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Consumer Benefits of Infrastructure Services

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Abstract

This paper provides methodologies for evaluating consumer benefits of infrastructure services using potentially observable information. We define benefit measures for consumers and, using general principles from the index number literature, derive alternative first and second order approximations to these measures under the assumption of fixed prices for market goods and services. We then describe how the benefit measures and their associated approximations can be used in quantifying the economic benefits when prices are allowed to change endogenously as the provision of infrastructure services changes. In addition, under quite unrestrictive assumptions, a measure of welfare change based only on potentially observable data is derived.

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1   Introduction

The number of infrastructure services provided by the public sector is great, ranging from utility services such as water, electricity and gas supplies, to communication (e.g., cable, internet and telephone services) and transportation services (e.g., roads, railways and airports). While the provision of these services benefits many households and firms in the region (and sometimes in neighboring regions), there is, at the same time, a substantial cost involved in providing them. Therefore, when deciding which and how much infrastructure should be provided to a given region it is important to be able to measure the benefits resulting from providing these services.

An extensive literature has mainly focused on the evaluation of the benefits of infrastructure services to the *production sector* of a country or a region; see for example Aschauer (1989), Berndt and Hansson (1992), Holtz-Eakin (1994), Seitz (1994, 1995), Morrison and Schwartz (1996), Boarnet (1998), Fernald (1999), Boisso, Grosskopf and Hayes (2000), Shanks and Barnes (2008) and Elnasri (2012). That is, by comparison, the impact of the provision of additional infrastructure services on households has been relatively unexplored. This is no doubt in part due to the complexities involved, yet an understanding of the impacts on households is key to an understanding of the political economy of public infrastructure investment; as Haughwout (2002, p. 426) notes, "residents vote and firms do not".

In this paper, we provide a methodology for evaluating the benefits of infrastructure services to the *consumers* in a region. In this sense, it is in the spirit of Roback (1982), Albouy (2008), Parry and Small (2009), Albouy, Leibovici and Warman (2013)and Haughwout (2002), who used a spatial general equilibrium model to assess whether consumers benefited more than firms from local price changes induced by public infrastructure. However, our approach is closer to the welfare analysis of Hicks (1940-41), Hicks (1941-42), Harberger (1971), and Diewert (1992). In particular, we are going to draw on the work of Diewert (1986) who developed methods for evaluating the benefits of infrastructure services to the production sector based on information on prices and quantities for the two situations that are being compared. Thus, we derive methodologies for estimating the benefits of infrastructure services to households based on potentially observable price and quantity data. In addition to a range of results on household benefit measures, which focus on "efficiency gains" net of redistribution effects, we derive a direct measure for pure welfare change.

The paper proceeds as follows. In the next section we define benefit measures for consumers in the region based on a fixed price approach (i.e., we assume constant reference prices for market goods and services). In section 3 we introduce the concept of consumers' willingness to pay for infrastructure services and apply a direct approach in deriving the approximations for the benefit measures. That is, the obtained approximations are based on infrastructure services provided to the consumers in the region and on consumers' valuations of these services. In section 4 we apply an indirect approach in deriving the approximations for consumers' benefit measures. Specifically, the benefit measures are approximated using indirect information, that is, information on consumers' consumption of market goods and services and their respective prices. We discuss in section 5 how the benefit measures and their associated approximations can be used in quantifying the economic benefits of infrastructure

services when the assumption of fixed prices is relaxed. Section 6 provides an expression, using only potentially available data, for the pure welfare effect of a change in infrastructure services. The paper concludes with a review of the alternative approaches and results.

2   Benefit Measures

In this section we define benefit measures for consumers assuming constant reference prices for market goods and services; we assume that the region under consideration is small and changes in the regional demand or supply for these goods and services do not affect the "world" prices. By holding the prices fixed we can focus on the pure efficiency effects of changes in infrastructure services and avoid contaminating these effects with the effects of exogenous changes in the region's terms of trade. In section 5, the prices for local market goods and services are allowed to change endogenously as the provision of infrastructure services changes and discuss how the benefit measures developed in this section can still be used in quantifying the economic benefits of infrastructure services.

We consider a region in which there are a finite number H of households. We assume that households in this regional economy consume two types of goods and services. The first type consists of N goods and services which can be bought by the households at the fixed positive prices $(p_1, ..., p_N)$ which we denote by the price vector p. We denote the consumption vector of these N goods and services for household h as $c^h \equiv (c_1^h, ..., c_N^h)$. We restrict the consumption vector of the N market goods and services of household h to be a nonnegative vector, i.e., $c^h \geq 0_N$ where 0_N denotes a vector of zero of dimension N.[1] Note that this restriction implies that the labor supply of household h is measured indirectly through its leisure consumption. For example, if household h provides l_n^h hours of labor service n then c_n^h is measured as $c_n^h \equiv 24 - l_n^h$.[2]

The second type of goods and services is a class of I infrastructure services (e.g., water, electricity, sewage disposal, airport services, etc) that are provided by all levels of government to the inhabitants of the region. Included in the list of infrastructure services are potential new services that might be provided by the government but are being provided at zero levels in the current period. The consumption by household h of the ith type of infrastructure service is denoted by the nonnegative number $S_i^h \geq 0$ for $i = 1, ..., I$ and $h = 1, ..., H$. The vector of infrastructure services utilized by household h will be denoted by the nonnegative vector $S^h \equiv (S_1^h, ..., S_I^h) \geq 0_I$ for $h = 1, ..., H$ where 0_I denotes a vector of zero of dimension I. The household may or may not be paying user fees for the use of these infrastructure services. If all of the infrastructure services were pure public goods, then we would have $S^h = S$, for $h = 1, ..., H$; i.e., each household can consume the common amount of each of the I types of infrastructure services. However, in general, we will assume that each household is utilizing a specific amount of water, electricity, natural gas, postal services, rail services, etc.

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[1] Notation: $y \geq 0_M$ means each component of the vector y is nonnegative, $y \gg 0_M$ means that each component is strictly positive, $y > 0_M$ means $y \geq 0_M$ but $y \neq 0_M$ and $p \cdot y$ denotes the inner product of the vectors p and y.

[2] Of course, if household h also consumes service n then this amount would be added to c_n^h.

We assume that households' preferences over different combinations of the N market goods and services and of infrastructure services can be represented by utility functions, $U^h(c^h, S^h)$ for $h = 1, ..., H$. We assume that the domain of definition for $U^h(c^h, S^h)$ is the set $V \equiv \{(c^h, S^h) : c^h \geq 0_N; S^h \geq 0_I\}$ and that $U^h(c^h, S^h)$ is continuous, non-decreasing[3] and quasi-concave in c^h. We further assume that $U^h(c^h, S^h)$ is twice continuously differentiable over its domain of definition. Note that except for differentiability we do not assume any regularity properties for $U^h(c^h, S^h)$ with respect to its S^h variable.

Under these assumptions, the restricted expenditure function of household h, denoted as e^h, is defined for $p \gg 0_N$ by minimizing the cost of achieving a given utility level $u^h > 0$, given that the household has at its disposal the vector S^h of infrastructure services. Formally, for $p \gg 0_N$ and S^h and u^h such that there exists a c^h satisfying $U^h(c^h, S^h) = u^h$ with $(c^h, S^h) \in V$, the restricted expenditure function for household h is defined by

$$e^h(u^h, p, S^h) \equiv \min_c \{p \cdot c : U^h(c, S^h) = u^h\} \quad \text{for } h = 1, ..., H. \tag{1}$$

Suppose c^h solves (1). Then we have defined the household's *restricted expenditure function* e^h by $e^h(u^h, p, S^h) = p \cdot c^h$; namely, the minimized expenditure of household h is a function of the household's given utility level u^h, the price vector p it faces for its consumption of the N market goods and services and the vector of infrastructure services S^h it has at its disposal. Note that the restricted expenditure function defined by (1) is linearly homogeneous and concave in p.

We are interested in the benefits that will accrue to household h if the government changes its infrastructure services vector from S^{h0} to S^{h1}. Before addressing this, we first note that changes in the provision of infrastructure services are likely to affect the distribution of income and welfare in the region. Therefore, when evaluating the benefits accruing to households due to changes in the provision of infrastructure services we have to be cautious to separate the redistributive effects of these changes from the efficiency effects and focus on the latter. To do that, we adopt the following approach to measuring the pure efficiency effects of changes in infrastructure services on household h: we freeze the household's utility level at its initial welfare level u^{h0} (i.e., before the changes in infrastructure services were introduced). We then ask how the minimum cost of achieving the initial welfare level will change for household h as a result of its change in infrastructure consumption. Formally, let us denote G^h as the measure of the household's gross benefits from the change in infrastructure services (S^{h0} to S^{h1}). Then we define G^h for $h = 1, ..., H$ as follows:

$$G^h(S^{h0}, S^{h1}, u^{h0}, p) \equiv -\{e^h(u^{h0}, p, S^{h1}) - e^h(u^{h0}, p, S^{h0})\}. \tag{2}$$

A few notes should be mentioned in regards to this gross benefit change measure. First, the benefits of the infrastructure change to household h are termed *gross* benefits because we do not net out any changes in user fees that may result from the infrastructure change. Second, G^h depends on the two infrastructure service vectors S^{h0} and S^{h1}, as well as on the household's initial level of welfare u^{h0} and the fixed price vector p. Third, the benefits to

---

[3]Formally, we will make the assumption that for every $(c^h, S^h) \in V$ we have $\nabla_{c^h} U^h(c^h, S^h) > 0_N$.

household h defined in (2) are equal to *minus* the change in the household's minimized cost of achieving its initial welfare level. That is, household h will be better off (worse off) due to changes in its infrastructure consumption if its minimized cost of achieving its initial welfare level is reduced (increased). Last, note that the household's gross benefit change measure is also a measure of gross benefit change to society since it represents a change in the household's consumption of the N market goods and services available in the region (evaluated at the constant reference prices $p \equiv (p_1, p_2, ..., p_N)$) while still maintaining its initial welfare level.

