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Forecast Combination, Non-linear Dynamics, and the Macroeconomy

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Abstract

This paper introduces the concept of a Forecast Combination Equilibrium to model boundedly rational agents who combine a menu of different forecasts using insights from the forecasting literature to mimic the behavior of actual forecasters. The equilibrium concept is consistent with rational expectations under certain conditions, while also permitting multiple, distinct, self-fulfilling equilibria, many of which are stable under least squares learning. The equilibrium concept is applied to a simple Lucas-type monetary model where agents engage in constant gain learning. The combination of multiple equilibria and learning is sufficient to replicate some key features of inflation data, such as time-varying volatility and periodic bouts of high inflation or deflation in a model that experiences only i.i.d. random shocks.

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1 Introduction

A key feature of modern macroeconomic models is the forward-looking agent who makes current period decisions based on expectations of the future. The standard modeling assumption for agents' expectations is of course rational expectations (RE), which assumes that agents understand the true structure of the model and use this knowledge to form model consistent expectations. However, the actual experience of econometricians, policymakers, firms, and consumers of forming expectations involves significant model uncertainty, where there exists many suitable models to forecast any variable of interest.

The model uncertainty problem is reflected at the professional forecasting and policy levels by the common use of combined forecasts for decision making that aggregate many different forecasts together. Prominent examples of these forecasts include the Federal Reserve's Green Book consensus forecasts, the Survey of Professional Forecasters, or the Blue Chip Economic Indicators consensus forecasts.[1] The model uncertainty problem is observed at the individual level in laboratory experiments where people are asked to form forecasts in a controlled environment. For example, Anufriev and Hommes (2012a, b) show that experimental data on forecasting asset prices in a laboratory can be explained by participants coordinating on a set of heuristic forecasting rules and then switching among those rules over time. Surveys of the experimental Learning-to-Forecast literature are found in Hommes (2011) and Hommes (2013). The model uncertainty problem is also widely studied in the forecasting literature.

The forecasting literature typically proposes two solutions to overcome model uncertainty. Either a fitness criterion can be adopted to distinguish and select among the forecasts or a strategy can be employed to combine the forecasts. The forecast combination solution is often found to be the most effective. The solution allows a forecaster to capture important information from many different forecasts while lowering the risk of choosing a poor forecast. The seminal paper demonstrating the efficacy of forecast combination is Bates and Granger (1969), who showed that weighted averages of competing forecasting methods consistently outperforms any of the individual forecasts considered. Surveys of the literature are found in Clemen (1989), Granger (1989), Timmermann (2006), and Wallis (2011).

Despite the dominance of forecast combination in the forecasting literature, theoretical models that have studied agents with model uncertainty overwhelmingly model agents that select, rather than combine forecasts. A brief list of examples are Brock and Hommes (1997 and 1998), Chiarella and He (2003), Branch and Evans (2006, 2007, and 2011), Branch and McGough (2008 and 2010), Brock, Hommes, and Wagener (2009), Gibbs (2012), and Anfriev et al (2013). The agents in these models select forecast rules by a process called Dynamic Predictor Selection (DPS), where agents use a fitness measure to distinguish and select among the rules.

This paper proposes a simple framework to model boundedly rational agents who face model uncertainty and employ forecast combination strategies, instead of model selection, to mimic the forecasting behavior of actual professional forecasters. One explanation for the preference in the literature for model selection, rather than com-

[1] See Robertson (2000) for a detailed description of common central bank forecasting practices.

bination, is that from an aggregative perspective the two strategies appear to be the same. The aggregate expectation in both cases is a linear combination of the menu of forecast rules. Therefore, it would be natural to conclude that there is nothing to gain by explicitly exploring forecast combination. However, I demonstrate that this perception is not true. Forecast combination with endogenously determined weights that depend on past information can introduce new stationary equilibria and qualitatively different dynamics than what is observed under DPS.

There is currently only one study that has considered agents using some form of forecast combination. Evans, Honkapohja, Sargent, and Williams (2013) explore the case of agents using Bayesian model averaging in a forward-looking model to determine the long run model selection outcome when agents contemplate slight deviations to the RE forecast.[2] They find that with a certain probability a simulated economy can converge to a non-rational equilibrium in the long-run. They, however, do not address the dynamics that may arise in response to model uncertainty or compare outcomes to the dynamics predictor selection literature.

1.1 Equilibrium Concept

I introduce the concept of a *Forecast Combination Equilibrium* (FCE), which posits that a continuum of identical agents possesses a menu of different forecast rules. The agents create combined forecasts using a weighted sum of all the forecasts generated by the rules in each period. The concept is an extension of the Restricted Perception Equilibrium concept used to study dynamic optimizing agents that possess limited information as in Sargent (2001), Evans and Honkapohja (2001), Branch (2004), and McGough (2006).

