$ in the simple case of Siegmund (1986, 1988). In our case, we calculate the expected size based on the distribution of $\max_v -\frac{1}{2}|v|+W(v)$ for $v\geq 0$ and $v<0$, respectively, and rescaled by $\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)$ for $\tau_j\geq\hat{\tau}_j$ and $\left(\Gamma_{2,j}^2/\Psi_{2,j}^2\right)$ for $\tau_j<\hat{\tau}_j$. Note that the scales for the break dates, $\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)$ and $\left(\Gamma_{2,j}^2/\Psi_{2,j}^2\right)$, used to derive the distribution do not affect the value of the likelihood ratio statistic directly, but they are scale factors for the position of a break date, which means they should be considered when calculating the expected length. See the proof for more details.

In the following two corollaries, we consider simplified cases for either breaks in conditional mean or breaks in variance and solve for the simplified asymptotic distribution of the likelihood ratio statistic for a break date, critical values, and expressions for expected length:

Corollary 1 *Under Assumptions 1-8 and additionally if (i) there are only changes in conditional mean and (ii) the errors form a martingale difference sequence, then for the jth break date*

$$lr_j(\hat{\tau}_j-\tau_j^0)\Rightarrow\max_v\ \left(-\tfrac{1}{2}|v|+W_j(v)\right)\quad for\quad v\in(-\infty,\infty)$$

where $W_j(v)$ is a standard Wiener processes defined on the real line. Also, as $\alpha \to 0$, the asymptotic critical value of a $1-\alpha$ likelihood-based confidence set is

$$\kappa_{\alpha,j} = -\ln\left(1-(1-\alpha)^{1/2}\right)$$

and the expected length of the confidence set is

$$\left(\frac{1}{\Delta\beta_j' Q_1 \Delta\beta_j} + \frac{1}{\Delta\beta_j' Q_2 \Delta\beta_j}\right) 2(1-exp(-\kappa_{\alpha,j}))\left\{\kappa_{\alpha,j} - \frac{1}{2}(1-exp(-\kappa_{\alpha,j}))\right\}.$$

or, equivalently,

$$\left(\frac{1}{\Delta\beta_j' Q_1 \Delta\beta_j} + \frac{1}{\Delta\beta_j' Q_2 \Delta\beta_j}\right) 2(1-\alpha)^{1/2}\{-\ln[1-(1-\alpha)^{1/2}] - \frac{1}{2}(1-\alpha)^{1/2}\}.$$

Remark 1 *If, in addition to Assumptions in Corollary 1, the distribution of the regressors is stable, $\Pi_{1,j} = Q_{1,j} = \Pi_{2,j} = Q_{2,j}$ and $\omega_j = 1$. Thus, the expected length of confidence set would further simplify to*

$$\left(\frac{1}{\Delta\beta_j' Q \Delta\beta_j}\right) 4(1-exp(-\kappa_{\alpha,j}))\left\{\kappa_{\alpha,j} - \frac{1}{2}(1-exp(-\kappa_{\alpha,j}))\right\}.$$

or, equivalently,

$$\left(\frac{1}{\Delta\beta_j' Q \Delta\beta_j}\right) 4(1-\alpha)^{1/2}\{-\ln[1-(1-\alpha)^{1/2}] - \frac{1}{2}(1-\alpha)^{1/2}\}.$$

Remark 2 *If we replace the assumption of martingale difference errors in Remark 1 with the assumption that the errors are identically distributed, $\Pi = lim_{T\to\infty}\, var\left\{T^{-1/2}\left[\sum_{t=1}^T x_t(\Sigma^0)^{-1/2}\eta_t\right]\right\}$, $Q = plim_{T\to\infty}\, T^{-1}\sum_{t=1}^T x_t(\Sigma^0)^{-1}x_t'$, and $\omega_j = \frac{\Delta\beta_j'\Pi\Delta\beta_j}{\Delta\beta_j' Q\Delta\beta_j}$. Then, the expected length of the confidence set is*

$$\frac{\Delta\beta_j'\Pi\Delta\beta_j}{(\Delta\beta_j' Q\Delta\beta_j)^2} 4(1-exp(-\kappa_{\alpha,j}/\omega_j))\left\{\kappa_{\alpha,j}/\omega_j - \frac{1}{2}(1-exp(-\kappa_{\alpha,j}/\omega_j))\right\}.$$

or, equivalently,

$$\frac{\Delta\beta_j'\Pi\Delta\beta_j}{(\Delta\beta_j'Q\Delta\beta_j)^2}4(1-\alpha)^{1/2}\{-\ln[1-(1-\alpha)^{1/2}]-\frac{1}{2}(1-\alpha)^{1/2}\}.$$

Corollary 2 *Under Assumptions 1-8 and additionally if (i) there are only changes in variance and (ii) the errors are identically Normally distributed, then for the jth break date*

$$lr_j(\hat{\tau}_j-\tau_j^0)\Rightarrow \max_{v}\ \left(-\tfrac{1}{2}|v|+W_j(v)\right)\quad \textit{for}\quad v\in(-\infty,\infty)$$

where $W_j(v)$ is a standard Wiener processes defined on the real line Also, as $\alpha\to 0$, the asymptotic critical value of a $1-\alpha$ likelihood-based confidence set is

$$\kappa_{\alpha,j}=-\ln\left(1-(1-\alpha)^{1/2}\right)$$

and the expected length of the confidence set is

$$\left(\frac{2}{tr(B_1^2)}+\frac{2}{tr(B_2^2)}\right)2(1-exp(-\kappa_{\alpha,j}))\left\{\kappa_{\alpha,j}-\frac{1}{2}(1-exp(-\kappa_{\alpha,j}))\right\}.$$

or, equivalently,

$$\left(\frac{2}{tr(B_1^2)}+\frac{2}{tr(B_2^2)}\right)2(1-\alpha)^{1/2}\{-\ln[1-(1-\alpha)^{1/2}]-\frac{1}{2}(1-\alpha)^{1/2}\}.$$

In the simplified cases of Corollaries 1 and 2, the critical values for the likelihood ratio statistic are the same as reported in Table 1 of Hansen (2000) (divided by 2 given our different scaling of the statistic). For example, the likelihood ratio statistic $lr_j(\hat{\tau}_j-\tau_j^0)$ has critical values of 2.97, 3.68 and 5.30 for a test at the 10%, 5%, and 1% levels, respectively. Meanwhile, the expected length expressions again make use of results in Siegmund (1986, 1988) and allow for easy comparison with the expected length of Bai's (1997) confidence intervals for the timing of structural break dates.

3 Monte Carlo Analysis

In this section, we present extensive Monte Carlo analysis of the finite-sample performance of competing methods for constructing confidence sets of structural break dates. In addition to the likelihood-based method proposed in the previous section, we also consider Bai's (1997) approach, a bootstrap version of Bai's approach, and Elliott and Müller's (2007) approach. For brevity, we omit many of the details of these widely-used methods and encourage interested readers to consult the original articles. However, we make a few comments about these two other methods in the following subsection to help motivate our Monte Carlo experiments and facilitate interpretation of our results.

3.1 Bai (1997) and Elliott and Müller (2007)

Bai (1997) solves for the asymptotic distribution of the least squares break date estimator under the asymptotic thought-experiment of a slowly shrinking magnitude of break. In terms of the notation in the previous section, he assumes that $\upsilon_T \to 0$ and $\upsilon_T T^{1/2-\epsilon} \to \infty$ for some $\epsilon \in (0, 1/2)$ when $\Delta\beta = \upsilon_T \delta$. His confidence intervals are constructed based on this estimator. Bai's approach is designed for univariate analysis under fairly general assumptions about the error term and even allowing for the possibility of a deterministic time trend regressor. His approach has been generalized to more complicated settings of multiple breaks and multivariate models (see Bai, Lumsdaine, and Stock (1998), Bai and Perron (1998, 2003), Bai (2000), and Qu and Perron (2007)).

In order to calculate confidence intervals, Bai (1997) constructs the following statistic with a non-standard distribution:

$$\frac{(\Delta\beta_1' Q_1 \Delta\beta_1)^2}{\Delta\beta_1' \Pi_1 \Delta\beta_1}(\hat{\tau} - \tau_0) \overset{d}{\Rightarrow} \arg\max_s Z(s), \tag{4}$$

where

$$Z(s) = \begin{cases} W_1(-s) - |s|/2 & \text{if } s \leq 0 \\ \sqrt{\varphi} W_2(s) - \varsigma |s|/2 & \text{if } s > 0, \end{cases} \tag{5}$$

with $W_i(s)$, $i = 1, 2$ denoting two independent standard Wiener processes defined on $[0, \infty]$,

$\varphi = \frac{\Delta\beta_1'\Pi_2\Delta\beta_1}{\Delta\beta_1'\Pi_1\Delta\beta_1}$, $\varsigma = \frac{\Delta\beta_1'Q_2\Delta\beta_1}{\Delta\beta_1'Q_1\Delta\beta_1}$, and $\Delta\beta_i$, Q_i, and Π_i $i = 1, 2$ are as defined in the previous section.[6] The confidence intervals are then constructed by using least squares estimates and equal-tailed quantile values:

$$\left[\hat{\tau} - \frac{\Delta\hat{\beta}_1'\hat{\Pi}_1\Delta\hat{\beta}_1}{(\Delta\hat{\beta}_1'\hat{Q}_1\Delta\hat{\beta}_1)^2} \times q(1-\alpha/2), \hat{\tau} - \frac{\Delta\hat{\beta}_1'\hat{\Pi}_1\Delta\hat{\beta}_1}{(\Delta\hat{\beta}_1'\hat{Q}_1\Delta\hat{\beta}_1)^2} \times q(\alpha/2)\right],$$

where $q(\cdot)$ is the quantile function for the non-standard distribution in (4).

When regressors and errors are stationary across regimes (i.e. $Q = Q_1 = Q_2$ and $\Pi = Q$), the asymptotic expected length of Bai's (1997) confidence interval is given by

$$2\frac{1}{(\Delta\beta' Q\Delta\beta)} \times q(1-\alpha/2),$$

where the quantile function $q(\cdot)$ is determined by (5) under simplifying conditions that $\varphi = 1$ and $\varsigma = 1$. For example, the length of the confidence set at 95% confidence level is approximately $22 \times \frac{1}{(\Delta\beta' Q\Delta\beta)}$. Notably, this is almost twice the asymptotic expected length of approximately $12 \times \frac{1}{(\Delta\beta' Q\Delta\beta)}$ for the equivalent 95% likelihood-based confidence set implied by Corollary 1 in the previous section.