We now aggregate over all households' benefit change measures in order to define the regional gross benefit change measure due to a change in the government's provision of infrastructure services from S^{h0} to S^{h1} for household h, $h = 1, ...H$:

$$GH(S^{10}, ..., S^{H0}; S^{11}, ..., S^{H1}; u^{10}, ..., u^{H0}; p) \equiv \sum_{h=1}^{H} G^h(S^{h0}, S^{h1}, u^{h0}, p)$$

$$\equiv -\sum_{h=1}^{H} \{e^h(u^{h0}, p, S^{h1}) - e^h(u^{h0}, p, S^{h0})\}. \quad (3)$$

Since prices for the N market goods and services that the region face are likely to change between period 0 and period 1 the gross benefit measure in (2) (and hence also in (3)) can be evaluated using either the prices that prevail in period 0 (denoted as p^0) or the prices that prevail in period 1 (denoted as p^1). Thus, there are two possible benefit measures for the household:

$$G^h(S^{h0}, S^{h1}, u^{h0}, p^0) \equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \quad (4)$$

and

$$G^h(S^{h0}, S^{h1}, u^{h0}, p^1) \equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\}. \quad (5)$$

The above two equations represent the change in the household's consumption of the N market goods and services that is needed to maintain its initial welfare level u^{h0} due to the change in infrastructure services, where in (4) the change in the household's consumption is valued at the fixed price vector prevailing in period 0 and in (5) the change in the household's consumption is valued at the fixed price vector prevailing in period 1.

Both of (4) and (5) share a key element: they are defined in terms of differences in the households' restricted expenditure functions. Thus, in subsequent sections of this paper, we develop methods for approximating these differences using potentially observable information.

3 Approximating the Benefit Measures: A Direct Approach

In this section we study various first and second order approximations to the household benefit measures (4) and (5). The approach used in deriving these approximations is a

direct one. That is, the benefit measures are going to be approximated using information on the infrastructure services provided to the households in the region and on the households' valuations of these services.

We begin by defining the concept of the household's "willingness to pay" for an extra unit of the ith type of infrastructure services. If a household's welfare level is u^h, then under the set of assumptions of section 2, the household would be willing to pay its reduction in expenditure on the N market goods and services that is due to an additional unit of S_i^h while still maintaining its standard of living at the utility level u^h. That is, the household should be willing to pay the following amount:

$$-\{e^h(u^h, p, S_1^h, ..., S_{i-1}^h, S_i^h + 1, S_{i+1}^h, ..., S_I^h) - e^h(u^h, p, S_1^h, ..., S_{i-1}^h, S_i^h, S_{i+1}^h, ..., S_I^h)\}. \quad (6)$$

We can approximate the difference in (6) by the partial derivative $\partial e^h(u^h, p, S^h)/\partial S_i^h$. Since this partial derivative represents the amount that a cost minimizing household is willing to pay for the use of the extra marginal unit of S_i^h, we define the household's *willingness to pay function* for marginal units of the ith infrastructure service as follows:[4]

$$W_i^h(u^h, p, S^h) \equiv -\partial e^h(u^h, p, S^h)/\partial S_i^h \quad i = 1, ..., I. \quad (7)$$

Now, let the data for the initial situation be $p^0 \equiv (p_1^0, ..., p_N^0)$, a positive price vector for market goods and services; $u^{h0} > 0$, the household's welfare level in period 0; $c^{h0} \equiv (c_1^{h0}, ..., c_N^{h0})$, the corresponding consumption vector of household h in period 0; $S^{h0} \equiv (S_1^{h0}, ..., S_I^{h0})$, a nonnegative vector of infrastructure services that are being consumed by household h in period 0, and $W^{h0} \equiv (W_1^{h0}, ..., W_I^{h0})$, the corresponding willingness to pay vector of household h in period 0; i.e., $W_i^{h0} \equiv -\partial e^h(u^{h0}, p^0, S^{h0})/\partial S_i^h$ for $i = 1, ..., I$.

Suppose the government changes the infrastructure services vector for household h to S^{h1} in period 1. We also allow for a change in the price vector for market goods and services, so the new period 1 price vector is p^1. The household's new willingness to pay vector is defined as $W^{h1} \equiv (W_1^{h1}, ..., W_I^{h1})$ where $W_i^{h1} \equiv -\partial e^h(u^{h0}, p^1, S^{h1})/\partial S_i^h$ for $i = 1, ..., I$. Note that W_i^{h1} is evaluated at (u^{h0}, p^1, S^{h1}). That is, W_i^{h1} reflects the amount that household h, when facing period 1 prices p^1 and when consuming S^{h1} infrastructure services, is willing to pay for an additional unit of S_i^h while still maintaining its standard of living at the *initial* utility level u^{h0}.[5]

We may write the two willingness to pay vectors as a gradient vector (i.e., a vector of first order partial derivatives) of the household expenditure function with respect to the components of S^h as follows:

---

[4]This derivation of willingness to pay functions in an expenditure function framework is due to Diewert (1986) but precursors of this concept may be found in Samuelson (1953-1954) and Diewert (1974). For further discussion on households' willingness to pay functions and their properties see Diewert (1986).

[5]W_i^{h1} is defined in terms of the initial utility level so as to focus on efficiency effects, rather than redistribution effects.

$$W^{h0} \equiv -\nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}); \qquad W^{h1} \equiv -\nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}). \tag{8}$$

We now turn to develop first order approximations to the household benefit measures (4) and (5).

3.1   First Order Approximations

We assume that in each period the household minimizes the cost of achieving its welfare level in that period. In particular, we assume that c^{h0} is the solution to the expenditure minimization problem (1) of household h in period 0 when $u^h = u^{h0}$, $p = p^0$ and $S^h = S^{h0}$. Thus we have the following equality:

$$e^h(u^{h0}, p^0, S^{h0}) = p^0 \cdot c^{h0} \equiv \sum_{n=1}^{N} p_n^0 c_n^{h0}. \tag{9}$$

Note that the expenditure $e^h(u^{h0}, p^0, S^{h1})$ in (4) is the hypothetical expenditure that is associated with the period 1 allocation of infrastructure services and period 0 prices and welfare level. That is, it is the expenditure that household h would have spent to achieve welfare level u^{h0} had it faced prices p^0 and had had at its disposal infrastructure services S^{h1}. This expenditure is not observable but we can approximate it by means of a first order Taylor series approximation as follows:

$$
\begin{aligned}
e^h(u^{h0}, p^0, S^{h1}) & \simeq e^h(u^{h0}, p^0, S^{h0}) + \sum_{i=1}^{I} [\partial e^h(u^{h0}, p^0, S^{h0})/\partial S_i^h][S_i^{h1} - S_i^{h0}] \\
& = p^0 \cdot c^{h0} + \nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) \quad \text{using (9)} \\
& = p^0 \cdot c^{h0} - W^{h0} \cdot (S^{h1} - S^{h0}) \quad \text{using (8).} \tag{10}
\end{aligned}
$$

Similarly, we may approximate the unobservable expenditure $e^h(u^{h0}, p^1, S^{h0})$ in (5) as follows:

$$
\begin{aligned}
e^h(u^{h0}, p^1, S^{h0}) & \simeq e^h(u^{h0}, p^1, S^{h1}) + \nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) \\
& = e^h(u^{h0}, p^1, S^{h1}) - W^{h1} \cdot (S^{h0} - S^{h1}) \quad \text{using (8).} \tag{11}
\end{aligned}
$$

Substituting (10) and (11) into the household benefit measures (4) and (5), we obtain the following approximate benefit measures (12) and (13):

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^0) & \equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \\
& \simeq -\{p^0 \cdot c^{h0} - W^{h0} \cdot (S^{h1} - S^{h0}) - p^0 \cdot c^{h0}\} \quad \text{using (9) and (10)} \\
& = W^{h0} \cdot (S^{h1} - S^{h0}) \tag{12}
\end{aligned}
$$

and

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^1) &\equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} \\
&\simeq -\{e^h(u^{h0}, p^1, S^{h1}) - [e^h(u^{h0}, p^1, S^{h1}) - W^{h1} \cdot (S^{h0} - S^{h1})]\} \quad \text{using (11)} \\
&= W^{h1} \cdot (S^{h1} - S^{h0}). \tag{13}
\end{aligned}
$$

Expressions (12) and (13) have the same form as the linear approximations to equivalent variation and compensation variation, respectively, as derived by Hicks (1941-42); see also Diewert (1992, p. 568, footnote 11). Each benefit measure can be calculate simply if data are available on the change in infrastructure services, $(S^{h1} - S^{h0})$ and the ex ante (W^{h0}) and ex post (W^{h1}) willingness to pay.

3.2   Second Order Approximations

We now turn to develop second order approximations to the household benefit measures (4) and (5). We assume that households' preferences are homothetic in the N market goods and services, conditional on any vector of infrastructure services. This assumption is equivalent to the following linear homogeneity assumption on $U^h(c^h, S^h)$:

$$
U^h(\lambda c^h, S^h) = \lambda U^h(c^h, S^h) \quad \text{for all } \lambda > 0; c^h \geq 0_N; S^h \geq 0_I. \tag{14}
$$

Next we define the *unit* (utility) *expenditure function* for household h. To do so, we first note that for $(c^h, S^h) \in V$, we have,

$$
\begin{aligned}
0 &\leq c^h \cdot \nabla_{c^h} U^h(c^h, S^h) \\
&= U^h(c^h, S^h), \tag{15}
\end{aligned}
$$

where the inequality follows from our assumption that for every $(c^h, S^h) \in V$ we have $\nabla_{c^h} U^h(c^h, S^h) > 0_N$ (non-decreasing in c^h utility function) and the equality follows from our assumption of linear homogeneity (14) and Euler's Theorem on homogeneous functions.[6] Thus (15) implies that for each fixed S^h, the range of $U^h(c^h, S^h)$ as c^h varies over the non-negative orthant is the set $R \equiv \{u^h : 0 \leq u^h \leq +\infty\}$. In particular, $u^h = 1$ belongs to this range set R. Hence we can define the *unit expenditure function* for household h as follows:

$$
\begin{aligned}
E^h(p, S^h) &\equiv \min_c \{p \cdot c : U^h(c, S^h) = 1\} \\
&= e^h(1, p, S^h) \quad \text{for } h = 1, ..., H, \tag{16}
\end{aligned}
$$

where the last equality follows from (1). Note that the household's unit expenditure function defined by (16) is linearly homogeneous and concave in p.