The menu of forecast rules the agents consider consists of different underparameterizations of the true data generating process. The use of underparameterized forecast rules mimics the standard practices in the forecasting literature, where parsimony is key to creating efficient forecasts. The use of parsimonious models avoids data overfitting that can lead to a substantial loss in out-of-sample forecasting accuracy.[3] The use of a menu of parsimonious forecasts is also the standard approach used in the Dynamic Predictor Selection literature.

The Forecast Combination Equilibrium concept is developed in a general reduced form macroeconomic model. The FCE concept is shown to have a unique equilibrium under broad assumptions if agents combine the forecasts using exogenously determined weights that do not evolve or respond to past data. The equilibrium concept, however, permits multiple equilibria when agents use past information to optimally choose combination weights to minimize expected squared forecast error. The set of possible equilibria may also include an FCE that is observationally equivalent to the rational expectations equilibrium (REE) under certain conditions. I call this FCE the Fundamental FCE.

[2] The paper is referred to as Evans et al (2013) for the remainder of the paper.

[3] Empirical examples of the efficacy of using parsimonious forecasts from the forecasting literature are Atkeson and Ohanian (2001), Stock and Watson (2004), and Ang, Bekaert, and Wei (2007).

The equilibrium concept is applied to a Lucas-type monetary model to study the properties of the FCEs under least squares learning. The application shows that there exist multiple E-stable equilibria, where the number and stability of the equilibria are dependent on the policy parameters of the models. In addition, when there exists a Fundamental FCE, the stability of this equilibrium directly relates to the existence of multiple equilibria. If the Fundamental FCE is stable, then it is the unique equilibrium. However, if the Fundamental FCE is unstable, then there exist multiple equilibria of which many are stable. Therefore, although the Fundamental FCE is an equilibrium of the economy under certain conditions, it is not the equilibrium outcome obtained in the presence of multiple equilibria.

I also use the Lucas-type monetary model example to compare the dynamics generated under FCE to the dynamics generated by DPS. DPS is shown by Branch and Evans (2007) to produce endogenous volatility, which is consistent with the volatility observed in actual inflation data. I show that endogenous weight forecast combination, combined with constant gain learning, can replicate their results, while adding endogenous breaks to trend inflation. The endogenous breaks to trend inflation resemble bouts of fast rising inflation or deflation, which are not present under DPS. These two sources of endogenous time-variation are sufficient to replicate some key properties of U.S. inflation dynamics in a simple model that is subject only to i.i.d. shocks.

The remainder of the paper proceeds as follows. Section 2 develops the equilibrium concept in a general reduced form macro model. Section 3 explores the existence of multiple equilibria in a Lucas-type monetary model and employs E-stability as a selection criterion for the equilibria. Section 4 compares the dynamics generated by forecast combination in the Lucas-type monetary model to DPS and to actual inflation. Section 5 concludes.

2 A General Framework

In this section I develop the equilibrium concept in a reduced form macroeconomic model that has a unique REE. I explore issues of existence and uniqueness and describe conditions under which the equilibrium concept can nest the REE of the model under study.

2.1 The Reduced Form Economy

The model I consider is a reduced form economy described by a self-referential stochastic process driven by a vector of exogenous shocks. The model takes the following form,

$$y_t = \mu + \alpha E_{t-1} y_t + \zeta' x_{t-1} + w_t, \tag{1}$$

where y_t is a scalar, x_{t-1} is a $n \times 1$ vector of exogenous and observable shocks that follows a stationary process with zero mean, and w_t is white noise. The model presented is the reduced form version of two well-known macroeconomic models depending on the value of α. For $\alpha < 0$ the model is the reduced form of the Cobweb model of Muth

(1961), while for $0 < \alpha < 1$ it is the reduced form of a Lucas-type aggregate supply model of Lucas (1973). The model has a unique REE, which can be written as

$$y_t = (1-\alpha)^{-1}\Omega' z_{t-1} + w_t, \tag{2}$$

where $\Omega = (\mu,\ \zeta')'$ and $z_{t-1} = (1,\ x'_{t-1})'$.

2.2 Misspecified Models

Instead of rational expectations, the agents in the economy are assumed to have uncertainty over the correct specification of the data generating process for y_t. The agents consider k different underparameterized laws of motion for the economy that each omit one or more of the exogenous observables in z_{t-1}. For the general case it is assumed that the agents' information set contains all of z_{t-1}.[4] Let the $k \geq 2$ possible underparameterized laws of motion or forecast rules be denoted as

$$\hat{y}_{i,t} = \phi_i' u_i z_{t-1} \tag{3}$$

for $i = 1, 2, ...k$, where u_i is an $m_i \times n+1$ selector matrix that picks out the included exogenous variables for each rule, and ϕ_i is an $m_i \times 1$ vector of parameter beliefs.[5]