Despite its asymptotic justification, poor finite-sample properties of Bai's asymptotic confidence intervals have often been noted (e.g., Elliott and Müller (2007)). Thus, we follow some of the applied literature and also consider a bootstrap version of Bai's approach. Specifically, we construct a bootstrap version of Bai's confidence intervals by using equal-tailed quantile values from the bootstrapped distribution of the statistic in (4) instead of its asymptotic distribution. We consider a parametric bootstrap based on parameter estimates and the assumption of Normal errors. The bootstrap quantiles are calculated using the sample statistic and 199 draws from the boostrap distribution.

We also consider Elliott and Müller's (2007) alternative approach to constructing a confidence set (not interval) for a break date based on the inversion of a sequence of tests for an additional break given a maintained break date. The validity of their approach is established under a different asymptotic thought-experiment of a quickly shrinking magnitude of break

[6]Note that Q_i and Π_i are normalized by the conditional variance, as in Qu and Perron (2007), but different to Bai (1997).

(i.e. $\Delta\beta = \delta T^{-1/2}$). They argue that Bai's approach has poor finite-sample performance due to his asymptotic thought-experiment of a slowly-shrinking magnitude being inappropriate for the moderately-sized breaks that appear to occur in practice. It should be noted, however, that Bai's thought-experiment (originally due to Picard (1985)) is widely-used in the literature on structural breaks, including by Qu and Perron (2007) and, therefore, in our asymptotic analysis in the previous section as well. Meanwhile, it should also be noted that, because Elliott and Müller's approach is based on tests for an additional break, it is only suitable for a one-time break and cannot be generalized to multiple breaks such as with Bai and Perron (1998) for Bai's approach or with the likelihood-based approach proposed here.

Another thing to note is that, even though the approaches of Bai (1997) and Elliott and Müller (2007) are designed for a break in conditional mean, the generality of their assumptions about the error distribution suggests that their approaches can be applied to absolute or squared errors to calculate confidence sets for a break in variance. We use absolute errors, following Stock and Watson (2002), in order to apply these standard methods to a break in variance. However, we do not consider the bootstrap version of Bai's approach in this case as the assumption of Normal errors for the parametric bootstrap would clearly be inappropriate.

3.2 Experiments

For our Monte Carlo experiments, we calculate the empirical coverage rates and average lengths of 95% confidence sets (or intervals) of break dates given data generating processes involving structural breaks. We first consider one break in mean and/or variance for a simple univariate model with i.i.d. Normal errors for sample sizes of $T = 60, 120, 240, 480$ observations. Then, we consider more complicated settings of serial correlation in the errors, multiple breaks, and a multivariate model, all with the sample size of $T = 240$ observations for each simulated series, which roughly corresponds to postwar quarterly observations for macroeconomic time series. We consider 10,000 replications for each experiment.

3.2.1 One break in mean and/or variance

For the experiments of one break in mean or variance, the general univariate model for our data generating processes is given by

$$y_t = z_t'\beta_1 + z_t'\Delta\beta\mathbf{1}[t > \tau] + u_t, \tag{6}$$

where $\mathbf{1}[\cdot]$ is an indicator function, $u_t = \left(\sqrt{\Sigma_1 + \Delta\Sigma\mathbf{1}[t > \tau]}\right) e_t$, and $e_t \sim i.i.d.\mathcal{N}(0, 1)$. To simplify comparisons for different sample sizes, we re-parameterize the structural break date and break magnitudes in terms of the sample size as follows: $\tau = [rT]$, $\Delta\beta = \beta_2 - \beta_1 = \delta T^{-\nu}$ and $\Delta\Sigma = \Sigma_2 - \Sigma_1 = \Phi T^{-\nu}$, where we set $\Phi = 0$ for a break in mean and $\delta = 0$ for a break in variance. For our simulations, we set $z_t = 1$, $\beta_1 = 0$, $\Sigma_1 = 1$, and the true break point fraction to the the midpoint of the sample, $r = 0.5$, although we consider breaks closer to the beginning and end of the sample when we consider multiple breaks in Section 3.2.3 below.

We make three different assumptions about the magnitude of the break in relation to the sample size. First, we assume $\nu = 1/4$, which corresponds to a slowly shrinking magnitude of break, as in our asymptotic analysis. For the respective break in mean or variance, we have $\delta = 4$ or $\Phi = 8$, which imply a moderately-sized break in the sense that the break is large enough to be detected with high probability by a test for structural instability in typical sample sizes available for macroeconomic data, but not so large that the exact break date is known with near certainty. Second, we assume $\nu = 1/2$, which corresponds to a quickly shrinking magnitude of break, exactly as in Elliott and Müller's (2007) asymptotic thought-experiment. Again, we have $\delta = 4$ or $\Phi = 8$, which corresponds to the smallest magnitude for the break in mean considered in their Monte Carlo analysis. Third, we assume $\nu = 0$, which corresponds to a fixed magnitude of break, such as what actually occurs in practice. In this case, we set $\delta = 1$ or $\Phi = 3$, as 4 or 8 would imply very large magnitudes of breaks when $\nu = 0$.

For each simulated sample, we estimate the parameters of the corresponding model in (6) via maximum likelihood using a sample trimming of the first and last 15% of observations for the possible break date. In the case of a break in variance for the approaches of Bai (1997) and Elliott and Müller (2007), quasi-maximum likelihood estimation based on Normal errors

Table 1: Coverage Rate and Length of Confidence Sets: Slowly Shrinking Magnitude of Break

(a) Break in Mean ($\delta = 4$ and $\Delta\beta = \delta/T^{1/4}$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.971	9.268	0.974	11.972	0.969	15.278	0.971	21.070
Bai	0.913	11.425	0.925	15.918	0.937	22.286	0.940	31.150
Bai Bootstrap	0.935	14.511	0.945	19.091	0.951	25.382	0.948	34.177
Elliott&Müller	0.948	16.224	0.951	23.192	0.950	35.198	0.952	54.754

(b) Break in Variance ($\Phi = 8$ and $\Delta\Sigma = \Phi/T^{1/4}$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.974	22.977	0.969	26.572	0.966	29.214	0.968	33.089
Bai	0.894	33.337	0.924	39.379	0.943	47.118	0.952	57.218
Elliott&Müller	0.949	33.925	0.954	46.905	0.948	59.321	0.952	79.524

is applied to the following model allowing for a break in mean of the absolute value of errors from a first-stage regression based on (6) with a constant mean:

$$\sqrt{\frac{\pi}{2}}|u_t| = \Sigma_1 + \Delta\Sigma\mathbf{1}[t > \tau]) + \epsilon_t. \tag{7}$$

Table 1 reports the coverage and length results for the slowly shrinking magnitude of break in mean or variance ($v_T = T^{-1/4}$ with $\delta = 4, \Phi = 0$ for break in mean and $\delta = 0, \Phi = 8$ for break in variance). For both a break in mean and a break in variance, the ILR confidence sets are somewhat conservative in the sense that the true break date is included in the set slightly more than 95% of the time.[7] However, this is clearly preferable to the undercovering the true break date, as occurs for Bai's (1997) approach in the smaller samples. The bootstrap version of Bai's approach does somewhat better in terms of coverage, but it does so at the cost of additional length. By contrast, even though the ILR confidence sets are conservative, the confidence sets are shorter in length than Bai's approach, even when

[7] Although we are considering a slowly-shrinking magnitude of break across different sample sizes in these Monte Carlo experiments, the break for any given sample size is, of course, of fixed magnitude. Thus, if the asympotitic distribution of the likelihood ratio statistic for a slowly-shrinking magnitude of break provides an upper bound on the distribution of a fixed magnitude of break, as it should according to Hansen (2000), at least under Normal errors (and assumed in (6)), we would expect the coverage to be conservative for any given experiment.

Table 2: Coverage Rate and Length of Confidence Sets: Quickly Shrinking Magnitude of Break

(a) Break in Mean ($\delta = 4$ and $\Delta\beta = \delta/T^{1/2}$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.961	35.463	0.961	69.640	0.952	136.862	0.951	273.012
Bai	0.867	57.669	0.866	114.157	0.859	220.442	0.852	434.885
Bai Bootstrap	0.783	83.186	0.785	170.525	0.779	338.456	0.778	679.379
Elliott&Müller	0.950	44.552	0.947	92.202	0.950	187.769	0.946	379.790

(b) Break in Variance ($\Phi = 8$ and $\Delta\Sigma = \Phi/T^{1/2}$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.971	37.328	0.968	69.505	0.959	130.766	0.957	249.193
Bai	0.880	197.081	0.883	145.857	0.883	275.740	0.886	480.221
Elliott&Müller	0.950	47.040	0.950	93.391	0.956	184.019	0.952	358.890

Bai's approach undercovers. Thus, there is no tradeoff involved in using the likelihood-based approach instead to Bai's approach. Meanwhile, Elliott and Müller's (2007) approach produces even longer confidence sets than Bai's approach, although its coverage is extremely accurate, as was found in their original study. Of course, it may not be fair to evaluate the performance of Elliott and Müller's approach in this setting of a slowly-shrinking magnitude of break because it was designed based on a different asymptotic thought-experiment.

With Elliott and Müller's (2007) thought-experiment in mind, Table 2 reports the coverage and length results for the quickly shrinking magnitude of break ($v_T = T^{-1/2}$, with $\delta = 4, \Phi = 0$ for break in mean and $\delta = 0, \Phi = 8$ for break in variance). For both a break in mean and a break in variance, the ILR confidence sets remain somewhat conservative, but still has the shortest average length. Bai's (1997) confidence interval performs particularly poorly in this setting, with considerable undercoverage even in large samples and longer average length than even Elliott and Müller's confidence sets.[8] The bootstrap version of Bai's approach fails to correct the coverage problem in this setting. Instead, bootstrapping

[8]This experiment is very similar to the experiment in Table 3 of Elliott and Müller (2007), although they consider $T = 100$. However, Bai's approach does not perform quite as poorly here as in Elliott and Müller (2007). The reason is that we trim the possible break dates to exclude the first and last 15% of the sample period, as is standard in the structural break literature, while Elliott and Müller (2007) only trim the first and last 5% of the sample period.