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[6] Using Euler's Theorems on homogeneous functions, our assumption of linear homogeneity implies that for every $(c^h, S^h) \in V$ we have $c^h \cdot \nabla_{c^h} U^h(c^h, S^h) = U^h(c^h, S^h)$.

Using the linear homogeneity assumption (14) we can express the household's restricted expenditure function in terms of its unit expenditure function. Specifically, for each $u^h > 0$, $p \gg 0_N$ and S^h such that there exists a c^h satisfying $U^h(c^h, S^h) = u^h$ with $(c^h, S^h) \in V$, we have

$$
\begin{aligned}
e^h(u^h, p, S^h) &\equiv \min_c \{p \cdot c : U^h(c, S^h) = u^h\} \\
&= \min_c \{p \cdot c : (1/u^h)U^h(c, S^h) = 1\} \\
&= \min_c \{p \cdot c : U^h(c/u^h, S^h) = 1\} \quad \text{using linear homogeneity (14)} \\
&= \min_c \{(u^h p \cdot c)/u^h : U^h(c/u^h, S^h) = 1\} \\
&= u^h \min_z \{p \cdot z : U^h(z, S^h) = 1\} \quad \text{letting } z \equiv c/u^h \\
&= u^h E^h(p, S^h) \quad \text{using definition (16)} \quad (17)
\end{aligned}
$$

for $h = 1, ..., H$. The advantage of relationship (17) is that it simplifies our search for a class of functional forms that can approximate to the second order households' restricted expenditure functions. Indeed, it suffices for us to find a class of functional forms that can approximate to the second order households' unit expenditure functions in order to derive second order approximations to households' benefit measures.

Consider the following (normalized) biquadratic functional form for the unit expenditure function for household h:

$$
E^h(p, S^h) \equiv \sum_{n=1}^{N} \gamma_n p_n + (1/2) \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} d_{mn} p_m p_n (p_N)^{-1}
$$
$$
+ \sum_{n=1}^{N} \sum_{i=1}^{I} f_{ni} p_n S_i^h + (1/2)(\sum_{n=1}^{N} \delta_n p_n)(\sum_{i=1}^{I} \sum_{j=1}^{I} g_{ij} S_i^h S_j^h) \quad (18)
$$

which can be written in matrix notation as,

$$
E^h(p, S^h) \equiv \gamma \cdot p + (1/2)(p_N)^{-1}(p' \cdot Dp') + p \cdot FS^h + (1/2)(\delta \cdot p)(S^h \cdot GS^h), \quad (19)
$$

where $p' \equiv (p_1, ..., p_{N-1})$, γ is an N dimensional vector with elements γ_n, D is an $N-1$ by $N-1$ symmetric and negative semi-definite matrix with elements d_{mn},[7] F is a N by I matrix with elements f_{ni}, $\delta > 0_N$ is a nonnegative, nonzero vector of *fixed constants* with elements δ_n and G is an I by I symmetric matrix with elements g_{ij}. To keep the number of unknown parameters to a minimum, and to make the household's expenditure function linear in the unknown parameters, γ_n, d_{mn}, f_{ni} and g_{ij}, we assume that the elements δ_n of the vector δ are *known* nonnegative numbers which are not all equal to zero. The chosen values for the δ_n have no impact on the properties of the expenditure function, and thus can be chosen freely.

---

[7]We require the matrix D to be negative semi-definite in order to ensure the global concavity of $E^h(p, S^h)$ in p. See Diewert and Wales (1987) and their proof of their Theorem 10.

For the purpose of obtaining second order approximations to household's benefit measures let us define the following normalized price and willingness to pay vectors,

$$\tilde{p}^t \equiv p^t/\delta \cdot p^t; \qquad \tilde{W}^{ht} \equiv W^{ht}/\delta \cdot p^t; \qquad t = 0, 1. \qquad (20)$$

As one can see, p^0 and W^{h0} in definition (20) are deflated by $\delta \cdot p^0 \equiv \sum_{n=1}^N \delta_n p_n^0$, where δ is the nonnegative, nonzero vector of fixed constants which appears in the functional form for the household's unit expenditure function (18). Similarly, p^1 and W^{h1} in definition (20) are deflated by $\delta \cdot p^1 \equiv \sum_{n=1}^N \delta_n p_n^1$ to form $\tilde{p}^1$ and $\tilde{W}^{h1}$.

Using definitions (2) and (20) we can define the following analogous measures to the household's benefit measures (4) and (5) as follows:

$$G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^0) \equiv -\{e^h(u^{h0}, \tilde{p}^0, S^{h1}) - e^h(u^{h0}, \tilde{p}^0, S^{h0})\}, \qquad (21)$$

and

$$G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^1) \equiv -\{e^h(u^{h0}, \tilde{p}^1, S^{h1}) - e^h(u^{h0}, \tilde{p}^1, S^{h0})\}. \qquad (22)$$

The household benefit measure defined by (21) is the same as the theoretical household benefit measure (4) only here the price vector p^0 is replaced by the normalized price vector $\tilde{p}^0$. Likewise, the household benefit measure defined by (22) is the same as the theoretical household benefit measure (5) where the price vector p^1 is replaced by the normalized price vectors $\tilde{p}^1$.

Then we can derive the following result.

Proposition 1 *Suppose the unit expenditure function for household h, E^h, is defined by (18). Then we have the following exact identity:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^0) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^1) = (1/2)[\tilde{W}^{h0} + \tilde{W}^{h1}] \cdot [S^{h1} - S^{h0}] \quad (23)$$

Proof. See the Appendix ∎

The left hand side of (23) is an average of the two theoretical household benefit measures defined by (21) and (22). The right hand side is an average of the two first order approximate benefit measures defined by (12) and (13), where the original willingness to pay vectors W^{h0} and W^{h1} are replaced with the normalized willingness to pay vectors $\tilde{W}^{h0}$ and $\tilde{W}^{h1}$.

The implication of the class of functional forms defined by (18) is that this class can approximate an arbitrary twice continuously differentiable unit expenditure function $E^{h*}(p, S^h)$ to the second order. Specifically we have the following result,

Proposition 2 *Let $p^* \gg 0_N$ and $S^{h*} \geq 0_I$ and let a given unit expenditure function E^{h*} be twice continuously differentiable at (p^*, S^{h*}). Then for any given nonnegative, nonzero vector $\delta > 0_N$, there exists a E^h in the class of functions defined by (18) (where the δ which appears in (18) is the same as the given δ) such that*

$$E^h(p^*, S^{h^*}) = E^{h^*}(p^*, S^{h^*}) \tag{24}$$

$$\nabla_z E^h(p^*, S^{h^*}) = \nabla_z E^{h^*}(p^*, S^{h^*}) \tag{25}$$

$$\nabla_{zz}^2 E^h(p^*, S^{h^*}) = \nabla_{zz}^2 E^{h^*}(p^*, S^{h^*}) \tag{26}$$

where $z \equiv (p, S^h)$.

Proof. See the Appendix ∎

i.e., the level, all $N + I$ first order partial derivatives and all $(N + I)^2$ second order partial derivatives of E^h and E^{h^*} coincide at the point (p^*, S^{h^*}).

Combining Propositions 1 and 2 we can conclude that for any nonnegative, nonzero vector δ, $(1/2)[\tilde{W}^{h0} + \tilde{W}^{h1}] \cdot [S^{h1} - S^{h0}]$ will approximate the average of the household's benefit measures (21) and (22) to the *second* order.[8]

4 Approximating the Benefit Measures: An Indirect Approach

The approach used to derive the approximations to the household benefit measures in the previous section was a direct one in the sense that the derived approximations involved direct information, that is, information on the infrastructure services provided to the household and the household's valuation over these services. In this section we are going to employ an indirect approach for approximating the household benefit measures, in a similar vein to the indirect approach to benefit measurement of Kanemoto (1980), Harris (1978) and Negishi (1972). In particular, the benefit measures are going to be approximated using indirect information, namely, information from the market of the N goods and services in the region.

While this approach obviates the need for knowledge of the household's willingness to pay for infrastructure services, which may be difficult to obtain, we will see that it will only work if *ex post* data are available; we will require price and quantity data for periods 0 and 1.

As before, we denote the data for the initial situation as: $p^0 \equiv (p_1^0, ..., p_N^0)$, a positive price vector for market goods and services; $u^{h0} > 0$, the household's welfare level in period

---

[8]Diewert (1986) derived related expressions from the producer side to those presented in this section. When approximating the benefit measures using the direct approach, similar approximations for firms and households are obtained. The first order approximation to the firm's period 0 (period 1) benefit measure is equal to the change in infrastructure services provided to the firm multiplied by the firm's willingness to pay for infrastructure services in period 0 (period 1). The second order approximation to the average of the firm's period 0 and period 1 benefit measures, evaluated at normalized prices for market goods and services, is equal to the change in infrastructure services provided to the firm multiplied by the firms's normalized average of period 0 and period 1 willingness to pay for infrastructure services.

0; $c^{h0} \equiv (c_1^{h0}, ..., c_N^{h0})$, the corresponding consumption vector of household h in period 0 and $S^{h0} \equiv (S_1^{h0}, ..., S_I^{h0})$, a nonnegative vector of infrastructure services that are being consumed by household h in period 0.

We assume that in each period the household minimizes the cost of achieving its welfare level in that period. That is, we assume that c^{h0} is the solution to the expenditure minimization problem (1) of household h in period 0 when $u^h = u^{h0}$, $p = p^0$ and $S^h = S^{h0}$. Similarly, c^{h1} is assumed to be the solution to the expenditure minimization problem (1) of household h in period 1 when $u^h = u^{h1}$, $p = p^1$ and $S^h = S^{h1}$. Thus we have the following equalities:

$$e^h(u^{h0}, p^0, S^{h0}) = p^0 \cdot c^{h0} \equiv \sum_{n=1}^{N} p_n^0 c_n^{h0}; \quad e^h(u^{h1}, p^1, S^{h1}) = p^1 \cdot c^{h1} \equiv \sum_{n=1}^{N} p_n^1 c_n^{h1} \quad (27)$$

Moreover, since $e^h(u^h, p, S^h)$ is assumed to be differentiable with respect to the components of the price vector p, then using Shephard's Lemma (Shephard (1953)),[9] we have that the household's consumption vector of market goods and services in period t, c^{ht}, is equal to the vector of first order partial derivatives of $e^h(u^{ht}, p^t, S^{ht})$ with respect to the components of p. That is,

$$c^{h0} \equiv \nabla_p e^h(u^{h0}, p^0, S^{h0}); \quad c^{h1} \equiv \nabla_p e^h(u^{h1}, p^1, S^{h1}). \quad (28)$$

With this information in hand we now turn to derive alternative first order approximations to the household benefit measures (4) and (5).