I adopt the term "parameter beliefs" for the ϕ_i's to denote that they represent the agents' perception of how $u_i z_{t-1}$ relates to y_t and the terms "expectation" or "forecast" to denote the implied value of y_t given by $\hat{y}_{i,t}$. Agents treat each considered forecast rule as if it described the true data generating process and form parameter beliefs $\{\phi_i\}_{i=1}^k$ as optimal linear projections of y_t on $u_i z_{t-1}$. This implies that each ϕ_i satisfies the following orthogonality condition:

$$E u_i z_{t-1}(y_t - \phi_i' u_i z_{t-1}) = 0. \tag{4}$$

The consideration of a list of misspecified models is a familiar setup in the DPS literature. The standard way to proceed at this point is to assume that agents choose a fitness criteria, such as past mean squared forecast error, and select a single rule to make a forecast. I deviate from this structure and instead assume that the agents follow standard practices in the forecast combination literature to create a single prediction by employing a weighted sum of all the individual forecast rules:

$$E^*_{t-1} y_t = \sum_{i=1}^{k} \gamma_i \phi_i' u_i z_{t-1}, \tag{5}$$

[4]This assumption is not crucial for most of the analysis and the agents' information set could contain only a portion of z_{t-1}.

[5]The m_i's correspond to the number of parameters included in each forecast rule so that $m_i \leq n$ for all i. The $m_i \times n+1$ selector matrices are $n+1 \times n+1$ identity matrices with rows that do not correspond to included exogenous variables deleted. An example is given in the appendix. Similar notation is used in Branch and Evans (2006).

where $\gamma_i \in \mathbb{R}$ is the weight given to i^{th} forecast rule and E^*_{t-1} denotes a combined expectation. The model is then closed by specifying how agents choose weights.

2.3 Combination Weights

I consider two ways to choose weights. The first is to simply assume exogenous weights that do not change with the evolution of the economy. This case is a catchall that nests all possible forecast combination strategies, since in a stationary equilibrium any endogenously chosen weights are fixed. The case also nests one of the most common and effective ways to create a combined forecast, which is to take a simple average of the forecasts. The simple average acts as a hedge against model uncertainty by remaining agnostic about the best forecast rule and placing equal weight on each rule. The strategy is often a key part of any discussion of forecast combination because it is routinely found to outperform more theoretically justified combination strategies. This finding is often referred to as the forecast combination puzzle.[6]

The principle case I consider is an endogenous weight case where agents pick weights to minimize the expected squared forecast error of the combined forecast:

$$\min_{\{\Gamma\}} E[(y_t - \Gamma' Y_t)^2], \tag{6}$$

where $Y_t = (\hat{y}_{1,t}, \ \dots \ , \hat{y}_{k,t})'$ and $\Gamma = (\gamma_i, ..., \gamma_k)'$. I refer to this case as the *optimal weights* case. The optimal weights case is similar to the weights proposed in Bates and Granger (1969) and Granger and Ramanathan (1984) for use in the actual empirical practice of forecasting and represents a common objective function considered in the literature.

Optimal weights of course is not the only *optimal* combination strategy considered in the forecasting literature. An attractive alternative to optimal weights is Bayesian model averaging. However, Evans et al (2013) show that the Bayesian case is largely intractable and requires simulations for most of the analysis. The optimal weights considered here in contrast allows for analytic solutions and provides intuition that may apply to other weighting strategies. In fact, many of the key results for optimal weights rely on the specification of the forecast rules and the underlying economic model and not the combination strategy. Therefore, the intuition behind many of the results presented in this paper may apply to other backward-looking weighting schemes.

It is also important to reiterate that the optimal weights are not always optimal in practice. It is a common finding that optimal weights underperforms relative to equal weights due to imprecision in estimating the weights as shown by Smith and Wallis (2009) or by Yang (2004), which is essentially the forecast combination puzzle mentioned previously. However, the strategy has been widely used throughout the forecasting literature and in many circumstances performs well. Alternative combination strategies or objective functions represent an interesting topic for future research.

[6]See Hendry and Clements (2004) for a more detailed discussion of the forecast combination puzzle.

2.4 Exogenous Weights

A combined forecast with exogenous weights $(\gamma_i, \ \dots\ , \gamma_k)' = \Gamma \in \mathbb{R}^k$ takes the form of equation (5). The combined forecast functions as a perceived law of motion (PLM) for the economy. The PLM can be substituted in for $E_{t-1}y_t$ in equation (1) to yield the actual law of motion (ALM) for economy

$$y_t = (\Omega' + \alpha \sum_{i=1}^{k} \gamma_i \phi_i' u_i) z_{t-1} + w_t. \tag{7}$$

The ALM describes the actual evolution of y_t, given agents' parameter beliefs $\{\phi_i\}_{i=1}^k$. Substituting the ALM into equation (4) and simplifying yields the equilibrium condition for the parameter beliefs for each of the considered forecast rules

$$\phi_i = [(1 - \alpha\gamma_i) u_i \Sigma_z u_i']^{-1} (u_i \Sigma_z \Omega + \alpha \sum_{j \neq i} \gamma_j u_i \Sigma_z u_j' \phi_j) \tag{8}$$

where $E z_{t-1} z_{t-1}' = \Sigma_z$.