Table 3: Coverage Rate and Length of Confidence Sets: Fixed Magnitude of Break

(a) Break in Mean ($\delta = 1$ and $\Delta\beta = \delta/T^0$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.967	18.949	0.966	17.926	0.970	16.296	0.974	15.484
Bai	0.888	21.654	0.919	22.736	0.933	22.976	0.946	23.050
Bai Bootstrap	0.908	31.618	0.943	30.177	0.944	26.298	0.952	24.600
Elliott&Müller	0.948	26.959	0.953	30.530	0.950	35.978	0.952	45.536

(b) : Break in Variance ($\Phi = 3$ and $\Delta\Sigma = \Phi/T^0$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.971	22.325	0.970	20.386	0.969	18.840	0.971	17.456
Bai	0.890	33.205	0.945	32.013	0.965	32.588	0.960	32.370
Elliott&Müller	0.947	47.330	0.962	39.434	0.949	43.577	0.946	53.236

makes the undercoverage worse and the average length of the intervals even longer. Indeed, the bootstrap version of Bai's approach accomplishes the remarkable feat of undercovering despite having an average length that is longer than the sample size! Yet, it is notable that the failure of Bai's approach appears not to be due to the asymptotic thought-experiment per se because the likelihood-based approach developed under the same asymptotic thought-experiment performs well in this setting even given the smallest magnitude of break ($\delta = 4$) considered by Elliott and Müller (2007).

In reality, the magnitude of a structural break is fixed. The idea of a shrinking break at a judiciously-chosen rate is a useful construct for asymptotic analysis because it allows for an analytical solution to the asymptotic distribution of the likelihood ratio statistic and it provides an upper bound on the asymptotic distribution under a fixed break, at least under Normal errors (see Hansen (2000)). Thus, it is interesting to consider how the different methods for constructing confidence sets perform given a fixed magnitude of break. Table 3 reports the coverage and length results in this case ($v_T = 1$, with $\delta = 1, \Phi = 0$ for break in mean and $\delta = 0, \Phi = 3$ for break in variance). Again, for both a break in mean and a break in variance, the ILR confidence set always has conservative coverage, but the shortest average length. Bai's approach undercovers in smaller samples, although the bootstrap version is reasonably accurate for $T = 120$ and above. For the likelihood-based approach and Bai's

Table 4: Coverage Rate and Length of Confidence Sets: Break in Mean and Variance with Slowly Shrinking Magnitude of Break ($\delta = 4$, $\Phi = 8$ and $\Delta\beta = \delta/T^{1/4}$)

	T=60		T=120		T=240		T=480	
	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	0.975	14.162	0.974	16.030	0.966	17.945	0.976	21.219

approach, the average lengths converge to a similar proportion as the asymptotic expected lengths calculated in the Section 3.1, although the convergence is slower for the break in mean than the break in variance. Meanwhile, Elliott and Müller's (2007) approach has extremely accurate coverage, but a large average length that increases with the sample size.

Despite a focus on a break in mean or variance in the preceding experiments, it is straightforward to consider a break in mean and variance for the likelihood-based approach. Table 4 reports the coverage and length results for the ILR confidence set given a slowly shrinking magnitude of break in mean and variance ($v_T = T^{-1/4}$, with $\delta = 4, \Phi = 8$). As before, the ILR confidence sets are conservative. Meanwhile, the average lengths are somewhere in between the results for a break in mean and a break in variance in Table 1, although it should be noted that the asymptotic expected length is not the same as for the cases of a break in mean or break in variance only. Importantly, the average length is always better than that for Elliott and Müller's (2007) approach in Table 1, while Elliott and Müller (2007) find that the average length of their approach worsens when the variance also undergoes a break (comparing Table 3 to Table 4 in their paper).

Overall, the likelihood-based approach provides the most precise inferences about the timing of break dates. It tends to have conservative coverage, consistent with the analysis in Hansen (2000) for ILR confidence sets of threshold parameters under Normal errors, as are assumed in these Monte Carlo experiments. Elliott and Müller's (2007) approach does remarkably well in terms of coverage rates in finite samples, but it produces much less precise confidence sets than the proposed likelihood-based approach. Bai's (1997) approach undercovers in smaller samples and has fairly wide confidence intervals. A bootstrap version of Bai's approach improves coverage for large breaks at the expense of even longer average length when considering large breaks, but its performance is worse on all dimensions given small breaks.

3.2.2 Serial correlation

In this subsection, we extend the univariate data generating process in (6) to allow for serial correlation. For simplicity, we focus on a break in mean ($\Phi = 0$) given sample size $T = 240$ and with magnitude of break $\delta = 4$ for a slowly-shrinking break, $v_T = T^{-1/4}$.

First, we assume an AR(1) error process:

$$u_t = \rho u_{t-1} + e_t, \quad e_t \sim i.i.d.\mathcal{N}(0, 1).$$

We consider two cases of low and high persistence in the errors: $\rho = 0.3$ (as in Table 5 of Elliott and Müller (2007)) and $\rho = 0.9$. For this setting, estimation of the parameters of the model in (6) is via quasi maximum likelihood and we employ a HAC estimator of the long-run variance of u_t in order to calculate scaled test statistics with asymptotically pivotal distributions for the purposes of constructing confidence sets. Following Elliott and Müller (2007) and Qu and Perron (2007), we consider the HAC estimator due to Andrews and Monahan (1992). Note that, in addition to serial correlation, this estimator would also address heteroskedasticity if it were present, although we focus on the problem of serially correlated errors in this Monte Carlo experiment.

Second, we consider an AR(1) model for y_t. Specifically, we set $z_t = (1 \ y_{t-1})$ and $\beta_1 = (0 \ \rho)'$ in (6) and, again, consider two cases of low and high persistence: $\rho = 0.3$ and $\rho = 0.9$. The structural break in mean corresponds to a change in the intercept such that $\Delta\beta = (\delta T^{-1/4} \ 0)'$. For this setting, estimation of parameters in (6) is via conditional maximum likelihood.

Table 5 reports the coverage and length results for a break in mean with serially-correlated errors. As in the previous subsection, the likelihood-based approach performs best. Indeed, in this case, the coverage is extremely accurate and the length is even better than in the equivalent case in Table 1. The other approaches perform more similarly to the equivalent case in Table 1, with Bai's (1997) approach undercovering and Elliott and Müller's (2007) approach having the longest average length.

Table 6 reports the coverage and length results for a break in mean for an AR(1) model. Again, the likelihood-based approach performs best, although it reverts to being conservative,

Table 5: Coverage Rate and Length of Confidence Sets: Break in Mean with Serially-Correlated Errors ($\delta = 4$ and $\Delta\beta = \delta/T^{1/4}$, $T = 240$)

	Low Persistence (ρ=0.3)		High Persistence (ρ=0.9)	
	Cov.	Lgth.	Cov.	Lgth.
ILR	0.950	13.910	0.950	13.841
Bai	0.934	22.385	0.930	21.982
Bai Bootstrap	0.943	25.526	0.942	24.991
Elliott&Müller	0.951	35.363	0.945	35.088

Table 6: Coverage Rate and Length of Confidence Sets: Break in Mean for an AR(1) Model ($\delta = 4$ and $\Delta\beta = (\delta/T^{1/4}\ 0)'$, $T = 240$)

	Low Persistence (ρ=0.3)		High Persistence (ρ=0.9)	
	Cov.	Lgth.	Cov.	Lgth.
ILR	0.967	15.746	0.943	17.213
Bai	0.924	21.345	0.846	16.541
Bai Bootstrap	0.948	26.964	0.926	29.330
Elliott&Müller	0.947	42.556	0.955	231.536

as in the previous subsection. The results given low persistence in the process are very similar to the equivalent results in Table 1, while the results given high persistence in the process show a worsening of the coverage properties of Bai's (1997) approach and dramatic increase in the average length of Elliott and Müller's (2007) approach.

Overall, serial correlation poses no problem for the likelihood-based approach, while it sometimes worsens the performance of the other methods when there is high persistence. Meanwhile, the assumption of high persistence is far from a theoretical curiousity, as it appears to be the relevant case for some of the data in our empirical application below.

3.2.3 Multiple breaks

Next, we consider a data generating process with multiple breaks:

$$y_t = \beta_1 + \Delta\beta_1 \mathbf{1}[t > \tau_1] + \Delta\beta_2 \mathbf{1}[t > \tau_2] + u_t, \tag{8}$$

Table 7: Coverage Rate and Length of Confidence Sets: Multiple Breaks in Mean and/or Variance ($r_1 = 0.3$ and $r_2 = 0.7$, $\delta = 4$ and $\Delta\beta = \delta/T^{1/4}$, $\Phi = 8$ and $\Delta\Sigma = \Phi/T^{1/4}$, $T = 240$)

		2 Breaks in Mean		2 Breaks in Var.		2 Breaks in Mean/Var.	
		Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
ILR	1st Break	0.969	16.455	0.968	17.364	0.972	12.563
	2nd Break	0.968	16.415	0.972	17.338	0.972	12.613

where $u_t = \left(\sqrt{\Sigma_1 + \Delta\Sigma_1 \mathbf{1}[t > \tau_1] + \Delta\Sigma_2 \mathbf{1}[t > \tau_2]}\right) e_t$ and $e_t \sim i.i.d.\mathcal{N}(0,1)$. Again, we re-parameterize the structural break dates and break magnitudes in terms of the sample size: $\tau_j = [r_j T]$, $\Delta\beta_j = \beta_{j+1} - \beta_j = \delta_j T^{-\nu}$, and $\Delta\Sigma_j = \Sigma_{j+1} - \Sigma_j = \Phi_j T^{-\nu}$, where $j = 1, 2$. As in the previous subsection, we focus on $\nu = 1/4$ and $T = 240$. In all cases, we set $\beta_1 = 0$ and $\Sigma_1 = 1$. For breaks in mean, $\delta_1 = \delta_2 = 4$ and $\Phi_1 = \Phi_2 = 0$. For breaks in variance, $\Phi_1 = \Phi_2 = 8$ and $\delta_1 = \delta_2 = 0$. For breaks in mean and variance, $\delta_1 = \delta_2 = 4$ and $\Phi_1 = \Phi_2 = 8$. The true break date fractions are $r_1 = 0.3$ and $r_2 = 0.7$. Estimation of (8) is via maximum likelihood.

Table 7 reports the coverage and length results for the likelihood-based approach in the three cases of two breaks in mean, two breaks in variance, and two breaks in mean and variance. Again, the coverage of the ILR confidence sets is conservative. The average length in the case of breaks in mean is slightly worse than the equivalent case in Table 1. However, the average length in the case of breaks in variance and breaks in mean and variance are better than in Tables 1 and 4. The general point is that there is no obvious deterioration in the performance of the likelihood-based confidence sets when break dates are closer to the beginning or end of the sample and when more than one break is considered.