4.1 First Order Approximations

Using equations (27) and (28), we can form the following first order Taylor series approximations:

$$
\begin{aligned}
e^h(u^{h0}, p^1, S^{h0}) &\simeq e^h(u^{h0}, p^0, S^{h0}) + \nabla_p e^h(u^{h0}, p^0, S^{h0}) \cdot (p^1 - p^0) \\
&= p^0 \cdot c^{h0} + c^{h0} \cdot (p^1 - p^0) \quad \text{using (27) and (28)} \\
&= p^1 \cdot c^{h0} \quad (29)
\end{aligned}
$$

$$
\begin{aligned}
e^h(u^{h0}, p^0, S^{h1}) &\simeq e^h(u^{h1}, p^1, S^{h1}) + \nabla_p e^h(u^{h1}, p^1, S^{h1}) \cdot (p^0 - p^1) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \\
&= p^1 \cdot c^{h1} + c^{h1} \cdot (p^0 - p^1) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \quad \text{using (27) and (28)} \\
&= p^0 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \quad (30)
\end{aligned}
$$

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[9]See also Appendix 3 in Diewert (1986).

12

$$e^h(u^{h0}, p^1, S^{h1}) \simeq e^h(u^{h1}, p^1, S^{h1}) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1})$$

$$= p^1 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \quad \text{using (27)} \qquad (31)$$

Now substituting (29), (30) and (31) into the benefit measures (4) and (5), and we obtain the approximate benefit measures in (32) and (33):

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^0) &\equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \\
&\simeq -\{p^0 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) - p^0 \cdot c^{h0}\} \quad \text{using (27) and (30)} \\
&= p^0 \cdot (c^{h0} - c^{h1}) - \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^1) &\equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} \\
&\simeq -\{p^1 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) - p^1 \cdot c^{h0}\} \quad \text{using (29) and (31)} \\
&= p^1 \cdot (c^{h0} - c^{h1}) - \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}). \qquad (33)
\end{aligned}
$$

The first term on the right-hand side of expressions (32) and (33) gives the household's consumption change evaluated at either period 0 or period 1 prices, respectively. The second term in each expression accounts for redistributive effects, since the use of period 1 consumption data implies the need to "net out" redistributive effects. In the absence of redistributive effects ($u^{h0} = u^{h1}$), we obtain benefit measures which depend only on prices and household's consumption of market goods and services. Importantly from a practical point of view, if the redistributive effects in the region cancel out in the aggregate, then we can evaluate these approximations using only aggregate price and consumption data for the region.

For cases where the assumption of no redistributive effects (whether at the household or at the aggregate level) is inappropriate, we can derive an explicit expression for the redistributive effects if we assume that households' preferences are homothetic in the N market goods and services, conditional on any vector of infrastructure services. Specifically, we assume that the set of assumptions used in deriving the direct second order approximations to households' benefit measures (section 3.2) holds so that a unit expenditure function exists and relation (17) holds. Then, by (17) we have $e^h(u^h, p, S^h) \equiv u^h E^h(p, S^h)$ and thus, $\frac{\partial e^h(u^h, p, S^h)}{\partial u^h} \equiv E^h(p, S^h)$. Using this relation between the restricted expenditure function and the unit expenditure function we can write the expression for the redistributive effects as:[10]

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[10]Another way to derive (34) without the use of the unit expenditure function is to use the fact that under this set of assumptions e^h is linearly homogeneous in u^h.

$$\frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) = E^h(p^1, S^{h1})(u^{h0} - u^{h1})$$

$$= u^{h1} E^h(p^1, S^{h1})(\frac{u^{h0}}{u^{h1}} - 1)$$

$$= e^h(u^{h1}, p^1, S^{h1})(\frac{u^{h0}}{u^{h1}} - 1) \quad \text{using (17)}$$

$$= (\frac{u^{h0} - u^{h1}}{u^{h1}})(p^1 \cdot c^{h1}) \quad \text{using (27).} \tag{34}$$

Thus, under the assumption of homothetic preferences the redistribution effects in (33) are equal to the relative change of the household's welfare level multiplied by household's period 1 expenditure. Since the redistribution effects are expressed in terms of *relative* change (and not absolute change) of the household's welfare level, we can use data on the relative change in the household's real income as a proxy for the relative change of the household's welfare level, i.e. money metric utility scaling.

4.2   Second Order Approximations

We now turn to derive indirect second order approximations to the household benefit measures (4) and (5). In doing so we are going to use the same set of assumptions on households' preferences which was used in deriving the direct second order approximations (section 3.2). Moreover, we are going to make use of the same class of functional form for the household's unit expenditure function as in (18).

We define the following household benefit measures using definition (2):

$$G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) \equiv -\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\}, \tag{35}$$

$$G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \equiv -\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\}. \tag{36}$$

The household benefit measure defined by (35) is the same as the theoretical household benefit measure (4) only here the price vector p^0 is replaced by the normalized price vector $p^0/p_N^0 \equiv (p_1^0/p_N^0, ..., p_{N-1}^0/p_N^0, 1)$. Likewise, the household benefit measure defined by (36) is the same as the theoretical household benefit measure (5) where the price vector p^1 is replaced by the normalized price vectors $p^1/p_N^1 \equiv (p_1^1/p_N^1, ..., p_{N-1}^1/p_N^1, 1)$. We can then obtain the following result.

Proposition 3 *Suppose the unit expenditure function for household h, E^h, is defined by (18) and that relations (27) and (28) hold. Then we have the following exact identity:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad (37)$$

Proof. See the Appendix ■

The left hand side of (37) is an average of the two theoretical household benefit measures (35) and (36). On the right hand side of (37), the first term is the average of the two first terms in the first order approximate benefit measures (32) and (33), where the price vectors p^0 and p^1 are replaced by the normalized price vectors p^0/p_N^0 and p^1/p_N^1. The second term ("netting out" redistributive effects), is the same as the second term in the first order approximate benefit measures (32) and (33), only here p^1 is replaced by the average of the normalized prices p^0/p_N^0 and p^1/p_N^1. In the absence of redistributive effects ($u^{h0} = u^{h1}$) the average of the two theoretical household benefit measures (35) and (36) equals the household's consumption change of market goods and services (a quantity change) multiplied by the average of the normalized prices p^0/p_N^0 and p^1/p_N^1. Note that this has the form of a Bennet (1920) quantity indicator; see also Diewert (2005).[11]

As was shown in Proposition 2, the class of functional forms defined by (18) can approximate an arbitrary twice continuously differentiable unit expenditure function $E^{h*}(p, S^h)$ to the second order. Thus, Propositions 2 and 3 imply that the right hand side of (37) is an approximate household benefit measure which approximates the average of the two theoretical household benefit measures (35) and (36) to the *second* order.

While the indirect approach yields approximations to benefit measures that do not rely on knowledge of the consumers' willingness to pay, household utility in both periods appear in expressions (34) and (37). As utility is usually not directly observable, we can assume e.g. money metric utility scaling, or attempt to establish a result based only on directly observable price and consumption data, as follows.

---

[11]In the producer context considered by Diewert (1986), the first order approximation to the firm's period t, $t = 0, 1$, benefit measure is equal to the firm's net output change where all outputs and inputs are evaluated at period t prices. The second order approximation to the average of the firm's period 0 and period 1 benefit measures is equal to the firm's net output change where all outputs and inputs are evaluated at a normalized average of period 0 and period 1 prices. While it is possible to use information on changes in quantities of market goods and services on the production side to implicitly infer benefits of changes in infrastructure provision to the production sector, we cannot do the same on the consumer side without further adjustment; the household's consumption data in period 1 implicitly conveys information on redistributive effects which we do not wish to include as part of the benefit measure. In the absence of redistributive effects we get similar approximations for both firms and households which consist of information on changes in quantities of market goods and services only.

Corollary 1 *Suppose the conditions stipulated in Proposition 3 hold. Then the average of the two theoretical household benefit measures (35) and (36) has the following upper bound:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \leq (1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0} \quad (38)$$

Proof. See the Appendix ∎

Note that this upper bound can be calculated with only information on prices and the household's consumption of market goods and services. Note also that we can simply aggregate over households by summing the upper bounds, leading to a benefit measure that can be used for cost-benefit analysis. Because of the need for period 1 prices in (38), it can perhaps be thought of as being of most use in *ex post* analysis of infrastructure projects.

5 An Endogenous Price Approach to Benefit Measures

As changes in infrastructure services in a region tend to be large discrete changes, these changes are likely to have relatively large effects on the regional economy, and therefore they may cause systematic (endogenous) changes in the prices of local goods and services. Thus, holding the prices of market goods and services fixed while comparing the benefits of different infrastructure provisions might not be a good strategy since some of these prices will change endogenously as the provision of infrastructure services changes.

To accommodate for local goods and services and their price endogeneity we can make use of the setup developed by Diewert (1986). In particular, we can treat the initial N market goods and services as inter-regionally traded goods and services which are supplied to or demanded from the region in a perfectly elastic manner, that is, their prices are assumed to be fixed. We then can introduce into the regional economy a second class of M market goods and service where the prices of this second class of goods and services will depend on local supply and demand conditions.

Diewert (1986) derived an endogenous price benefit measure of changes in the provision of infrastructure services for the region as a whole. He showed that under certain conditions, the endogenous price benefit measure lies below the constant price benefit measure evaluated at period 0 prices. Hence, an estimate of period 0 price benefit measure for the whole region will provide an upper bound to the endogenous price benefit measure. Moreover, in Diewert's period 0 price benefit measure for the whole region, the benefits to consumers' are measure by (4). Thus, to estimate consumers' benefits in the period 0 price benefit measure we can use the first order approximations, (12) and (32), which were derived using the direct approach and the indirect approach, respectively.[12]

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[12]The first order approximation (32) can be used together with (34) so that the redistribution effects can be identified.