Definition 1: An *Exogenous Weight Forecast Combination Equilibrium* (EW-FCE) is a set of parameter beliefs $\{\phi_i^*\}_{i=1}^k$ and weights Γ that satisfies the system of equations given by (8) for all i.

In equilibrium, the ALM of the economy and the i parameter beliefs can be written as

$$y_t = D' z_{t-1} + w_t \text{ and } \phi_i = (u_i \Sigma_z u_i')^{-1} u_i \Sigma_z D.$$

this implies that an EW-FCE must satisfy

$$D' = \Omega' + \alpha \sum_{i=1}^{k} \gamma_i D' \Sigma_z u_i' (u_i \Sigma_z u_i')^{-1} u_i. \tag{9}$$

Remark 1: *There exists a unique Exogenous Weight Forecast Combination Equilibrium if and only if* $\Delta(\Gamma)$ *is invertible, where*

$$\Delta(\Gamma) = I - \alpha \sum_{i=1}^{k} \gamma_i \Sigma_z u_i' (u_i \Sigma_z u_i')^{-1} u_i$$

and I *is* $n + 1 \times n + 1$.

The same condition given in Remark 1 is derived by Branch and Evans (2006) to show the existence of a Restricted Perceptions Equilibrium in a model with heterogeneous agents. They show that the condition of Remark 1 is always met if Γ is in the unit simplex and α is sufficiently small. Under these assumptions, $\Delta(\Gamma)$ is a diagonal dominant matrix and has a non-zero determinant. Thus, for the aforementioned special case of equal weights, there always exists an α small enough such that an EW-FCE exists.

Remark 2: *There exist multiple equilibria if* $\alpha \sum_{i=1}^{k} \gamma_i = 1$.

This remark is due to the fact that each individual forecast rule has at least an intercept belief in common. The common intercept beliefs become completely self-confirming when $\alpha \sum_{i=1}^{k} \gamma_i = 1$ because any belief is perfectly reflected in the realizations of the data generating process.

The EW-FCE definition also nests an equilibrium that is observationally equivalent to the REE of the model. This FCE is of particular interest because it represents the conditions under which the REE predictions are robust to the boundedly rational behavior proposed in this paper. I call this equilibrium the Fundamental FCE.

Definition 2: A *Fundamental FCE* is an FCE that is observationally equivalent to the REE given by equation (2).

A Fundamental FCE will exist assuming the following three conditions:

A1: $\mu = 0$.

A2: The exogenous observables are uncorrelated, i.e. Σ_z *is a diagonal matrix.*

A3: Each forecast rule uses a mutually exclusive set of the exogenous observable shocks, which implies

$$\sum_{i=1}^{k} u_i' u_i = \begin{bmatrix} \omega & 0 \\ 0 & I \end{bmatrix},$$

where $0 \leq \omega \leq k$ *and* I *is* $n \times n$ *identity matrix.*

Proposition 1: *Assume A1, A2, A3,* $\Gamma = (1, \ \ldots \ , 1)'$, *and* $\alpha \neq \frac{1}{\omega}$, *then there exists a unique EW-FCE given by* $\phi_i^* = (1-\alpha)^{-1} u_i \Omega$ *for all* $i = 1, ..., k$ *which is the Fundamental FCE.*

The three conditions are motivated by standard practices in either the macroeconomic theory literature or the forecasting literature, and importantly, do not place restriction on the number of forecast rules agents may consider. The A1 condition is equivalent to redefining the reduced form model in terms of deviations from steady state. The A2 condition eliminates the possibility of omitted variable bias in the equilibrium beliefs for the menu of considered forecast rules by requiring that all the exogenous shocks are uncorrelated. The A3 condition imposes that each forecast rule draws upon a disjoint information set from all other forecast rules. This condition is akin to assuming that all forecast rules are non-encompassing, where an encompassed forecast is a forecast that includes no new information with respect to the other forecasts rules considered.

It is often recommended that only non-encompassed forecasts be included in combined forecasts (see Harvey and Newbold (2000)).

The $0 \leq \omega \leq k$ condition of A3 implies that each forecast rules does not need to include an intercept belief in its specifications since A1 already assumes that the process is mean zero. However, it is standard to always include an intercept and it will be useful in the next section to define a strengthened version of A3.

A3s: Each forecast rule includes an intercept and employs a mutually exclusive set of the exogenous observable shocks, which implies

$$\sum_{i=1}^{k} u_i' u_i = \begin{bmatrix} k & 0 \\ 0 & I \end{bmatrix},$$

where I is $n \times n$ identity matrix.