3.2.4 Multivariate setting

Our last Monte Carlo experiment involves a multivariate setting for a structural break, as in Bai, Lumsdaine, and Stock (1998), Bai (2000), and Qu and Perron (2007). For our data generating processes, we consider a bivariate model with correlation between errors across equations:

Table 8: Coverage Rate and Length of Confidence Sets: Break in Mean for Bivariate Model with Correlated Errors across Equations ($\delta = 4$ and $\Delta\beta = \delta/T^{1/4}$, $T = 240$)

	Break for One Variable		Break for Both Variables	
	Cov.	Lgth.	Cov.	Lgth.
ILR	0.969	13.158	0.954	9.237

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \beta_{1,1} + \Delta\beta_{1,1} 1[t > \tau_0] \\ \beta_{2,1} + \Delta\beta_{2,1} 1[t > \tau_0] \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}, \quad \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \sim N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \tag{9}$$

where $\Delta\beta_{i,1} = \beta_{i,2} - \beta_{i,1} = \delta_i T^{-\nu}$, $i = 1, 2$. As in the previous two subsections, we focus on $\nu = 1/4$ and $T = 240$. For our simulations, we set $\beta_{i,1} = 0$, $i = 1, 2$, and $\rho = 0.3$. Then, we consider two cases of i) a break in the mean of the first series only, $\delta_1 = 4$ and $\delta_2 = 0$, and ii) a break in the mean of both series, $\delta_1 = \delta_2 = 4$. Estimation of (9) is via maximum likelihood.

Table 8 reports the coverage and length results for the likelihood-based approach in the multivariate setting. For both a break in mean of one series and a break in mean of both series, the ILR confidence sets are conservative, as in most of the earlier experiments. However, the notable result is that the average lengths are shorter than in the univariate case in Table 1. This is especially true in the case of a break in mean for both variables, which is perhaps not so surprising. However, there is also an improvement in the average length just by including information from a second variable that does not undergo a break, but has a stable correlation with the first variable. These results are consistent with the findings in Bai, Lumsdaine, and Stock (1998) and Qu and Perron (2007) that adding equations to a multivariate model can produce more precise inferences.

4 Structural Breaks in Postwar U.S. Real GDP and Consumption

We apply our proposed likelihood-based method of constructing confidence sets to investigate structural breaks in postwar quarterly U.S. real GDP and consumption of nondurables and services. We first consider univariate models of the growth rates of output and consumption and then we consider a multivariate model that imposes balanced long-run growth between output and consumption. The data for real GDP and consumption were obtained from the BEA website for the sample period of 1947Q1 to 2012Q1.[9] Annualized quarterly growth rates are calculated as 400 times the first differences of the natural logarithms of the levels data.

4.1 Univariate Models

The typical approach to investigating structural breaks in a time series is to consider a univariate model. Although this can be less efficient than considering a multivariate model, as we found in our Monte Carlo analysis, it has the benefit of making the interpretation of estimated breaks straightforward. Thus, we begin our analysis with univariate models of output growth and consumption growth, respectively, as the results will help with understanding the results for the multivariate model presented below.

For the univariate analysis, we assume that log output has a stochastic trend with drift and a finite-order autoregressive representation. Specifically, our model for quarterly output growth is an AR(p) process:

$$\Delta y_t = \gamma_y + \sum_{j=1}^{p} \zeta_{y,j} \Delta y_{t-j} + e_{yt}, \quad e_{yt} \sim i.i.d.\mathcal{N}(0, \sigma_y^2) \tag{10}$$

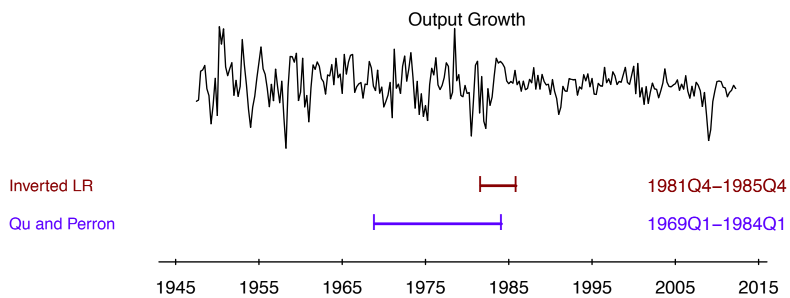
Similarly, we assume log consumption has a stochastic trend with drift and a finite-order autoregressive representation. Thus, our model for quarterly consumption growth is also an AR(p) process:

$$\Delta c_t = \gamma_c + \sum_{j=1}^{p} \zeta_{c,j} \Delta y_{t-j} + e_{ct}, \quad e_{ct} \sim i.i.d.\mathcal{N}(0, \sigma_c^2) \tag{11}$$

For lag selection, we employ Kurozumi and Tuvaandorj's (2011) modified BIC to account for the possibility of multiple structural breaks. Given an upper-bound of four lags and four breaks, with the common adjusted sample of 1948Q2 to 2012Q1, we find that the highest

[9] The raw data are from the BEA Tables 1.1.5 and 1.1.6 for the vintage of April 27, 2012. We need both real and nominal measures for total consumption and consumption of durables in order to construct a chain-weighted measure of real consumption of nondurables and services based on Whelan's (2000) suggestion of a Tornqvist approximation to the ideal Fisher index.

Figure 1: U.S. Real GDP Growth and Confidence Sets for AR(1) Model



lag order selected is $p = 1$ for output growth and $p = 2$ for consumption growth.

Figure 1 plots the output growth series over the postwar period. Although the series clearly resembles the realization of a low-order autoregressive process with fairly low persistence, the parameters for this process may have changed over time. Applying Qu and Perron's (2007) testing procedure to an AR(1) model estimated over the longest available sample period for conditional maximum likelihood of 1947Q3 to 2012Q1, we find evidence of one break (the same as the number of breaks chosen by the modified BIC statistic mentioned above). The break is estimated to occur in 1983Q2, which corresponds closely to the timing of the so-called "Great Moderation" widely reported in the past literature (e.g., Kim and Nelson (1999) and McConnell and Perez-Quiros (2000)). The break is significant at the 5% level and there is no support for an additional break, even at the 10% level. Estimates for the long-run growth rate, largest eigenvalue measure of persistence, and the conditional standard deviation are reported in Table 9.[10] Likelihood ratio tests of parameter restrictions suggest that the break corresponds to a change only in the conditional standard deviation, which is estimated to have dropped by more than 50%.[11]

The likelihood-based confidence set based on inverting a likelihood ratio test for a break date is also reported in Figure 1. The confidence set is the relatively short interval of 1981Q4-1985Q4. Notably, as mentioned in the introduction, this interval is similar in length to the

[10]For easy comparison across models, we measure persistence by the (modulus of the) largest eigenvalue of the companion matrix for the stationary representations of an autoregressive model or a vector error correction model. For the AR(1) model, this is simply the autoregressive coefficient.

[11]Note that, for simplicity, we always consider the unrestricted model when constructing confidence sets, as this allows for a more straightforward comparison of results across models when certain parameter restrictions are rejected for only one model, but not for another.

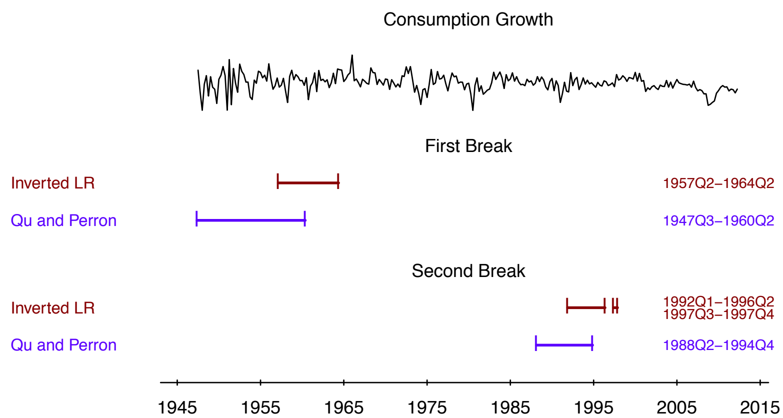
Table 9: Autoregressive Model of U.S. Real GDP Growth: 1947:Q3-2012:Q1

Regime	Break Date	LR Growth Rate	Largest Eig.	Cond. SD
1		3.458	0.330	4.487
2	1983:Q2	2.714	0.538	2.045

67% interval for the Great Moderation reported in Stock and Watson (2002) based on Bai's (1997) approach. For illustration, we compare our confidence set to the 95% confidence interval calculated by Qu and Perron's (2007) procedures using the same model and data. This confidence interval is based on the distribution of the break date estimator, as in Bai (1997), but is also applicable in the multivariate setting that we consider in the second part of our application (see Qu and Perron (2007) for more details). Notably, the Qu and Perron confidence interval is much wider, running from 1969Q1-1984Q1, thus also including the possible "productivity growth slowdown" in the early 1970s (see, for example, Perron (1989) and Hansen (2001)). Thus, the interval is much less informative about when the structural break occurred, including whether it was abrupt.

Figure 2 plots the consumption growth series. Although consumption is by far the largest expenditure component of U.S. real GDP, it is not as important for quarterly fluctuations in output given the volatility of other components, especially investment. Thus, it is not automatic that consumption growth will exhibit the same volatility reduction in the mid-1980s. Instead, it appears that there are breaks in consumption growth that do not manifest themselves in the overall behaviour of aggregate output. Indeed, applying Qu and Perron's (2007) testing procedure to an AR(2) model estimated over the longest available sample period for conditional maximum likelihood of 1947Q4 to 2012Q1, we find evidence of two breaks (again the same as the number chosen by the modified BIC statistic) that are estimated to have occurred in 1958Q3 and 1993Q3, respectively. The breaks are significant at the 5% level and there is no support for additional breaks at the 10% level. Estimates for the long-run growth rate, largest eigenvalue measure of persistence, and the conditional standard deviation are reported in Table 10. Likelihood ratio tests of parameter restrictions for this model suggest that these are both breaks in the conditional standard deviation of consumption growth, with the second break also corresponding to a decrease in the long-run growth rate and an increase in persistence.

Figure 2: U.S. Consumption Growth and Confidence Sets for AR(2) Model



The confidence sets for the two structural breaks in consumption growth are also reported in Figure 2. As with output growth, the ILR confidence sets are shorter than those based on Qu and Perron (2007). Also, the ILR confidence sets reject earlier possible break dates compared to Qu and Perron (2007), again similar to the case of output growth. Notably, the confidence sets exclude the periods of a possible productivity growth slowdown in the early 1970s and the Great Moderation in the mid-1980s that correspond to the most widely-hypothesized breaks in U.S. economic activity. Given these apparently different breaks from output growth, it is an open question of whether a multivariate model of output and consumption would lead to different or more precise inferences about structural breaks in these two series, as found, for example, by Bai, Lumsdaine, and Stock (1998). We turn to this question next.