Diewert (1986) also showed that the first order approximation to the endogenous price benefit measure around the period 0 allocation of infrastructure services coincides with the *direct approach* first order approximation to the period 0 price benefit measure for the whole region. Therefore, an estimate of the direct approach first order approximation to the period 0 price benefit measure for the whole region will provide a first order approximation to the endogenous price benefit measure. Once again, the direct approach first order approximation (12) can be used to estimate the consumers' part.

6   Households' Welfare and the Provision of Infrastructure Services

In the previous sections we have focussed on benefit measures in terms of reduced household expenditure. In the following result we derive an explicit expression for the household's change of welfare level from period 0 to period 1, which includes redistribution effects. As expected, the derived expression is a function of the amount of infrastructure services provided to the household, the household's valuation of these services (household's willingness to pay for infrastructure services), as well as the household's consumption of market goods and services and their respective prices.

Proposition 4 *Suppose the unit expenditure function for household h, E^h, is defined by (18) and that relations (27) and (28) hold with $c^{h1} \neq 0_N$. Suppose further that δ in (18) is chosen so that $\delta \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] = 0$. Then the ratio of the household's welfare levels in period 0 and 1, $\frac{u^{h0}}{u^{h1}}$, is given by the following expression:*

$$\frac{u^{h0}}{u^{h1}} = \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} \quad (39)$$

Proof. See the Appendix ∎

As the parameter vector δ can be freely chosen (see section 3.2), the assumption on its value is unrestrictive. Inverting (39) obviously gives the change in the utility level going from period 0 to period 1, giving a measure of the welfare change for the household. Note that when we substitute (39) into (37) then we are back to the direct measure of section 3; we have a variant of (23) but with different normalization. That is, we get that the average of the two theoretical household benefit measures (35) and (36) is equal to the change in infrastructure services evaluated at the average of period 0 and period 1 willingness to pay functions.

The following result provides sufficient conditions under which households in the region are better off, from an efficiency point of view (although not necessarily with respect to the redistribution of income), due to the government's change in the provision of infrastructure services.

Proposition 5 *Suppose the conditions stipulated in Proposition 4 hold. Suppose further that $W^{h0}, W^{h1} \geq 0_I$ and $(S^{h1} - S^{h0}) \geq 0_I$. Then the average of the two theoretical household benefit measures (35) and (36) is non-negative, i.e.,:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \geq 0 \qquad (40)$$

Proof. See the Appendix ∎

The intuition is as follows. The condition $(S^{h1} - S^{h0}) \geq 0_I$ means that the government has increased the amount of infrastructure services provided to the household. The condition $W^{h0}, W^{h1} \geq 0_I$ means that the household's willingness to pay functions are non-negative. This condition can be satisfied if we assume that the household's utility function satisfies the free disposal property; this implies that if the household gets more infrastructure services then its members cannot be made worse off. Thus these two conditions combined together ensure that when the government increases the level of infrastructure services provided to the household then the household is better off from an efficiency point of view, i.e. the average of the two theoretical household benefit measures (35) and (36) is non-negative.

7   Conclusion

This paper has presented a range of results on methods for the measurement of consumer benefits from changes in infrastructure services, with an emphasis on using only potentially observable data. The key results can be concisely summarized as follows.

In section 3, we found that when approximating benefit measures for households using a direct approach, the first order approximation to the period t benefit measure is equal to the change in infrastructure services that are provided to the household valued at the household's willingness to pay for infrastructure services in period t. The second order approximation to the average of period 0 and period 1 benefit measures is equal to the change in infrastructure services that are provided to the household valued at the household's normalized average of period 0 and period 1 willingness to pay for infrastructure services. Thus, with access to information on willingness to pay, we have easily implementable measures of benefits from changes in the provision of infrastructure services.

In section 4 we considered an alternative approach with different data requirements. When approximating the benefit measures for households using an "indirect approach," utilizing changes in prices of marketed goods and services, the first order approximation to the household's period t benefit measure is equal to the change in the household's consumption of market goods and services evaluated at period t prices less redistributive effects that are due to the change in the provision of infrastructure services. Under the assumption of homothetic preferences, the household's redistribution effects equal the relative change of the household's welfare level evaluated in terms of the household's expenditure in period 1. In the absence of more attractive alternatives, the relative change in the household's real income can be used as a proxy for the relative change of the household's welfare level.

The second order approximation to the average of the household's period 0 and period 1 benefit measures for the indirect approach consisted of two expressions; the change in the household's consumption of market goods and services evaluated at a normalized average of period 0 and period 1 prices, less redistributive effects. The expression for the redistributive effects is the same as for the first order approximations, which was derived under the homotheticity assumption, only here period 1 prices are replaced by a normalized average of the prices in period 0 and period 1. The utility level in each period features in this expression for the redistributive effects, yet utility is typically not observable. However, an upper bound to the average of the household's period 0 and period 1 benefit measures is established, which is equal to the household's consumption of market goods and services in period 0 evaluated at a normalized average of period 0 and period 1 prices. Thus, it provides an upper bound on benefits using only potentially observable data.

In section 5, we considered the relaxation of the assumption of fixed prices, to acknowledge the possibility of endogenous changes in local prices with changes to the provision of infrastructure services. We found that our fixed price results can be used to provide to provide first order approximations to the endogenous price benefit measures.

Finally, in section 6, under quite unrestrictive assumptions, we derived an expression for the relative change of the household's welfare level from period 0 to period 1 to be a function of the amount of infrastructure services provided to the household, the household's valuation of these services, as well as the household's consumption of market goods and services and their respective prices. Thus, we have an expression of welfare change that can be calculated using only potentially available information. Using this result, it is shown that sufficient conditions exist under which households in the region are better off, from an efficiency point of view, due to the government's change in the provision of infrastructure services.

By presenting practical methods that can be implemented with only potentially observable price and quantity information, we believe we have expanded the range of implementable methods and advanced the understanding of issues involved in assessing consumer benefits from infrastructure services, a key area of public policy interest.

Appendix: Proofs of Propositions

Proof of Proposition 1

Under the assumptions made in the setup for the second order approximation we have,

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \tag{A1}$$

and thus,

$$\nabla_{S^h} e^h(u^h, p, S^h) = u^h \nabla_{S^h} E^h(p, S^h) \tag{A2}$$

Now using (A1) we have,

$$-\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} = -\{u^{h0}E^h(p^0, S^{h1}) - u^{h0}E^h(p^0, S^{h0})\}$$
$$= -u^{h0}\{E^h(p^0, S^{h1}) - E^h(p^0, S^{h0})\} \quad \text{(A3)}$$

and

$$-\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} = -\{u^{h0}E^h(p^1, S^{h1}) - u^{h0}E^h(p^1, S^{h0})\}$$
$$= -u^{h0}\{E^h(p^1, S^{h1}) - E^h(p^1, S^{h0})\} \quad \text{(A4)}$$

Also, since $E^h(p, S^h)$ defined by (18) is quadratic in S^h for each fixed p, its second order Taylor series expansion will be exact. Hence, we have,

$$E^h(p^0, S^{h1}) - E^h(p^0, S^{h0}) = \nabla_{S^h}E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})$$
$$+ (1/2)(S^{h1} - S^{h0}) \cdot \nabla^2_{S^h S^h}E^h(p^0, S^{h0})(S^{h1} - S^{h0}) \quad \text{(A5)}$$

and

$$E^h(p^1, S^{h0}) - E^h(p^1, S^{h1}) = \nabla_{S^h}E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1})$$
$$+ (1/2)(S^{h0} - S^{h1}) \cdot \nabla^2_{S^h S^h}E^h(p^1, S^{h1})(S^{h0} - S^{h1}) \quad \text{(A6)}$$

Moreover, by (18) we have $\nabla^2_{S^h S^h}E^h(p, S^h) = (\delta \cdot p)G$. Therefore, (A5) and (A6) can be written, respectively, as,

$$E^h(p^0, S^{h1}) - E^h(p^0, S^{h0}) = \nabla_{S^h}E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})$$
$$+ (\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \quad \text{(A7)}$$

and

$$E^h(p^1, S^{h0}) - E^h(p^1, S^{h1}) = \nabla_{S^h}E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1})$$
$$+ (\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) \quad \text{(A8)}$$

Re-arranging (A8) we get,

$$E^h(p^1, S^{h1}) - E^h(p^1, S^{h0}) = \nabla_{S^h}E^h(p^1, S^{h1}) \cdot (S^{h1} - S^{h0})$$
$$- (\delta \cdot p^1)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \quad \text{(A9)}$$

Substituting (A7) and (A9) in (A3) and (A4), respectively, we get,

$$
-\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} = -u^{h0}\{E^h(p^0, S^{h1}) - E^h(p^0, S^{h0})\}
$$
$$
= -u^{h0}\{\nabla_{s^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) + (\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\}
$$
$$
= -u^{h0}\nabla_{s^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) - (\delta \cdot p^0)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\}
$$
$$
\text{(A10)}
$$

and

$$
-\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} = -u^{h0}\{E^h(p^1, S^{h1}) - E^h(p^1, S^{h0})\}
$$
$$
= -u^{h0}\{\nabla_{s^h} E^h(p^1, S^{h1}) \cdot (S^{h1} - S^{h0}) - (\delta \cdot p^1)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\}
$$
$$
= -u^{h0}\nabla_{s^h} E^h(p^1, S^{h1}) \cdot (S^{h1} - S^{h0}) + (\delta \cdot p^1)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\}
$$
$$
\text{(A11)}
$$

Using (A2), (A10) and (A11) can be written as,

$$
-\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} = -\nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}) \cdot (S^{h1} - S^{h0})
$$
$$
- (\delta \cdot p^0)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A12)}
$$

and

$$
-\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} = -\nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}) \cdot (S^{h1} - S^{h0})
$$
$$
+ (\delta \cdot p^1)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A13)}
$$

Using definition (8), (A12) and (A13) can be written as,

$$
-\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} = W^{h0} \cdot (S^{h1} - S^{h0})
$$
$$
- (\delta \cdot p^0)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A14)}
$$

and

$$
-\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} = W^{h1} \cdot (S^{h1} - S^{h0})
$$
$$
+ (\delta \cdot p^1)(1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A15)}
$$