2.5 Optimal Weights

The first order condition for the optimal Γ from the minimization problem given by (6) can be expressed as an orthogonality condition similar to the one given for agents' beliefs,

$$EY_t(y_t - \Gamma' Y_t) = 0. \tag{10}$$

The orthogonality condition and the ALM given by (7) define a system of equations for the optimal weights as a function of the agents' parameter beliefs $\{\phi_i\}_{i=1}^k$, where the equation for the i^{th} weight is

$$\gamma_i = [(1-\alpha)\phi_i' u_i \Sigma_z u_i' \phi_i]^{-1} (\phi_i' u_i \Sigma_z (\Omega + (\alpha - 1) \sum_{j \neq i} \gamma_j u_j' \phi_j)). \tag{11}$$

Definition 3: An *Optimal Weight Forecast Combination Equilibrium* (OW-FCE) is set of parameter beliefs $\{\phi_i^*\}_{i=1}^k$ and vector of weights Γ^* that solves the system of equations given by (8) and (11) for all i.

The equilibrium in this case is defined by a system of polynomial equations, which may have many real solutions. The degree of the system of polynomial equations can grow with the number of forecast rules considered and with respect to the specification of the rules.[7] Therefore, in the general case, there is often not an algebraically tractable solution. However, the solution is tractable in one special case of interest. The special case is the one that permits the Fundamental FCE.

Proposition 2: *Assume A1, A2, and A3, then parameter beliefs $\phi_i^* = (1-\alpha)^{-1} u_i \Omega$ for all $i = 1, ..., k$ and combination weights $\Gamma^* = (1, \ \ldots \ , 1)'$ constitute a Fundamental*

[7] See Sturmfels (2002) for further explanation on the complexity of systems of polynomial equations and applications to economic problems.

OW-FCE.

For tractability it is also necessary to assume:

A4: $\Omega' u_i' u_i \Sigma_z u_i' u_i \Omega = C$ *for all* $i = 1, ..., k$, *where* C *is a constant scalar.*

This assumption has no real economic interpretation and again is only imposed to simplify the derivation of the multiple equilibria result. The existence of multiple equilibria is shown to be robust to this assumption and assumptions A1 and A2 in the application of the equilibrium concept in Section 3.2.1.

Proposition 3: *Assume A1, A2, A3s, A4, and* $\frac{1}{k} < \alpha < 1$, *then there exist multiple OW-FCE.*

Proof: From Proposition 2 it immediately follows that there exists at least one OW-FCE: the Fundamental FCE. Therefore, to prove the proposition I show the existence of other distinct equilibria under the same conditions.

The multiplicity of equilibria in this case is driven by the intercept beliefs. We know from Remark 2 that any intercept belief is self-confirming if $\alpha \sum_{i=1}^{k} \gamma_i = 1$. Therefore, to consider this case, write the equilibrium parameter beliefs as $\phi_i^* = \{(a_i, (1-\alpha\gamma_i)^{-1}\zeta_1, ..., (1-\alpha\gamma_i)^{-1}\zeta_m)' \mid m < n\}$ for $i = 1, .., k$, where a_i is free variable. These beliefs are obtained using equation (8) and applying assumptions A1, A2, and A3s.

The equilibrium condition for the i^{th} optimal weight is then given by

$$\gamma_i = [(1-\alpha)\phi_i' u_i \Sigma_z u_i' \phi_i]^{-1}(\phi_i' u_i \Sigma_z(\Omega + (\alpha - 1)\sum_{j \neq i} \gamma_j u_j' \phi_j)).$$

Substituting the equilibrium parameter beliefs $\{\phi_i^*\}_{i=1}^{k}$ into the condition and rearranging yields

$$\gamma_i(1-\alpha)\Psi = \Xi + (\alpha - 1)\sum_{j \neq i} \Lambda_j,$$

where without loss of generality

$$\begin{aligned} \Psi &= \phi_i' u_i \Sigma_z u_i' \phi_i = a_i^2 + (1-\alpha\gamma_i)^{-2}(\zeta_1^2\sigma_1^2 + ... + \zeta_{m_i}^2\sigma_{m_i}^2) \\ \Xi &= \phi_i' u_i \Sigma_z \Omega = (1-\alpha\gamma_i)^{-1}(\zeta_1^2\sigma_1^2 + ... + \zeta_{m_i}^2\sigma_{m_i}^2) \\ \Lambda_j &= \gamma_j a_i a_j. \end{aligned}$$

Now imposing $a_i = a_j$ for all i and j, $\alpha \sum_{i=1}^{k} \gamma_i = 1$, and A4 it follows that the equilibrium condition for γ_i simplifies further to

$$a^2 = \frac{\alpha C(1-\gamma_i)}{(1-\alpha)(1-\alpha\gamma_i)^2}.$$

One possible set of equilibrium optimal weights are thus given by

$$\gamma_i = \frac{C + 2a^2(\alpha - 1) \pm \sqrt{C\left(C - 4a^2(\alpha - 1)^2\right)}}{2a^2(\alpha - 1)\alpha}.$$