4.2 Multivariate Model

Following Cochrane (1994), we assume that real GDP and consumption of nondurables and services have balanced long-run growth due to a common stochastic trend, possibly reflecting common shocks to productivity as suggested by a stochastic neoclassical growth model (see Bai, Lumsdaine, and Stock (1998) for a full theoretical motivation of this assumption). The empirical justification for the balanced-growth assumption comes from the apparent cointegrating relationship between these particular measures of consumption and output. If

Table 10: Autoregressive Model of U.S. Consumption Growth: 1947:Q4-2012:Q1

Regime	Break Date	LR Growth Rate	Largest Eig.	Cond. SD
1		3.067	-0.044	3.092
2	1958:Q3	3.264	0.511	1.841
3	1993:Q3	1.965	0.900	0.951

we impose a balanced long-run relationship corresponding to a cointegrating vector of $(1\ -1)$ for the natural logarithms of consumption and output from 1947Q1 to 2012Q1, we find that we can reject a unit root with a p-value of 0.008 for an ADF test for the consumption rate, $c_t - y_t$ with a constant in the test regression and BIC for lag selection. Thus, there is empirical support for the idea that output and consumption (appropriately measured) have a balanced long-run relationship.

Assuming log output and consumption have a finite-order vector autoregressive representation, cointegration with known cointegrating vector $(1\ -1)$ implies that the growth rates of output and consumption can be captured by the following VECM(p) model:

$$\Delta y_t = \gamma_y + \sum_{j=1}^{p} \zeta_{yy,j} \Delta y_{t-j} + \sum_{j=1}^{p} \zeta_{yc,j} \Delta c_{t-j} + \pi_y (c_{t-1} - y_{t-1}) + e_{yt}, \tag{12}$$

$$\Delta c_t = \gamma_c + \sum_{j=1}^{p} \zeta_{cy,j} \Delta y_{t-j} + \sum_{j=1}^{p} \zeta_{cc,j} \Delta c_{t-j} + \pi_c (c_{t-1} - y_{t-1}) + e_{ct}, \tag{13}$$

where $e_t \sim \mathcal{N}(0, \Omega)$. This form of cointegration also directly implies that the long-run consumption rate is constant and consumption and output share the same long-run growth rate. We parameterize these two long-run rates as follows:

$$E[c_t - y_t] = \kappa,$$

$$E[\Delta y_t] = E[\Delta c_t] = \mu.$$

It is possible then to solve for these two long-run parameters given estimates of the VECM parameters in (12) and (13) as follows:

$$\begin{bmatrix} \kappa \\ \mu \end{bmatrix} = \begin{bmatrix} -\pi_y & 1 - \sum_{j=1}^{p}(\zeta_{yy,j} + \zeta_{yc,j}) \\ -\pi_c & 1 - \sum_{j=1}^{p}(\zeta_{cy,j} + \zeta_{cc,j}) \end{bmatrix}^{-1} \begin{bmatrix} \gamma_y \\ \gamma_c \end{bmatrix}. \tag{14}$$

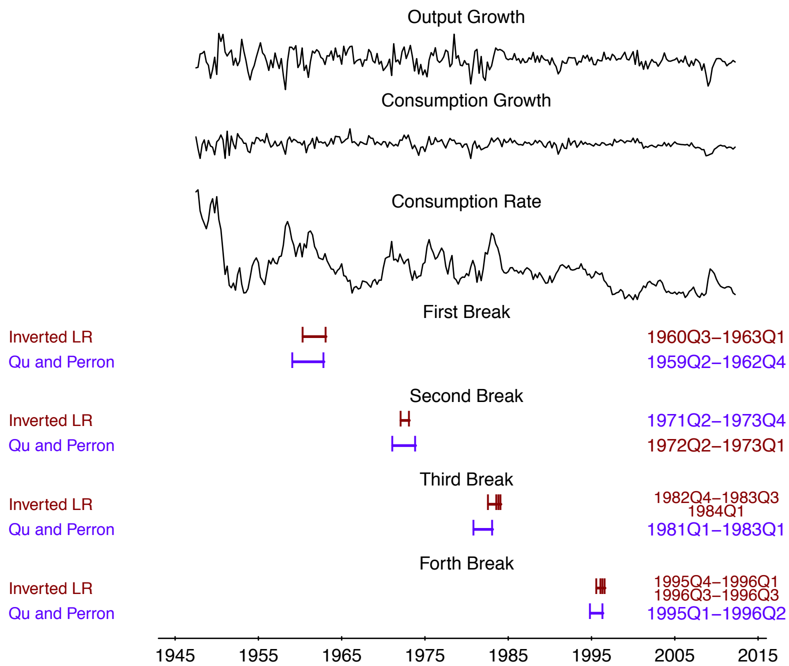
Thus, using the relationship in (14), we can uncover structural breaks in the long-run consumption rate and the long-run growth rate by testing for structural breaks in the conditional mean parameters of the VECM. Bai, Lumsdaine, and Stock (1998) emphasize that this is a test for break in the long-run growth rate, μ, under the assumption of no break in unconditional mean of the cointegrating relationship, κ. However, we leave it as an empirical issue whether a common break in the conditional mean parameters corresponds to a break in the long-run consumption rate, long-run growth, or both.[12]

As with the univariate model for output growth, we find that the highest lag order selected by the modified BIC is $p = 1$. However, under the assumption of no breaks, the second lags of the growth rates are jointly significant at 5% level based on a likelihood ratio test (notably, the second lag of consumption growth in (13) has t-statistic of 2.1). Therefore, to avoid under-fitting, we consider a VECM(2), which we estimate by conditional maximum likelihood for the sample period of 1947Q4 to 2012Q1. Note that $p = 2$ is also consistent with the lag order selected by modified BIC for the univariate model of consumption growth.

Applying Qu and Perron's (2007) testing procedure for structural breaks to the VECM(2) model estimated over the longest available sample period for conditional maximum likelihood of 1947Q4 to 2012Q1, we find evidence of three breaks estimated in 1958Q1, 1982Q4, and 1996Q1 at the 5% level. The estimated timing of these breaks corresponds closely to the timing for the breaks in the univariate models of output growth and consumption growth. Thus, the first and third break likely correspond to a change in the behaviour of consumption, while the second break corresponds to the Great Moderation. However, in contrast to the univariate results, we now find evidence of four breaks estimated in 1961Q3, 1972Q4, 1982Q3, and 1996Q1 at the 10% level. Presumably the first, third, and fourth breaks again

[12] In a related study, Cogley (2005) considers a time-varying parameter version of Cochrane's (1994) VECM model of output and consumption to investigate changes in the long-run growth rate and long-run consumption rate for the U.S. economy. He finds a gradual decline in the long-run growth rate from the mid-1960s to the early 1990s, followed by a gradual increase in long-run growth in the 1990s. He also finds that the consumption rate is very stable over the postwar period, although it gradually declines in the 1990s. However, Bayesian estimation of the time-varying parameter model imposes the strong prior that structural change is gradual, precluding the possibility of large, abrupt changes that are considered and found in this paper.

Figure 3: U.S. Real GDP Growth, Consumption Growth, Log Consumption Rate, and Confidence Sets for VECM(2) Model



correspond closely to the breaks found in the univariate models. But the second break estimated in 1972Q4 appears to conform, at least in its timing, to the widely-hypothesized productivity growth slowdown that should affect both output and consumption and may be better identified by the consideration of a multivariate model that imposes the same long-run growth rate for the two series.

Figure 3 plots the output growth, consumption growth, and the consumption rate series over the postwar period. Visually, it is difficult to detect whether the estimated break in 1972Q4 corresponds to a break in the long-run growth rate or the long-run consumption rate. However, it is easier to see that the estimated break in 1996Q1 corresponds to a reduction in the long-run consumption rate, in addition to a change in the behaviour of consumption growth detected in the univariate analysis. Indeed, the reasonable clarity of this change could explain the slight change in timing of the estimated break date from 1993Q3 for a change in consumption behaviour in the univariate analysis.

Table 11 reports the estimates of the long-run growth rate, long-run consumption rate, largest eigenvalue measure of persistence, and conditional standard deviations of output

Table 11: Vector Error Correction Model of U.S. Real GDP and Consumption Growth: 1947:Q4-2012:Q1

Regime	Break Date	LR Growth Rate	LR Con. Rate	Largest Eig.	Cond.	SDs
1		3.188	-181.218	0.776	4.566	2.719
2	1961:Q3	4.225	-188.367	0.783	3.068	1.571
3	1972:Q4	2.826	-184.370	0.803	3.713	1.666
4	1982:Q4	2.875	-188.825	0.807	1.496	1.273
5	1996:Q1	1.585	-196.138	0.746	1.842	0.948

growth and consumption growth for the VECM(2) model with four structural breaks. Consistent with the univariate results, the first break in the early 1960s corresponds clearly to a reduction in consumption growth volatility. The second break in the early 1970s corresponds to a reduction in the long-run growth rate of 1.4 annualized percentage points, consistent with the productivity growth slowdown, more than to a change in the long-run consumption rate or a change in volatility or persistence. The third break in the mid-1980s corresponds quite clearly to a reduction in output growth volatility, consistent with the Great Moderation. The fourth break in the mid-1990s corresponds to an additional reduction in the long-run growth rate of 1.3 annualized percentage points, as well as to the reduction in the long-run consumption rate evident in Figure 3.[13] Interestingly, the largest eigenvalue measure of persistence remains remarkably stable over the full sample period. Likelihood ratio tests of parameter restrictions generally support our interpretation of the breaks, although it can be harder to relate how rejections of restrictions on intercept, slope, and/or the conditional variance/covariance parameters map into some of the parameters of interest. Thus, we report confidence sets for the unrestricted model that allows all parameters to change with each break.

The most striking result for the multivariate model is how precise the confidence sets are in Figure 3. This is consistent with our Monte Carlo results for the multivariate model and with the analysis in Bai, Lumsdaine, and Stock (1998) and Qu and Perron (2007) on the usefulness of multivariate inference about break dates. All four breaks have very short

[13]Given the assumption of balanced growth for output and consumption, the magnitude of the estimated reduction in the long-run growth rate in Table 11 reflects changes in the average growth rates for both output and consumption. However, it should be noted that, on its own, the average growth rate for output declined by 1.1 annualized percentage points between regimes 4 and 5, quite consistent with a sizeable growth slowdown of 1.3 annualized percentage points reported in the table.