21

Dividing both sides of (A14) by $\delta \cdot p^0 \neq 0$ and both sides of (A15) by $\delta \cdot p^1 \neq 0$ we get,[13]

$$-\{(\delta \cdot p^0)^{-1}e^h(u^{h0}, p^0, S^{h1}) - (\delta \cdot p^0)^{-1}e^h(u^{h0}, p^0, S^{h0})\} = (\delta \cdot p^0)^{-1}W^{h0} \cdot (S^{h1} - S^{h0})$$
$$- (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A16)}$$

and

$$-\{(\delta \cdot p^1)^{-1}e^h(u^{h0}, p^1, S^{h1}) - (\delta \cdot p^1)^{-1}e^h(u^{h0}, p^1, S^{h0})\} = (\delta \cdot p^1)^{-1}W^{h1} \cdot (S^{h1} - S^{h0})$$
$$+ (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A17)}$$

Since $e^h(u^h, p, S^h)$ is linearly homogeneous in p (see definition (1)), (A16) and (A17) can be written as,

$$-\{e^h(u^{h0}, (\delta \cdot p^0)^{-1}p^0, S^{h1}) - e^h(u^{h0}, (\delta \cdot p^0)^{-1}p^0, S^{h0})\} = (\delta \cdot p^0)^{-1}W^{h0} \cdot (S^{h1} - S^{h0})$$
$$- (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A18)}$$

and

$$-\{e^h(u^{h0}, (\delta \cdot p^1)^{-1}p^1, S^{h1}) - e^h(u^{h0}, (\delta \cdot p^1)^{-1}p^1, S^{h0})\} = (\delta \cdot p^1)^{-1}W^{h1} \cdot (S^{h1} - S^{h0})$$
$$+ (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A19)}$$

Using definition (20), (A18) and (A19) are equivalent to,

$$-\{e^h(u^{h0}, \tilde{p}^0, S^{h1}) - e^h(u^{h0}, \tilde{p}^0, S^{h0})\} = \tilde{W}^{h0} \cdot (S^{h1} - S^{h0})$$
$$- (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A20)}$$

and

$$-\{e^h(u^{h0}, \tilde{p}^1, S^{h1}) - e^h(u^{h0}, \tilde{p}^1, S^{h0})\} = \tilde{W}^{h1} \cdot (S^{h1} - S^{h0})$$
$$+ (1/2)(u^{h0})(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})\} \quad \text{(A21)}$$

Taking the average of the two equations (A20) and (A21) and using definitions (21) and (22) yields the desired result,

---

[13]Note that $\delta > 0_N$ and $p^0, p^1 \gg 0_N$.

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^0) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^1) \equiv$$
$$- (1/2)\{e^h(u^{h0}, \tilde{p}^0, S^{h1}) - e^h(u^{h0}, \tilde{p}^0, S^{h0})\} - (1/2)\{e^h(u^{h0}, \tilde{p}^1, S^{h1}) - e^h(u^{h0}, \tilde{p}^1, S^{h0})\}$$
$$= (1/2)[\tilde{W}^{h0} + \tilde{W}^{h1}] \cdot [S^{h1} - S^{h0}] \quad (A22)$$

which is (23). □

Proof of Proposition 2

Ignoring any restrictions on E^h and E^{h*}, to solve the system of equations (24)-(26), we would require E^h to have at least $1 + (N + I) + (N + I)^2$ independent parameters. However, if E^h and E^{h*} are both twice continuously differentiable at (p^*, S^{h*}), Young's Theorem on the symmetry of second order partial derivatives reduces the number of independent second order derivatives from $(N + I)^2$ to $N(N + 1)/2 + NI + I(I + 1)/2$. Also, the linear homogeneity in p of E^h and E^{h*} imply the following additional $1 + N + I$ restrictions on the derivatives of E^h and E^{h*}:

$$E^h(p^*, S^{h*}) = p^* \cdot \nabla_p E^h(p^*, S^{h*}) \tag{A23}$$

$$[\nabla^2_{pp} E^h(p^*, S^{h*})]p^* = 0_N \tag{A24}$$

$$[\nabla^2_{S^h p} E^h(p^*, S^{h*})]p^* = \nabla_{S^h} E^h(p^*, S^{h*}) \tag{A25}$$

In light of these restrictions, we see that E^h will have to have at least $N(N + 1)/2 + NI + I(I + 1)/2$ independent parameters.

Now let $\delta > 0_N$ be any given vector which is nonnegative and nonzero, and let us consider the functional form defined by (18) where the δ which appears in (18) is the same as the given δ. Note first that E^h defined by (18) is twice continuously differentiable and linearly homogeneous in p. Moreover, since the δ_n parameters are already determined, E^h has N independent γ_n parameters, $N(N - 1)/2$ independent d_{mn} parameters, NI independent f_{ni} parameters and $I(I + 1)/2$ independent g_{ij} parameters. This is the minimal number of parameters required to satisfy equations (24)-(26).

Now we are remained to find the values of the independent parameters of the functional form defined by (18) which will satisfy equations (24)-(26). To do that we first solve the following system of equations for g_{ij}, $1 \le i \le j \le I$:

$$\partial^2 E^h(p^*, S^{h*})/\partial S_i^h \partial S_j^h = g_{ij}(\sum_{n=1}^{N} \delta_n p_n^*) = \partial^2 E^{h*}(p^*, S^{h*})/\partial S_i^h \partial S_j^h \tag{A26}$$

23

Note that the resulting G matrix with elements g_{ij} as determined above is a symmetric matrix due to the symmetry of the second order partial derivatives of E^{h^*}. With the elements g_{ij} determined, we can then solve (A27) for f_{ni}, $n = 1, ..., N$, $i = 1, ..., I$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_n \partial S_i^h = f_{ni} + \delta_n \sum_{j=1}^{I} g_{ij} S_j^{h^*} = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_n \partial S_i^h \qquad \text{(A27)}$$

Next we solve the following system of equations for d_{mn}, where $1 \leq m < n \leq N - 1$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_m \partial p_n = (p_N^*)^{-1} d_{mn} = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_m \partial p_n \qquad \text{(A28)}$$

Note that the symmetry of the second order partial derivatives of E^{h^*} will ensure the symmetry of the matrix D. Now using the results of (A28) we can then solve the following $N - 1$ equations (A29) for d_{nn}, $n = 1, ..., N - 1$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_N \partial p_n = -(p_N^*)^{-2} \sum_{m=1}^{N-1} d_{mn} p_m^* = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_N \partial p_n \qquad \text{(A29)}$$

Finally, we can use (A30) below to solve for γ_n, $n = 1, ..., N - 1$:

$$\partial E^h(p^*, S^{h^*})/\partial p_n = \gamma_n + \sum_{m=1}^{N-1} d_{mn} p_m^* (p_N^*)^{-1} + \sum_{i=1}^{I} f_{ni} S_i^{h^*} +$$
$$(1/2)\delta_n \sum_{i=1}^{I} \sum_{j=1}^{I} g_{ij} S_i^{h^*} S_j^{h^*} = \partial E^{h^*}(p^*, S^{h^*})/\partial p_n \quad \text{(A30)}$$

and (A31) below to solve for γ_N:

$$\partial E^h(p^*, S^{h^*})/\partial p_N = \gamma_N - (1/2) \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} d_{mn} p_m^* p_n^* (p_N^*)^{-2} + \sum_{i=1}^{I} f_{Ni} S_i^{h^*} +$$
$$(1/2)\delta_N \sum_{i=1}^{I} \sum_{j=1}^{I} g_{ij} S_i^{h^*} S_j^{h^*} = \partial E^{h^*}(p^*, S^{h^*})/\partial p_N \quad \text{(A31)}$$

Observe that all the parameters of E^h defined by (18) have now been determined. Moreover, the system of equations (24)-(26) is now fully satisfied. In particular, (A30) and (A31) together with (A23) ensure that equation (24) is satisfied. Also, (A27) and (A25) ensure that the first order partial derivatives of E^h and E^{h^*} with respect to S^h coincide. This together with equations (A30) and (A31) ensure that the system of equations (25) is satisfied. Lastly, note that (A24), (A26), (A27), (A28) and (A29)

ensure that the second order partial derivatives of E^h and E^{h^*} coincide, thus satisfying the system of equations (26).[14] $\square$

Proof of Proposition 3

Under the setup of the second order approximation we have,[15]

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \tag{A32}$$

and thus,

$$\nabla_p e^h(u^h, p, S^h) = u^h \nabla_p E^h(p, S^h) \tag{A33}$$

Now using (A32) we have,

$$-\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} = -\{u^{h0} E^h(\frac{p^0}{p_N^0}, S^{h1}) - u^{h0} E^h(\frac{p^0}{p_N^0}, S^{h0})\}$$
$$= -u^{h0}\{E^h(\frac{p^0}{p_N^0}, S^{h1}) - E^h(\frac{p^0}{p_N^0}, S^{h0})\} \tag{A34}$$

and

$$-\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\} = -\{u^{h0} E^h(\frac{p^1}{p_N^1}, S^{h1}) - u^{h0} E^h(\frac{p^1}{p_N^1}, S^{h0})\}$$
$$= -u^{h0}\{E^h(\frac{p^1}{p_N^1}, S^{h1}) - E^h(\frac{p^1}{p_N^1}, S^{h0})\} \tag{A35}$$

If p_N is fixed, then $E^h(p, S^h)$ defined by (18) is quadratic in $p_1, ..., p_{N-1}$ and thus its second order Taylor series expansion in p will be exact for each fixed S^h. Hence we have,

$$E^h(\frac{p^1}{p_N^1}, S^{h0}) = E^h(\frac{p^0}{p_N^0}, S^{h0}) + \nabla_p E^h(\frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})]$$
$$+ (1/2)[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] \cdot \nabla_{pp}^2 E^h(\frac{p^0}{p_N^0}, S^{h0})[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] \tag{A36}$$

and

---
[14](A24), (A28) and (A29) ensure that $\nabla_{pp}^2 E^h(p^*, S^{h^*}) = \nabla_{pp}^2 E^{h^*}(p^*, S^{h^*})$.
[15]See equation (17).