Finally, picking either solution for γ_i, substituting it into $\alpha \sum_{i=1}^{k} \gamma_i = 1$ for all i, and solving the equation for a yields

$$a = \pm \frac{\sqrt{k}\sqrt{C(1 - k\alpha)}}{(k-1)\sqrt{\alpha - 1}}. \tag{12}$$

The solutions for a are real and non-zero if $\frac{1}{k} < \alpha < 1$. Therefore, there exist multiple distinct OW-FCEs if $\frac{1}{k} < \alpha < 1$. $\square$

There are three interesting takeaways from Proposition 3 and the OW-FCE case. The first is that the Fundamental OW-FCE coexists with other non-fundamental equilibria. Therefore, the economy may obtain equilibria that depart from the rational predictions of the model even when the Fundamental FCE is a possible outcome. The second is that the range of parameter values for which multiple equilibria exist is dependent on the number of forecast rules considered. As the number of forecast rules considered increases, the range of the parameter space for which there is a unique OW-FCE shrinks.[8] This implies that multiple equilibria can arise in any parameterization of the model with positive feedback as long as agents consider enough different forecast rules. And finally, the multiplicity result does not rely on agents holding some type of flawed belief. For example, none of the beliefs that parameterize the forecasts rules exhibit omitted variable bias due to assumption A2. And there is no bias present in any of the individual forecasts or in the combined forecast by definition of the equilibrium.[9] Thus, although the agents are boundedly rational, the multiple equilibria result does not require agents to entertain beliefs that an econometrician could readily fault.[10]

3 Multiple Equilibria and Expectational Stability in a Simple Example

I apply the FCE concept in this section to a simple Lucas-type monetary model to provide examples of the equilibria and to study the stability of the equilibria under least squares learning. The model I consider follows Branch and Evans (2007). The economy is described by an aggregate supply equation (AS), an aggregate demand equation (AD),

$$AS: \quad y_t = \xi(p_t - E_{t-1}^{*} p_t) + \rho_1' x_t \tag{13}$$

$$AD: \quad y_t = m_t - p_t + \rho_2' x_t + v_t, \tag{14}$$

[8] See equation (12) in the proof of Proposition 3.

[9] The existence of these biases also do not rule out the existence of multiple equilibria.

[10] A remaining restriction that an econometrician may want to impose is that the weights sum to one. This case is considered in Section 3.2.1.

and a monetary policy rule (MP) given by

$$MP: \; m_t = p_{t-1} + \rho_3' x_t + \eta_t. \tag{15}$$

The variables p_t and m_t are logs of the price level and money supply receptively, y_t is the log deviation of output from a deterministic trend, x_t is a 2×1 vector of serially correlated shocks, and v_t and η_t are i.i.d. white noise. The shocks follow a stationary VAR(1) process

$$x_t = Bx_{t-1} + \epsilon_t. \tag{16}$$

Inflation in the economy is defined as $\pi_t = p_t - p_{t-1}$.

Imposing the inflation definition and combining equations (13), (14), and (15) yields an expectation augmented Phillips curve in the familiar reduced form:

$$\pi_t = \alpha E_{t-1}^* \pi_t + \zeta' x_{t-1} + w_t, \tag{17}$$

where $\alpha = \frac{\xi}{1+\xi}$, $\psi = \frac{(\rho_2' + \rho_3' - \rho_1')}{1+\xi}$, $\zeta' = \psi B$ and $w_t = \psi \epsilon_t + \frac{\eta_t + v_t}{1+\xi}$. The agents consider all non-trivial underparameterizations of the Phillips curve as forecast rules:

$$\begin{aligned} \hat{\pi}_{1,t} &= a_1 + b_1 x_{1,t-1} \\ \hat{\pi}_{2,t} &= a_2 + b_2 x_{2,t-1}. \end{aligned}$$

The agents are assumed to create optimal combined forecasts that minimize the expected squared forecast error.

3.1 Expectational Stability

A natural assumption in this boundedly rational environment is that agents form their expectations using recursive least squares learning. Following Evans and Honkapohja (2001), least squares learning replaces the parameter and weight beliefs, $(\phi_1', \phi_2', \Gamma')$, with estimated beliefs, $(\hat{\phi}_{1,t}', \hat{\phi}_{2,t}', \hat{\Gamma}_t')$, that are updated recursively using past data.