ILR confidence sets, suggesting that the structural changes were abrupt. Even Qu and Perron's (2007) confidence intervals are reasonably short, although they are less precise than the ILR confidence sets. It is, perhaps, not surprising that confidence sets for breaks in parameters that are common to both output and consumption in the VECM model, such as the long-run growth rate, are more precise. But, notably, the confidence sets for the Great Moderation, which appears to be a much more important phenomenon for output growth than for consumption growth, also become a lot more precise, with the length of the ILR confidence set shrinking from 20 quarters to just 5 quarters. Thus, the improvement in inferences arises from both the model structure and from the additional multivariate information.

5 Conclusion

We have proposed a likelihood-based approach to constructing confidence sets for the timing of structural breaks. The confidence sets include all possible break dates that cannot be rejected based on a likelihood ratio test. The asymptotical validity for this approach is established for a broad setting of a system of multivariate linear regression equations under the asymptotic thought-experiment of a slowly-shrinking magnitude of a break, with the asymptotic expected length of the likelihood-based confidence sets being about half that of the standard method employed in the literature. Monte Carlo analysis supports the finite-sample performance of the proposed approach in a number of realistic experiments, including given small breaks. An application to U.S. real GDP and consumption demonstrates the relevance of the performance gains of the proposed approach relative to the standard method. Specifically, the empirical analysis provides much more precise inferences about the timing of the "productivity growth slowdown" in the early 1970s and the "Great Moderation" in the mid-1980s than previously found. It also suggests the presence of an additional large, abrupt decline in the long-run growth rate of the U.S. economy in the mid-1990s, at least when taking cointegration between output and consumption into account.

A Appendix

Proof of Proposition 1. Qu and Perron (2007, Theorem 1) show that the estimates of the break dates $(\hat{\tau}_0, \ldots, \hat{\tau}_j)$ and the coefficients $(\hat{\tilde{\beta}}, \hat{\tilde{\Sigma}})$ are asymptotically independent so that the distribution of the estimates of break dates conditional on the true values of the coefficients $(\tilde{\beta}^0, \tilde{\Sigma}^0)$ are not affected by the restrictions imposed on the coefficients as long as we restrict our analysis to values of the parameters in the set C_M where

$$\begin{aligned} C_M &= \{(\Upsilon, \tilde{\beta}, \tilde{\Sigma}) : v_T^2 |\tau_j - \tau_j^0| \leq M \text{ for } j = 1, \ldots, m, \\ & \quad |\sqrt{T}(\beta_j - \beta_j^0)| \leq M, |\sqrt{T}(\Sigma_j - \Sigma_j^0)| \leq M \text{ for } j = 1, \ldots, m+1\} \end{aligned}$$

and M is a fixed positive number which is large enough so that the estimates fall in this set with probability arbitrarily close to 1.

Without loss of generality, consider the j-th break date and the estimate of the break date is before the true break date. For $\tau_j - \tau_j^0 = r = -1, -2, \ldots,$

$$\begin{aligned} lr_j(r) &= -\frac{r}{2}\left(\ln|\Sigma_j^0| - \ln|\Sigma_{j+1}^0|\right) \\ & \quad -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0} u_t'\left((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1}\right)u_t \\ & \quad -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0} (\beta_{j+1}^0 - \beta_j^0)' x_t (\Sigma_{j+1}^0)^{-1} x_t' (\beta_{j+1}^0 - \beta_j^0) \\ & \quad + \sum_{t=\tau_j^0+r}^{\tau_j^0} (\beta_{j+1}^0 - \beta_j^0)' x_t (\Sigma_{j+1}^0)^{-1} u_t \end{aligned} \tag{A.1}$$

Qu and Perron (2007) show that for the first term on RHS of equation (A.1)

$$
\begin{aligned}
-\frac{r}{2}\left(\ln|\Sigma_j^0| - \ln|\Sigma_{j+1}^0|\right) &= -\frac{r}{2}\ln|(\Sigma_j^0 - \Sigma_{j+1}^0 + \Sigma_{j+1}^0)(\Sigma_{j+1}^0)^{-1}| \\
&= -\frac{r}{2}\ln|I + (\Sigma_j^0 - \Sigma_{j+1}^0)(\Sigma_{j+1}^0)^{-1}| \\
&= -\frac{r}{2}\ln|I + (-\Phi_j v_T)(\Sigma_{j+1}^0)^{-1}| \\
&= \frac{r}{2}v_T tr\left(\Phi_j(\Sigma_{j+1}^0)^{-1}\right) + \frac{r}{4}v_T^2 tr\left((\Phi_j(\Sigma_{j+1}^0)^{-1})^2\right) + o(v_T^2) \quad (A.2)
\end{aligned}
$$

From the third line to the fourth line, we approximate it by using Taylor expansion around $v_T = 0$ up to the second order since $v_T \to 0$. Also, for the second term in equation (A.1)

$$
\begin{aligned}
&-\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0} u_t'\left((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1}\right)u_t \\
=\ & -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0} tr\left\{((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1})(u_t u_t' - \Sigma_j^0 + \Sigma_j^0)\right\} \\
=\ & -\frac{1}{2}tr\left\{\sum_{t=\tau_j^0+r}^{\tau_j^0}((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1})((\Sigma_j^0)^{1/2}\eta_t\eta_t'(\Sigma_j^0)^{1/2} - \Sigma_j^0) + ((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1})\Sigma_j^0\right\} \\
=\ & -\frac{1}{2}tr\left\{\sum_{t=\tau_j^0+r}^{\tau_j^0}(\Sigma_j^0)^{1/2}((\Sigma_{j+1}^0)^{-1} - (\Sigma_j^0)^{-1})(\Sigma_j^0)^{1/2}(\eta_t\eta_t' - I)\right\} - \frac{1}{2}tr\left\{\sum_{t=\tau_j^0+r}^{\tau_j^0}(\Sigma_{j+1}^0)^{-1}\Sigma_j^0 - I\right\} \\
=\ & -\frac{1}{2}tr\left\{(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}(I - \Sigma_{j+1}^0(\Sigma_j^0)^{-1})(\Sigma_j^0)^{1/2}\sum_{t=\tau_j^0+r}^{\tau_j^0}(\eta_t\eta_t' - I)\right\} + \frac{r}{2}tr\left\{(\Sigma_{j+1}^0)^{-1}(\Sigma_j^0 - \Sigma_{j+1}^0)\right\} \\
=\ & -\frac{1}{2}tr\left\{(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}(\Sigma_j^0 - \Sigma_{j+1}^0)(\Sigma_j^0)^{-1/2}\sum_{t=\tau_j^0+r}^{\tau_j^0}(\eta_t\eta_t' - I)\right\} + \frac{r}{2}tr\left\{(\Sigma_{j+1}^0)^{-1}(\Sigma_j^0 - \Sigma_{j+1}^0)\right\} \\
=\ & \frac{1}{2}tr\left\{(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Phi_j(\Sigma_j^0)^{-1/2}v_T\sum_{t=\tau_j^0+r}^{\tau_j^0}(\eta_t\eta_t' - I)\right\} - \frac{r}{2}v_T tr\left((\Sigma_{j+1}^0)^{-1}\Phi_j\right). \qquad (A.3)
\end{aligned}
$$

From equations (A.2) and (A.3), the sum of the first two terms in equation (A.1) is given by

$$
\begin{aligned}
& -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0} u_t'\left((\Sigma_{j+1}^0)^{-1}-(\Sigma_j^0)^{-1}\right)u_t-\frac{r}{2}\left(\ln|\Sigma_{j+1}^0|-\ln|\Sigma_j^0|\right) \\
= \quad & \frac{r}{2}v_T tr\left(\Phi_j(\Sigma_{j+1}^0)^{-1}\right)+\frac{r}{4}v_T^2 tr\left((\Phi_j(\Sigma_{j+1}^0)^{-1})^2\right)+o(v_T^2) \\
& +\frac{1}{2}tr\left\{(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Phi_j(\Sigma_j^0)^{-1/2}v_T\sum_{t=\tau_j^0+r}^{\tau_j^0}(\eta_t\eta_t'-I)\right\}-\frac{r}{2}v_T tr\left(\Phi_j(\Sigma_{j+1}^0)^{-1}\right) \\
= \quad & \frac{1}{2}tr\left\{(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Phi_j(\Sigma_j^0)^{-1/2}v_T\sum_{t=\tau_j^0+[s/v_T^2]}^{\tau_j^0}(\eta_t\eta_t'-I)\right\}+\frac{r}{4}v_T^2 tr\left((\Phi_j(\Sigma_{j+1}^0)^{-1})^2\right)+o(v_T^2) \\
\Rightarrow \quad & \frac{1}{2}tr\left((\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Phi_j(\Sigma_j^0)^{-1/2}\zeta_{1,j}(s)\right)+\frac{s}{4}tr\left((\Phi_j(\Sigma_{j+1}^0)^{-1})^2\right) \\
= \quad & \frac{1}{2}tr\left(A_{1,j}\zeta_{1,j}(s)\right)+\frac{s}{4}tr\left(A_{1,j}^2\right) \qquad \text{(A.4)}
\end{aligned}
$$

where $A_{1,j}=(\Sigma_j^0)^{1/2}(\Sigma_{j+1}^0)^{-1}\Phi_j(\Sigma_j^0)^{-1/2}$ and $A_{2,j}=(\Sigma_{j+1}^0)^{1/2}(\Sigma_j^0)^{-1}\Phi_j(\Sigma_{j+1}^0)^{-1/2}$.

Since $v_t\to 0$ and $T^{1/2}v_T/(lnT)^2\to\infty$, $rv_T^2=[sv_T^{-2}]v_T^2\to s$ uniformly over bounded s and for $s<0$

$$
\frac{1}{v_T^{-1}}\sum_{t=\tau_j^0+[sv_T^{-2}]}^{\tau_j^0}(\eta_t\eta_t'-I)\Rightarrow\zeta_{1,j}(s)
$$

by the functional central limit theorem ($E[\eta_t\eta_t']=I$) where the weak convergence is in the space $D[0,\infty)^{n^2}$ and where the entries of the $n\times n$ matrix $\zeta_{1,j}(s)$ is a nonstandard Brownian motion process with $var\left\{vec(\zeta_{1,j}(s))\right\}=\Omega_{1,j}^0$.