$$E^h(\frac{p^0}{p_N^0}, S^{h1}) = E^h(\frac{p^1}{p_N^1}, S^{h1}) + \nabla_p E^h(\frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})]$$

$$+ (1/2)[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] \cdot \nabla_{pp}^2 E^h(\frac{p^1}{p_N^1}, S^{h1})[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] \quad (A37)$$

Using definition (18), the second order partial derivatives of $E^h(p, S^h)$ with respect to prices are given by,

$$\partial^2 E^h(p, S^h)/\partial p_i \partial p_j = (\frac{1}{p_N}) d_{ij} \quad 1 \le i, j \le N - 1 \quad (A38)$$

$$\partial^2 E^h(p, S^h)/\partial p_N \partial p_j = -(\frac{1}{p_N})^2 \sum_{i=1}^{N-1} d_{ij} p_i \quad j = 1, ..., N - 1 \quad (A39)$$

$$\partial^2 E^h(p, S^h)/\partial p_N \partial p_N = (\frac{1}{p_N})^3 \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i p_j \quad (A40)$$

Using (A38)-(A40), it is straight forward to evaluate the matrix of second order partial derivatives of $E^h(p, S^h)$ with respect to prices, $\nabla_{pp}^2 E^h(p, S^h)$, at the points $(p^0/p_N^0, S^{h0})$ and $(p^1/p_N^1, S^{h1})$, and verify that,

$$[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] \cdot \nabla_{pp}^2 E^h(\frac{p^0}{p_N^0}, S^{h0})[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] =$$

$$\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij}[(\frac{p_i^1}{p_N^1}) - (\frac{p_i^0}{p_N^0})][(\frac{p_j^1}{p_N^1}) - (\frac{p_j^0}{p_N^0})] \equiv \phi \quad (A41)$$

and

$$[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] \cdot \nabla_{pp}^2 E^h(\frac{p^1}{p_N^1}, S^{h1})[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] =$$

$$\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij}[(\frac{p_i^1}{p_N^1}) - (\frac{p_i^0}{p_N^0})][(\frac{p_j^1}{p_N^1}) - (\frac{p_j^0}{p_N^0})] \equiv \phi \quad (A42)$$

Substituting (A41) and (A42) into (A36) and (A37), respectively, we get,

$$E^h(\frac{p^1}{p_N^1}, S^{h0}) = E^h(\frac{p^0}{p_N^0}, S^{h0}) + \nabla_p E^h(\frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)\phi \quad (A43)$$

and

$$E^h(\frac{p^0}{p_N^0}, S^{h1}) = E^h(\frac{p^1}{p_N^1}, S^{h1}) + \nabla_p E^h(\frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] + (1/2)\phi \qquad \text{(A44)}$$

Now substituting (A43) into (A35) and (A44) into (A34) we have,

$$-\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} = -u^{h0}\{E^h(\frac{p^0}{p_N^0}, S^{h1}) - E^h(\frac{p^0}{p_N^0}, S^{h0})\}$$

$$= -u^{h0}\{E^h(\frac{p^1}{p_N^1}, S^{h1}) + \nabla_p E^h(\frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})]$$

$$+ (1/2)\phi - E^h(\frac{p^0}{p_N^0}, S^{h0})\} \quad \text{(A45)}$$

and

$$-\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\} = -u^{h0}\{E^h(\frac{p^1}{p_N^1}, S^{h1}) - E^h(\frac{p^1}{p_N^1}, S^{h0})\}$$

$$= -u^{h0}\{E^h(\frac{p^1}{p_N^1}, S^{h1}) - E^h(\frac{p^0}{p_N^0}, S^{h0})$$

$$- \nabla_p E^h(\frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] - (1/2)\phi\} \quad \text{(A46)}$$

Using (A32) and (A33), we can write (A45) and (A46) as,

$$-\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})u^{h1}E^h(\frac{p^1}{p_N^1}, S^{h1})$$

$$- (\frac{u^{h0}}{u^{h1}})u^{h1}\nabla_p E^h(\frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + u^{h0}E^h(\frac{p^0}{p_N^0}, S^{h0})$$

$$= -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1}) - (\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})]$$

$$- (1/2)u^{h0}\phi + e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0}) \quad \text{(A47)}$$

and

$$- \{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})u^{h1}E^h(\frac{p^1}{p_N^1}, S^{h1})$$

$$+ u^{h0}E^h(\frac{p^0}{p_N^0}, S^{h0}) + u^{h0}\nabla_p E^h(\frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi$$

$$= -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1}) + e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})$$

$$+ \nabla_p e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi \quad \text{(A48)}$$

Since $e^h(u^h, p, S^h)$ is linearly homogeneous in p, its first order partial derivatives with respect to prices, $\partial e^h(u^h, p, S^h)/\partial p_i$ for $i = 1, ..., N$, are homogeneous functions of degree 0 in p, and therefore,

$$e^h(u^h, \frac{p}{p_N}, S^h) = (\frac{1}{p_N})e^h(u^h, p, S^h) \quad \text{(A49)}$$

and

$$\nabla_p e^h(u^h, \frac{p}{p_N}, S^h) = \nabla_p e^h(u^h, p, S^h) \quad \text{(A50)}$$

Using (A49) and (A50), (A47) and (A48) can be written as,

$$- \{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1})$$

$$- (\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})e^h(u^{h1}, p^1, S^{h1}) - (\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1}, p^1, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})]$$

$$- (1/2)u^{h0}\phi + (\frac{1}{p_N^0})e^h(u^{h0}, p^0, S^{h0}) \quad \text{(A51)}$$

and

$$- \{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1}, \frac{p^1}{p_N^1}, S^{h1})$$

$$+ e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0}) + \nabla_p e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})e^h(u^{h1}, p^1, S^{h1}) + (\frac{1}{p_N^0})e^h(u^{h0}, p^0, S^{h0})$$

$$+ \nabla_p e^h(u^{h0}, p^0, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi \quad \text{(A52)}$$

As relations (27) and (28) are assumed to hold, (A51) and (A52) are equivalent to,

$$-\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})e^h(u^{h1}, p^1, S^{h1})$$

$$-(\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1}, p^1, S^{h1}) \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + (\frac{1}{p_N^0})e^h(u^{h0}, p^0, S^{h0})$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})(p^1 \cdot c^{h1}) - (\frac{u^{h0}}{u^{h1}})c^{h1} \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + (\frac{1}{p_N^0})(p^0 \cdot c^{h0})$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p_N^1} \cdot c^{h1}) - (\frac{u^{h0}}{u^{h1}})c^{h1} \cdot [(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + (\frac{p^0}{p_N^0} \cdot c^{h0})$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{p^0}{p_N^0} \cdot c^{h1}) - (1/2)u^{h0}\phi + (\frac{p^0}{p_N^0} \cdot c^{h0})$$

$$= -(\frac{p^0}{p_N^0} \cdot c^{h1}) - (\frac{u^{h0}}{u^{h1}} - 1)(\frac{p^0}{p_N^0} \cdot c^{h1}) - (1/2)u^{h0}\phi + (\frac{p^0}{p_N^0} \cdot c^{h0})$$

$$= \frac{p^0}{p_N^0} \cdot [c^{h0} - c^{h1}] - (\frac{u^{h0}}{u^{h1}} - 1)(\frac{p^0}{p_N^0} \cdot c^{h1}) - (1/2)u^{h0}\phi \quad \text{(A53)}$$

and

$$-\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})e^h(u^{h1}, p^1, S^{h1})$$

$$+ (\frac{1}{p_N^0})e^h(u^{h0}, p^0, S^{h0}) + \nabla_p e^h(u^{h0}, p^0, S^{h0}) \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})(p^1 \cdot c^{h1}) + (\frac{1}{p_N^0})(p^0 \cdot c^{h0}) + c^{h0} \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p_N^1} \cdot c^{h1}) + (\frac{p^0}{p_N^0} \cdot c^{h0}) + c^{h0} \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi$$

$$= -(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p_N^1} \cdot c^{h1}) + (c^{h0} \cdot \frac{p^1}{p_N^1}) + (1/2)u^{h0}\phi$$

$$= -(\frac{p^1}{p_N^1} \cdot c^{h1}) - (\frac{u^{h0}}{u^{h1}} - 1)(\frac{p^1}{p_N^1} \cdot c^{h1}) + (c^{h0} \cdot \frac{p^1}{p_N^1}) + (1/2)u^{h0}\phi$$

$$= \frac{p^1}{p_N^1} \cdot [c^{h0} - c^{h1}] - (\frac{u^{h0}}{u^{h1}} - 1)(\frac{p^1}{p_N^1} \cdot c^{h1}) + (1/2)u^{h0}\phi \quad \text{(A54)}$$

Taking the average of the two equations (A53) and (A54) and using definitions (35) and (36) gives the desired result,

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \equiv$$

$$-(1/2)\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\} - (1/2)\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\}$$

$$= (1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad (A55)$$

which is (37). $\square$

Proof of Corollary 1

From Proposition 3 we have,

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0} - (1/2)(\frac{u^{h0}}{u^{h1}})[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad (A56)$$

Now since $(1/2)(\frac{u^{h0}}{u^{h1}})[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq 0$ we get,[16]

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \leq (1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0}$$
$$(A57)$$

which is (38). $\square$

Proof of Proposition 4

Under the setup of the second order approximation we have,[17]

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \quad (A58)$$

and thus,

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[16]Note that $u^{h0}, u^{h1} > 0$; $c^{h1} \geq 0_N$; and $p \gg 0_N$.

[17]See equation (17).

$$\nabla_p e^h(u^h, p, S^h) = u^h \nabla_p E^h(p, S^h) \tag{A59}$$

Combining (A59) together with (28), which is assumed to hold, we get

$$c^{h0} = \nabla_p e^h(u^{h0}, p^0, S^{h0}) = u^{h0} \nabla_p E^h(p^0, S^{h0}) \tag{A60}$$

and

$$c^{h1} = \nabla_p e^h(u^{h1}, p^1, S^{h1}) = u^{h1} \nabla_p E^h(p^1, S^{h1}) \tag{A61}$$

To reduce clutter in notation let us re-write the class of functional forms for the household's unit expenditure function defined by (18) as follows:

$$E^h(p, S^h) \equiv \sum_{i=1}^{N} \gamma_i p_i + (1/2) \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i p_j (p_N)^{-1} + \sum_{i=1}^{N} p_i \Phi_i(S^h) \tag{A62}$$

where $\Phi_i(S^h) \equiv \sum_{j=1}^{I} f_{ij} S_j^h + \delta_i (1/2)(\sum_{j=1}^{I} \sum_{m=1}^{I} g_{jm} S_j^h S_m^h)$ and thus $\sum_{i=1}^{N} p_i \Phi_i(S^h) = \sum_{i=1}^{N} \sum_{j=1}^{I} f_{ij} p_i S_j^h + (1/2)(\sum_{i=1}^{N} \delta_i p_i)(\sum_{j=1}^{I} \sum_{m=1}^{I} g_{jm} S_j^h S_m^h)$.