In the example proposed, the agents' recursively estimates three regressions: two regressions to estimate the coefficients of $\hat{\pi}_{1,t}$ and $\hat{\pi}_{2,t}$ and a third to estimate Γ. The estimation can be written jointly and recursively as

$$\begin{aligned} \Theta_t &= \Theta_{t-1} + \kappa_t R_{t-1}^{-1} \mathbf{z_{t-1}} (\mathbf{y_t} - \mathbf{z_{t-1}'} \Theta_{t-1}) \\ R_t &= R_{t-1} + \kappa_t (\mathbf{z_{t-1}} \mathbf{z_{t-1}'} - R_{t-1}), \end{aligned} \tag{18}$$

where the first equations governs the evolution of the belief and weight coefficients $\Theta_t = (\hat{\phi}_{1,t}', \hat{\phi}_{2,t}', \hat{\Gamma}_t')'$, the second equation is the estimated second moments matrix, and κ_t is the gain sequence that governs the weight given to new observations. The recursive form uses a block matrix structure to estimate all coefficients simultaneously, where $\mathbf{y_t} = (\pi_t \ \pi_t \ \pi_t)'$, the regressors are stacked into the matrix

$$\mathbf{z_t} = \begin{pmatrix} u_1 z_t & 0 & 0 \\ 0 & u_2 z_t & 0 \\ 0 & 0 & \Pi_{t+1} \end{pmatrix}, \tag{19}$$

and $\Pi_t = (\hat{\pi}_{1,t}, \hat{\pi}_{2,t})'$. The equilibrium conditions of an OW-FCE are equivalent to the least squares criterion and thus the fixed points of (18) correspond to the possible OW-FCEs of the model.

The convergence to a given OW-FCE from nearby initial beliefs can be determined by appealing to the E-stability principle. The E-stability principle states that the stability of a fixed point of the stochastic recursive algorithm (18) is determined by the stability of an associated differential equation in notional time

$$\dot{\Theta} = T(\Theta) - \Theta,$$

where $\dot{\Theta}$ is given by rearranging equations (8) and (11), the equilibrium conditions for an OW-FCE. The function $T(\Theta)$ is known as the T-map.[11]

The E-stability of a fixed point of the model under study is usually dependent on the value of α, the feedback parameter on expectations. The standard result in the literature is that if agents consider a single correctly specified PLM, then the resulting fixed point is the REE and it is E-stable provided that $\alpha < 1$.[12]

3.2 Existence and Stability

To show the existence and E-stability of multiple equilibria in the Lucas-monetary model, I first consider the case that permits the Fundamental OW-FCE by assuming A1, A2, and A3. The T-map for optimal weights in this case is given by

$$T\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha(a_1\gamma_1 + a_2\gamma_2) \\ \alpha\gamma_1 b_1 + \zeta_1 \\ \alpha(a_1\gamma_1 + a_2\gamma_2) \\ \alpha\gamma_2 b_2 + \zeta_2 \\ \frac{a_1^2\alpha\gamma_1 + a_1a_2\gamma_2(\alpha-1) + b_1\sigma_1^2(b_1\alpha\gamma_1+\zeta_1)}{a_1^2+b_1^2\sigma_1^2} \\ \frac{a_2^2\alpha\gamma_2 + a_1a_2\gamma_1(\alpha-1) + b_2\sigma_2^2(b_2\alpha\gamma_2+\zeta_2)}{a_2^2+b_2^2\sigma_2^2} \end{pmatrix}. \tag{20}$$

The mapping is non-linear and has seven distinct fixed points. The algebraic representations of the solutions, however, are immense and are impractical to list here. Therefore, I use a bifurcation argument to show existence and E-stability for a subset of the equilibria.

Proposition 3 says that multiple equilibria will arise in this model when $\alpha > 1/2$.

[11]The derivation of the T-map is given in the appendix. Guse (2008) shows that the technical conditions for the stochastic recursive algorithm theorems that underpin the E-stability principle are satisfied for the block recursive least squares algorithm.

[12]See Evans and Honkapohja (2001) for a complete analysis and discussion of this result.

This conditions also identifies the point at which the Fundamental OW-FCE fixed point of the differential equation

$$\dot{\Theta} = T(\Theta) - \Theta \tag{21}$$

experiences a bifurcation, when α is treated as the bifurcation parameter. By characterizing the type of bifurcation that occurs at this point, the number of equilibria and the E-stability of those equilibria can be determined.

Lemma 1: *The Fundamental OW-FCE steady state of the dynamic system given by (20) and (21) experiences a supercritical pitchfork bifurcation at $\alpha = 1/2$.*

A supercritical pitchfork bifurcation occurs when a fixed point switches stability from stable to unstable and two new stable equilibria come into existence.

Proposition 4: *For the economy under study (17) represented by (20)*

1. *There exist at least three OW-FCEs for $\frac{1}{2} < \alpha < 1$.*
2. *The Fundamental OW-FCE is E-stable for $\alpha < \frac{1}{2}$*
3. *At least two non-fundamental OW-FCE are E-stable for some $\frac{1}{2} < \alpha < 1$.*

Proposition 4 shows that although the Fundamental OW-FCE is a possible equilibrium, it is not the equilibrium the economy will achieve under learning when $\frac{1}{2} < \alpha < 1$.[13] The economy instead will obtain other non-fundamental equilibria that are distinct from the RE predictions of the model.