Then, for the third term in equation (A.1)

$$
\begin{aligned}
& -\frac{1}{2}\sum_{t=\tau_j^0+r}^{\tau_j^0}(\beta_j^0-\beta_{j+1}^0)'x_t(\Sigma_{j+1}^0)^{-1}x_t'(\beta_j^0-\beta_{j+1}^0) \\
= \quad & -\frac{1}{2}v_T^2\sum_{t=\tau_j^0+[s/v_T^2]}^{\tau_j^0}\delta_j'x_t(\Sigma_{j+1}^0)^{-1}x_t'\delta_j \\
\to_p \quad & \frac{s}{2}\delta_j'Q_{1,j}\delta_j. \qquad \text{(A.5)}
\end{aligned}
$$

and for the fourth term in equation (A.1)

$$\begin{aligned}
& \sum_{t=\tau_j^0+r}^{\tau_j^0} (\beta_{j+1}^0 - \beta_j^0)' x_t (\Sigma_{j+1}^0)^{-1} u_t \\
= \quad & v_T \sum_{t=\tau_j^0+[s/v_T^2]}^{\tau_j^0} \delta_j' x_t (\Sigma_{j+1}^0)^{-1} (\Sigma_j^0)^{1/2} \eta_t \\
\Rightarrow \quad & \delta_j' (\Pi_{1,j})^{1/2} U_{1,j}(s) \qquad\qquad (A.6)
\end{aligned}$$

where the weak convergence is in the space $D[0,\infty)^p$ and where the entries of the p vector $U_{1,j}(s)$ are Wiener processes. Note that the assumption $E[\eta_{tk}\eta_{tl}\eta_{th}] = 0$ for all k, l, h and for all t ensures that $U_{1,j}(s)$ and $\zeta_{1,j}(s)$ are independent.

Combining all the terms in equation (A.1) from the derived results in (A.4)-(A.6) shows that for $s < 0$,

$$lr_j([\frac{s}{v_T^2}]) \Rightarrow \frac{1}{2} tr\left(A_{1,j}\zeta_{1,j}(s)\right) + \frac{s}{4} tr\left(A_{1,j}^2\right) + \frac{s}{2}\delta_j' Q_{1,j}\delta_j + \delta_j'(\Pi_{1,j})^{1/2} U_{1,j}(s). \qquad (A.7)$$

We define

$$\begin{aligned}
\Lambda_{1,j} &= \left(\frac{1}{4} vec(A_{1,j})'\Omega_{1,j}^0 vec(A_{1,j}) + \delta_j'\Pi_{1,j}\delta_j\right)^{1/2} \\
\Lambda_{2,j} &= \left(\frac{1}{4} vec(A_{2,j})'\Omega_{2,j}^0 vec(A_{2,j}) + \delta_j'\Pi_{2,j}\delta_j\right)^{1/2} \\
\Xi_{1,j} &= \left(\frac{1}{2} tr(A_{1,j}^2) + \delta_j' Q_{1,j}\delta_j\right) \\
\Xi_{2,j} &= \left(\frac{1}{2} tr(A_{2,j}^2) + \delta_j' Q_{2,j}\delta_j\right).
\end{aligned}$$

The sum of the second and third terms in (A.7) is

$$\frac{s}{4} tr\left(A_{1,j}^2\right) + \frac{1}{2} s\delta_j' Q_{1,j}\delta_j = -\frac{|s|}{2}\left(\frac{1}{2} tr(A_{1,j}^2) + \delta_j' Q_{1,j}\delta_j\right) \equiv -\frac{|s|}{2}\Xi_{1,j}$$

Note that $\frac{1}{2} tr\left(A_{1,j}\zeta_{1,j}(s)\right) = \frac{1}{2} vec(A_{1,j})' vec(\zeta_{1,j}(s)) \overset{d}{=} \left(\frac{1}{4} vec(A_{1,j})'\Omega_{1,j}^0 vec(A_{1,j})\right)^{1/2} V_{1,j}(s)$ and $\delta_j'(\Pi_{1,j})^{1/2} U_{1,j}(s) \overset{d}{=} \left(\delta_j'\Pi_{1,j}\delta_j\right)^{1/2} U_{1,j}(s)$ where $V_{1,j}(s)$ and $U_{1,j}(s)$ are independent standard Wiener processes. Then,

$$
\begin{aligned}
&\left(\frac{1}{4}vec(A_{1,j})'\Omega^0_{1,j}vec(A_{1,j})\right)^{1/2} V_{1,j}(s) + \left(\delta_j'\Pi_{1,j}\delta_j\right)^{1/2} U_{1,j}(s) \\
\overset{d}{=}\ & \left(\frac{1}{4}vec(A_{1,j})'\Omega^0_{1,j}vec(A_{1,j}) + \delta_j'\Pi_{1,j}\delta_j\right)^{1/2} W_{1,j}(s) \\
\equiv\ & \Lambda_{1,j}W_{1,j}(s)
\end{aligned}
$$

where $W_{1,j}(s)$ is a standard Wiener process.

In sum, for $s \leq 0$,

$$lr_j([\frac{s}{v_T^2}]) \Rightarrow -\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_{1,j}(s)$$

and for $s > 0$ similarly

$$lr_j([\frac{s}{v_T^2}]) \Rightarrow -\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_{2,j}(s).$$

Since $W_{1,j}(s)$ and $W_{2,j}(s)$ are independent and starting at $s = 0$ where $W_{1,j}(0) = W_{2,j}(0) = 0$,

$$lr_j(\hat{\tau}_j - \tau_j^0) \Rightarrow \max_s \begin{cases} -\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s) & \text{for} \quad s \leq 0 \\ -\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s) & \text{for} \quad s > 0. \end{cases}$$

Now, let $\xi = lr_j(\hat{\tau}_j - \tau_j^0) = \max[\xi_1, \xi_2]$ where $\xi_1 = \sup_{s\leq 0}\left(-\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s)\right)$ and $\xi_2 = \sup_{v>0}\left(-\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s)\right)$.

By a change in variables $s = (\Lambda_{1,j}^2/\Xi_{1,j}^2)v$ and the distributional equality with $W(a^2x) \equiv aW(x)$, for $s \leq 0$

$$\xi_1 = \sup_{s\leq 0}\left(-\frac{|s|}{2}\Xi_{1,j} + \Lambda_{1,j}W_j(s)\right) = \sup_{v\leq 0}\frac{\Lambda_{1,j}^2}{\Xi_{1,j}}\left(-\frac{|v|}{2} + W_j(v)\right) = \omega_{1,j} \times \overline{\xi}_1 \tag{A.8}$$

where $\overline{\xi}_1 = \sup_{v\leq 0}\left(-\frac{|v|}{2} + W_j(v)\right)$ and

$$\frac{\Lambda_{1,j}^2}{\Xi_{1,j}} = \frac{\Lambda_{1,j}^2 v_T^2}{\Xi_{1,j} v_T^2} = \frac{\Gamma_{1,j}^2}{\Psi_{1,j}} \equiv \omega_{1,j}.$$

Similarly, for $s > 0$ with $s = (\Lambda_{2,j}^2/\Xi_{2,j}^2)v$

$$\xi_2 = \sup_{s>0}\left(-\frac{|s|}{2}\Xi_{2,j} + \Lambda_{2,j}W_j(s)\right) = \sup_{v>0}\frac{\Lambda_{2,j}^2}{\Xi_{2,j}}\left(-\frac{|v|}{2} + W_j(v)\right) = \omega_{2,j} \times \overline{\xi}_2 \tag{A.9}$$

where $\overline{\xi}_2 = \sup_{v<0}\left(-\frac{|v|}{2} + W_j(v)\right)$ and

$$\frac{\Lambda_{2,j}^2}{\Xi_{2,j}} = \frac{\Lambda_{2,j}^2 v_T^2}{\Xi_{2,j} v_T^2} = \frac{\Gamma_{2,j}^2}{\Psi_{2,j}} \equiv \omega_{2,j}.$$

■

Proof of Proposition 2. Bhattacharya and Brockwell (1976) show that $\overline{\xi}_1$ and $\overline{\xi}_2$ in (A.8) and (A.9) are iid exponential random variables with distribution function $P(\overline{\xi}_1 \leq x) = 1 - \exp(-x)$ and $P(\overline{\xi}_2 \leq x) = 1 - \exp(-x)$ for $x > 0$ respectively. Thus,

$$\begin{aligned} P(\xi \leq \kappa) &= P(\max[\omega_{1,j}\overline{\xi}_1, \omega_{2,j}\overline{\xi}_2] \leq \kappa) \\ &= P(\overline{\xi}_1 \leq \kappa/\omega_{1,j})P(\overline{\xi}_2 \leq \kappa/\omega_{2,j}) \\ &= (1 - \exp(-\kappa/\omega_{1,j}))\,(1 - \exp(-\kappa/\omega_{2,j})) \end{aligned}$$

Let $C(y) = \{t| \max_{\tau_j} \ln L(\tau_j|y) - \ln L(t|y) \leq \kappa_{\alpha,j}\}$. We construct a $1-\alpha$ confidence set for $\tau_{j-1}+1 \leq t < \tau_j$, $C(y)$, by inverting the likelihood ratio test. The probability of coverage of $C(y)$ for any τ_j^0 is given by $P_{\tau_j^0}\left(\tau_j^0 \in C(y)\right)$.

We can easily find a unique $\kappa_{\alpha,j}$ such that

$$P_{\tau_j^0}\left(\tau_j^0 \in C(y)\right) = (1 - \exp(-\kappa_{\alpha,j}/\omega_{1,j}))\,(1 - \exp(-\kappa_{\alpha,j}/\omega_{2,j})) = 1 - \alpha \tag{A.10}$$

since for all $\kappa > 0$,

$$\frac{d\,(1 - \exp(-\kappa/\omega_{1,j}))\,(1 - \exp(-\kappa/\omega_{2,j}))}{d\kappa} > 0.$$

■

Lemma 1 *For* $v \in (-\infty, \infty)$, *if* $\overline{\xi}(\hat{v}) \overset{d}{\Rightarrow} \max_v\left(-\frac{1}{2}|v| + W(v)\right)$ *and* $\alpha \to 0$, $E\left[\lambda\{v|\overline{\xi}(v) \leq \kappa_\alpha\}\right] = 4(1 - exp(-\kappa_\alpha))\left\{\kappa_\alpha - \frac{1}{2}(1 - exp(-\kappa_\alpha))\right\}$.

Proof of Lemma 1. Let us start with a simple case in which the magnitude of mean shift is known and variance of regression errors is normalized to one. Siegmund (1986, 1988) shows that in this case

$$\ln L(\hat{\tau}) - \ln L(\tau^0) \overset{d}{\Rightarrow} \max_v \left(-\frac{1}{2}|v| + W(v) \right) \tag{A.11}$$

and $\hat{v} = \hat{\tau} - \tau^0 = \arg\max_v \left(-\frac{1}{2}|v| + W(v)\right)$ where $W(v)$ is a standard Wiener process.