Differentiating (A62) with respect to prices and making use of (A60) and (A61) yields the following equations:

$$c^{h0} = \begin{bmatrix} u^{h0}[\gamma_1 + (p_N^0)^{-1} \sum_{j=1}^{N-1} d_{1j} p_j^0 + \Phi_1(S^{h0})] \\ \vdots \\ u^{h0}[\gamma_{N-1} + (p_N^0)^{-1} \sum_{j=1}^{N-1} d_{(N-1)j} p_j^0 + \Phi_{N-1}(S^{h0})] \\ u^{h0}[\gamma_N - (1/2)(p_N^0)^{-2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^0 p_j^0 + \Phi_N(S^{h0})] \end{bmatrix} \tag{A63}$$

and

$$c^{h1} = \begin{bmatrix} u^{h1}[\gamma_1 + (p_N^1)^{-1} \sum_{j=1}^{N-1} d_{1j} p_j^1 + \Phi_1(S^{h1})] \\ \vdots \\ u^{h1}[\gamma_{N-1} + (p_N^1)^{-1} \sum_{j=1}^{N-1} d_{(N-1)j} p_j^1 + \Phi_{N-1}(S^{h1})] \\ u^{h1}[\gamma_N - (1/2)(p_N^1)^{-2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^1 p_j^1 + \Phi_N(S^{h1})] \end{bmatrix} \tag{A64}$$

Using (A63) and (A64), it is straight forward to verify that,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^0 + p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i + p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i$$

$$+ p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h0}) + p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h0})] \quad \text{(A65)}$$

and

$$p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1}) = u^{h1}[\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^0 + p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i + p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i$$

$$+ p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h1}) + p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h1})] \quad \text{(A66)}$$

where (A66) is derived by also using the symmetry of the D matrix.

Now using (A62), it is also straight forward to show that,

$$p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h0}) = p_N^1 E^h(p^0, S^{h0}) - p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i - (1/2)(\frac{p_N^1}{p_N^0})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^0 p_j^0 \quad \text{(A67)}$$

$$p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h0}) = p_N^0 E^h(p^1, S^{h0}) - p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i - (1/2)(\frac{p_N^0}{p_N^1})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^1 \quad \text{(A68)}$$

$$p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h1}) = p_N^0 E^h(p^1, S^{h1}) - p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i - (1/2)(\frac{p_N^0}{p_N^1})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^1 \quad \text{(A69)}$$

$$p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h1}) = p_N^1 E^h(p^0, S^{h1}) - p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i - (1/2)(\frac{p_N^1}{p_N^0})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^0 p_j^0 \quad \text{(A70)}$$

Substituting (A67) and (A68) into (A65) and substituting (A69) and (A70) into (A66) we get,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h0}) + p_N^0 E^h(p^1, S^{h0})] \quad \text{(A71)}$$

and

$$p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1}) = u^{h1}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})] \tag{A72}$$

where

$$\lambda \equiv \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^1 p_j^0 - (1/2)(\frac{p_N^1}{p_N^0}) \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^0 p_j^0 - (1/2)(\frac{p_N^0}{p_N^1}) \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^1 p_j^1 \tag{A73}$$

Now since $E^h(p, S^h)$ defined by (18) is quadratic in S^h for each fixed p, its second order Taylor series expansion will be exact. Hence, we have,

$$E^h(p^0, S^{h1}) - E^h(p^0, S^{h0}) =$$
$$\nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) + (1/2)(S^{h1} - S^{h0}) \cdot \nabla^2_{S^h S^h} E^h(p^0, S^{h0})(S^{h1} - S^{h0}) =$$
$$\nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) + (\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \tag{A74}$$

and

$$E^h(p^1, S^{h0}) - E^h(p^1, S^{h1}) =$$
$$\nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) + (1/2)(S^{h0} - S^{h1}) \cdot \nabla^2_{S^h S^h} E^h(p^1, S^{h1})(S^{h0} - S^{h1}) =$$
$$\nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) + (\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) \tag{A75}$$

Multiplying (A74) by p_N^1 and re-arranging yields,

$$p_N^1 E^h(p^0, S^{h0}) = p_N^1 E^h(p^0, S^{h1}) - p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})$$
$$- p_N^1(\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \tag{A76}$$

Similarly, multiplying (A75) by p_N^0 and re-arranging we get,

$$p_N^0 E^h(p^1, S^{h0}) = p_N^0 E^h(p^1, S^{h1}) + p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1})$$
$$+ p_N^0(\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) \tag{A77}$$

Substituting (A76) and (A77) into (A71) we obtain,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]$$
$$+ u^{h0}[p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})]$$
$$+ u^{h0}[p_N^0(\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) - p_N^1(\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0})]$$
$$\tag{A78}$$

Note that,

$$u^{h0}[p_N^0(\delta\cdot p^1)(1/2)(S^{h0}-S^{h1})\cdot G(S^{h0}-S^{h1})-p_N^1(\delta\cdot p^0)(1/2)(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})] =$$
$$u^{h0}(1/2)(S^{h0}-S^{h1})\cdot G(S^{h0}-S^{h1})[p_N^0(\delta\cdot p^1)-p_N^1(\delta\cdot p^0)] =$$
$$u^{h0}(1/2)(S^{h0}-S^{h1})\cdot G(S^{h0}-S^{h1})(p_N^1 p_N^0)(\delta\cdot[(\frac{p^1}{p_N^1})-(\frac{p^0}{p_N^0})]) = 0 \quad \text{(A79)}$$

where the last equality follows from the condition that $\delta\cdot[(\frac{p^1}{p_N^1})-(\frac{p^0}{p_N^0})]=0$.

Thus we get that,

$$p_N^1(p^0\cdot c^{h0})+p_N^0(p^1\cdot c^{h0}) = u^{h0}[\lambda+p_N^1 E^h(p^0,S^{h1})+p_N^0 E^h(p^1,S^{h1})]$$
$$+u^{h0}[p_N^0\nabla_{S^h}E^h(p^1,S^{h1})\cdot(S^{h0}-S^{h1})-p_N^1\nabla_{S^h}E^h(p^0,S^{h0})\cdot(S^{h1}-S^{h0})]$$
$$\text{(A80)}$$

Furthermore,

$$u^{h0}[p_N^0\nabla_{S^h}E^h(p^1,S^{h1})\cdot(S^{h0}-S^{h1})-p_N^1\nabla_{S^h}E^h(p^0,S^{h0})\cdot(S^{h1}-S^{h0})] =$$
$$p_N^0 u^{h0}\nabla_{S^h}E^h(p^1,S^{h1})\cdot(S^{h0}-S^{h1})-p_N^1 u^{h0}\nabla_{S^h}E^h(p^0,S^{h0})\cdot(S^{h1}-S^{h0}) =$$
$$p_N^0\nabla_{S^h}e^h(u^{h0},p^1,S^{h1})\cdot(S^{h0}-S^{h1})-p_N^1\nabla_{S^h}e^h(u^{h0},p^0,S^{h0})\cdot(S^{h1}-S^{h0}) =$$
$$-p_N^0\nabla_{S^h}e^h(u^{h0},p^1,S^{h1})\cdot(S^{h1}-S^{h0})-p_N^1\nabla_{S^h}e^h(u^{h0},p^0,S^{h0})\cdot(S^{h1}-S^{h0}) =$$
$$p_N^0(W^{h1}\cdot(S^{h1}-S^{h0}))+p_N^1(W^{h0}\cdot(S^{h1}-S^{h0})) \quad \text{(A81)}$$

where the second equality follows from the fact that under the assumptions made in the setup for the second order approximation $e^h(u^h,p,S^h)=u^h E^h(p,S^h)$ and hence, $\nabla_{S^h}e^h(u^h,p,S^h)=u^h\nabla_{S^h}E^h(p,S^h)$. The last equality is achieved using definition (8).

Substituting (A81) into (A80) and re-arranging yields,

$$p_N^1(p^0\cdot c^{h0})+p_N^0(p^1\cdot c^{h0})-p_N^1(W^{h0}\cdot(S^{h1}-S^{h0}))-p_N^0(W^{h1}\cdot(S^{h1}-S^{h0})) =$$
$$u^{h0}[\lambda+p_N^1 E^h(p^0,S^{h1})+p_N^0 E^h(p^1,S^{h1})]$$
$$\text{(A82)}$$

Combining (A82) together with (A72) we obtain the desired result,[18]

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[18]Note that $p_N^1(p^0\cdot c^{h1})+p_N^0(p^1\cdot c^{h1})\neq 0$ as $c^{h1}>0_N$ and $p\gg 0_N$.

$$\frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} =$$

$$\frac{u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]}{u^{h1}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]} = \frac{u^{h0}}{u^{h1}} \quad \text{(A83)}$$

which is (39). $\square$

Proof of Proposition 5

Under the conditions of Proposition 4 we have,

$$\frac{u^{h0}}{u^{h1}} = \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})}$$

(A84)

Now since $W^{h0}, W^{h1} \geq 0_I$ and $(S^{h1} - S^{h0}) \geq 0_I$ we get,

$$\frac{u^{h0}}{u^{h1}} \leq \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0})}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})}$$

(A85)

and hence,

$$(1/2)(\Upsilon - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad \text{(A86)}$$

where Υ is defined as,

$$\Upsilon \equiv \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0})}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})}$$

(A87)

This in turn implies that,

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\Upsilon - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} = 0 \quad \text{(A88)}$$

where the last equality is derived after substituting for Υ (equation (A87)) and pre-forming some algebraic manipulations. $\square$

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