He also shows that when $\alpha \to 0$ the expected length for a $1-\alpha$ confidence set for the case is given by

$$\begin{aligned} E_{\tau^0}[\lambda\{C(Y)_{1-\alpha}\}] &= E_{\tau^0}[\lambda\{t | t \in C(Y)_{1-\alpha}\}] \\ &= \int_{-\infty}^{\infty} P_{\tau^0}\left(t \in C(Y)_{1-\alpha}\right) dt \\ &= 4(1-\alpha)^{1/2} \left\{ -\ln[1-(1-\alpha)^{1/2}] - \frac{1}{2}(1-\alpha)^{1/2} \right\} \end{aligned} \tag{A.12}$$

where λ denotes Lebesque measure. See Siegmund (1986) for more details.

As shown in the proof of Proposition 2, we can similarly find a critical value κ_α such that

$$P\left(\overline{\xi}(\hat{v}) \le \kappa_\alpha\right) = (1 - exp(-\kappa_\alpha))(1 - exp(-\kappa_\alpha)) = 1 - \alpha$$

and it implies that

$$\kappa_\alpha = -\ln[1 - (1-\alpha)^{1/2}]. \tag{A.13}$$

By substituting (A.13) into (A.12), we can express the expected length for the $1-\alpha$ confidence set as a function of the critical value κ_α rather than the level $(1-\alpha)$ as follows.

$$E_{\tau^0}[\lambda\{C(Y)_{1-\alpha}\}] = 4(1 - exp(-\kappa_\alpha)) \left\{ \kappa_\alpha - \frac{1}{2}(1 - exp(-\kappa_\alpha)) \right\}. \tag{A.14}$$

Using (A.14) we can show that if $\overline{\xi}(\hat{v}) \overset{d}{\Rightarrow} \max_v \left(-\frac{1}{2}|v| + W(v)\right)$ for $v \in (-\infty, \infty)$ in (A.11), the expected length of v such that $\overline{\xi}(v) \le \kappa_\alpha$ is $4(1 - exp(-\kappa_\alpha)) \left\{\kappa_\alpha - \frac{1}{2}(1 - exp(-\kappa_\alpha))\right\}$.

■

Proof of Proposition 3. For the general case, as in our setup under Assumptions 1-8, first consider the period before the true jth break date, $\tau_j - \tau_j^0 \leq 0$ (i.e. $v \leq 0$).

Given a critical value $\kappa_{\alpha,j}$, the expected length of the $1-\alpha$ confidence set in the segment $\tau_j - \tau_j^0 \leq 0$ can be shown as follows.

$$\begin{aligned}
& E\left[\lambda\{\tau_j | lr_j(\tau_j - \tau_j^0) \leq \kappa_{\alpha,j}, \quad \tau_j - \tau_j^0 \leq 0\}\right] \\
= \; & E\left[\lambda\{\tau_j \mid \frac{lr_j(\tau_j - \tau_j^0)}{\omega_{1,j}} \leq (\kappa_{\alpha,j}/\omega_{1,j}), \quad \tau_j - \tau_j^0 \leq 0\}\right] \\
= \; & \underbrace{\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)}_{(i)} \underbrace{E\left[\lambda\{v \mid \overline{\xi}(v) \leq (\kappa_{\alpha,j}/\omega_{1,j}), \quad v \leq 0\}\right]}_{(ii)} \\
= \; & \left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right) \underbrace{2(1 - exp(-\kappa_{\alpha,j}/\omega_{1,j})) \left\{\kappa_{\alpha,j}/\omega_{1,j} - \frac{1}{2}(1 - exp(-\kappa_{\alpha,j}/\omega_{1,j}))\right\}}_{(ii)'}
\end{aligned} \tag{A.15}$$

In (A.15), the second line is from Proposition 1

$$lr_j(\hat{\tau}_j - \tau_j^0) \overset{d}{\Rightarrow} \max_v \; \omega_{1,j}\left(-\tfrac{1}{2}|v| + W_j(v)\right) \quad \text{for} \quad v \leq 0$$

where $W_j(v)$ is a standard Wiener processes and

$$P\left(\max_v \; \omega_{1,j}\left(-\frac{1}{2}|v| + W_j(v)\right) \leq \kappa_{\alpha,j}\right) = P\left(\max_v \left(-\frac{1}{2}|v| + W_j(v)\right) \leq \kappa_{\alpha,j}/\omega_{1,j}\right) = 1 - \alpha.$$

(ii) in the third line is the expected length of the confidence set measured on $v \in (-\infty, 0]$ so that it is multiplied by (i), $\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)$, since v is defined by

$$\begin{aligned}
\tau_j - \tau_j^0 &= r = s/v_T^2 = (\Lambda_{1,j}^2/\Xi_{1,j}^2)v/v_T^2 = (\Lambda_{1,j}^2 v_T^2/\Xi_{1,j}^2 v_T^4)v \\
&= \left\{\frac{\frac{1}{4}vec\left(B_{1,j}\right)' \Omega_j vec\left(B_{1,j}\right) + \Delta\beta_j' \Pi_{1,j} \Delta\beta_j}{\left(\frac{1}{2}tr(B_{1,j}^2) + \Delta\beta_j' Q_{1,j} \Delta\beta_j\right)^2}\right\} v \\
&= \left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right) v.
\end{aligned} \tag{A.16}$$

We can find the scale $\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)$ in (A.16) from the change in variables used in the proof of Proposition 1. The critical value for $\overline{\xi}(v)$ is $\kappa_{\alpha,j}/\omega_{1,j}$. Then, the fourth line is derived from Lemma 1. Substituting $\kappa_{1,\alpha} = \kappa_{\alpha,j}/\omega_{1,j}$ into the calculated expected length, the expected length is calculated for the segment $\tau_j - \tau_j^0 \leq 0$ so that $(ii)'$ in (A.15) is a half of the expected length in Lemma 1.

Similarly, the expected length for $\hat{\tau}_j - \tau_j^0 > 0$ is given by

$$2\left(\Gamma_{2,j}^2/\Psi_{2,j}^2\right)(1 - exp(-\kappa_{\alpha,j}/\omega_{2,j}))\left\{\kappa_{\alpha,j}/\omega_{2,j} - \frac{1}{2}(1 - exp(-\kappa_{\alpha,j}/\omega_{2,j}))\right\}.$$

Then, the expected length for a $1-\alpha$ likelihood-based confidence set is given by

$$\begin{aligned} & 2\left(\Gamma_{1,j}^2/\Psi_{1,j}^2\right)(1 - exp(-\kappa_{\alpha,j}/\omega_{1,j}))\left\{\kappa_{\alpha,j}/\omega_{1,j} - \frac{1}{2}(1 - exp(-\kappa_{\alpha,j}/\omega_{1,j}))\right\} \\ + \; & 2\left(\Gamma_{2,j}^2/\Psi_{2,j}^2\right)(1 - exp(-\kappa_{\alpha,j}/\omega_{2,j}))\left\{\kappa_{\alpha,j}/\omega_{2,j} - \frac{1}{2}(1 - exp(-\kappa_{\alpha,j}/\omega_{2,j}))\right\}. \end{aligned}$$

Note that when either $\omega_{1,j}$ or $\omega_{2,j}$ gets bigger (i.e. the magnitude of a structural change is bigger), the expected length gets smaller since we can have more precise information about the timing of structural break. When κ gets smaller, the expected length gets smaller since the cut-off value for the likelihood ratio gets smaller. ■

Proof of Corollary 1. If there is no break in variance, $B_{1,j} = B_{2,j} = 0$. In addition, if the distribution of the regressors is stable,

$$\Pi_{1,j} = \Pi_{2,j} = \Pi = \underset{T\to\infty}{lim}\, var\left\{T^{-1/2}\left[\sum_{t=1}^{T} x_t(\Sigma^0)^{-1/2}\eta_t\right]\right\}$$

and

$$Q_{1,j} = Q_{2,j} = Q = \underset{T\to\infty}{plim}\, T^{-1}\sum_{t=1}^{T} x_t(\Sigma^0)^{-1}x_t'.$$

Thus, this implies that

$$\omega_{1,j} = \omega_{2,j} = \omega = \frac{\Delta\beta_j'\Pi\Delta\beta_j}{\Delta\beta_j'Q\Delta\beta_j}$$

and

$$\frac{\Gamma_{1,j}^2}{\Psi_{1,j}^2} = \frac{\Gamma_{2,j}^2}{\Psi_{2,j}^2} = \frac{\Delta\beta_j'\Pi\Delta\beta_j}{(\Delta\beta_j'Q\Delta\beta_j)^2}.$$

By substituting the results above into the critical value in Proposition 2 and the expected length in Proposition 3, we can find the results in Corollary 1. The results in Remarks 1 and 2 follow in the same way. ■

Proof of Corollary 2. If there is no break in conditional mean, $\Delta\beta_j = 0$. In addition, if the standardized errors, η_t, are identically Normally distributed, $\eta_t\eta_t'$ has a Wishart distribution with $var(vec(\eta_t\eta_t')) = I_{n^2} + K_n$ where K_n is the commutation matrix. Then, $\Omega_{1,j} = \Omega_{2,j} = \Omega = I_{n^2} + K_n$. Further, since K_n is an idempotent matrix,

$$\begin{aligned} & vec(B_{1,j})'\Omega^0 vec(B_{1,j})/4 \\ = \; & vec(B_{1,j})'(I_{n^2} + K_n)vec(B_{1,j})/4 \\ = \; & vec(B_{1,j})'vec(B_{1,j})/2. \end{aligned}$$

Thus,

$$\begin{aligned} \omega_{1,j} &= \frac{\Gamma_{1,j}^2}{\Psi_{1,j}} \\ &= \frac{\frac{1}{4}vec(B_{1,j})'\Omega_{1,j}^0 vec(B_{1,j})}{\frac{1}{2}tr(B_{1,j}^2)} \\ &= \frac{\frac{1}{2}vec(B_{1,j})'vec(B_{1,j})}{\frac{1}{2}tr(B_{1,j}^2)} \\ &= 1 \end{aligned}$$

since $vec(B_{1,j})'vec(B_{1,j}) = tr(B_{1,j}^2)$. Similarly, $\omega_{2,j} = 1$. Then, $\frac{\Gamma_{1,j}^2}{\Psi_{1,j}^2} = \frac{2}{tr(B_1^2)}$, and $\frac{\Gamma_{2,j}^2}{\Psi_{2,j}^2} = \frac{2}{tr(B_2^2)}$. ■

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