Figure 1 illustrates the full set of OW-FCEs using a pseudo-bifurcation diagram. The diagram is "pseudo" because it illustrates consequences of the bifurcation that occur in multiple dimensions in a two-dimensional picture by utilizing the implied forecast of the equilibrium beliefs. The diagram is constructed numerically by calculating the full set of OW-FCEs for a given value of α and then plotting the implied forecasts of the equilibrium beliefs for a fixed realization of x_t. The parameter values for the diagram are $\zeta_1 = .9$, $\zeta_2 = -.9$, $Ex_t x_t' = I$, and $\hat{x}_{t-1} = (1,\ 1)'$.[14] E-stability is indicated by the solid lines in the figure.

Figure 1 shows that the Fundamental FCE is the unique equilibrium for $\alpha < \frac{1}{2}$. Then at $\alpha = \frac{1}{2}$, the predicted pitchfork bifurcation occurs and two new equilibrium come into existence. A second bifurcation occurs at $\alpha = \frac{3}{4}$ and results in a total of seven OW-FCEs.

[13] Evans et al (2013) also find that $\alpha = \frac{1}{2}$ is the boundary for non-rational equilibria to exist in their simulations.

[14] The range of the figures includes negative values of α to illustrate the Cobweb case for the reduced form model as well as the Lucas-type aggregate supply case of interest. Note that there is a unique OW-FCE if the model exhibits negative feedback. An example in the literature of the Cobweb case is Branch and Evans (2006).

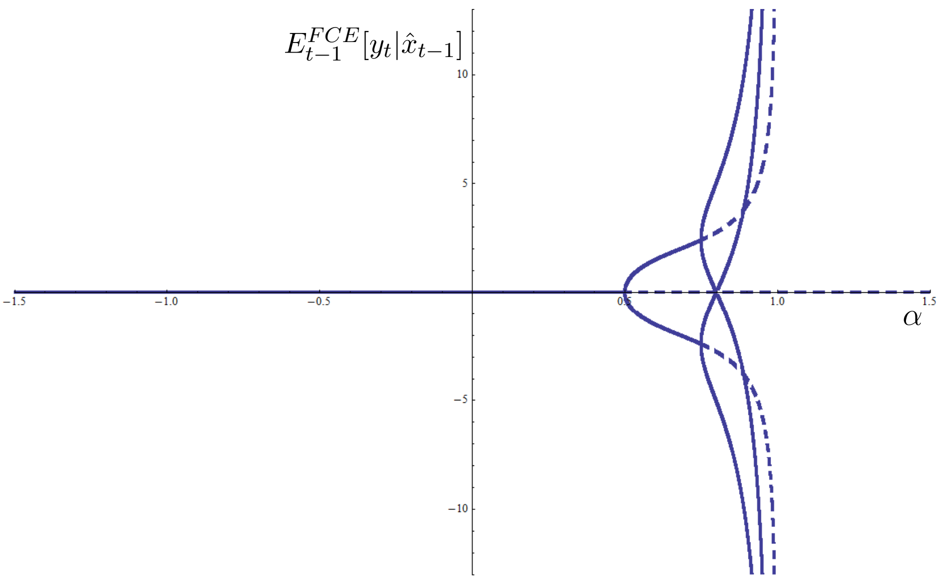

Figure 1: Pseudo-bifurcation diagram of the associated differential equation (20). E-stability is indicated by the solid lines. Parameter values are given in the text.

3.2.1 Exploration and Robustness

The key to multiple equilibria in OW-FCEs is retaining the non-linearity in the weights. Any modification that retains the non-linearity will produce multiple OW-FCEs that follow a similar pattern to the one depicted in Figure 1, where there exists a single stable equilibrium for small and negative α and multiple stable equilibria for larger $\alpha < 1$.

Figure 2 provides three pseudo-bifurcations examples to illustrate the claim. The first two plots show that the existence and stability results extend to cases where the Fundamental OW-FCE assumptions are not met. The first plot shows the equilibria and stability results when there is non-zero covariance between the elements of x_t ($\sigma_{12} = .1$), while the second plot shows the equilibria and stability results when there is a non-zero intercept ($\mu = 1$).[15]

The third plot in Figure 2 depicts the case where the optimal weights are restricted to sum to one. This case is of particular interest because this restriction is often imposed in the forecasting literature. The restricted optimal weights imply the following orthogonality conditions

$$E(\hat{\pi}_{1,t} - \hat{\pi}_{2,t})[(\pi_t - \hat{\pi}_{2,t}) - \gamma_1(\hat{\pi}_{1,t} - \hat{\pi}_{2,t})] = 0. \tag{22}$$

The conditions follows directly from solving the constrained version of equation (6).

[15]The two assumptions violate A2 and A1, respectively.

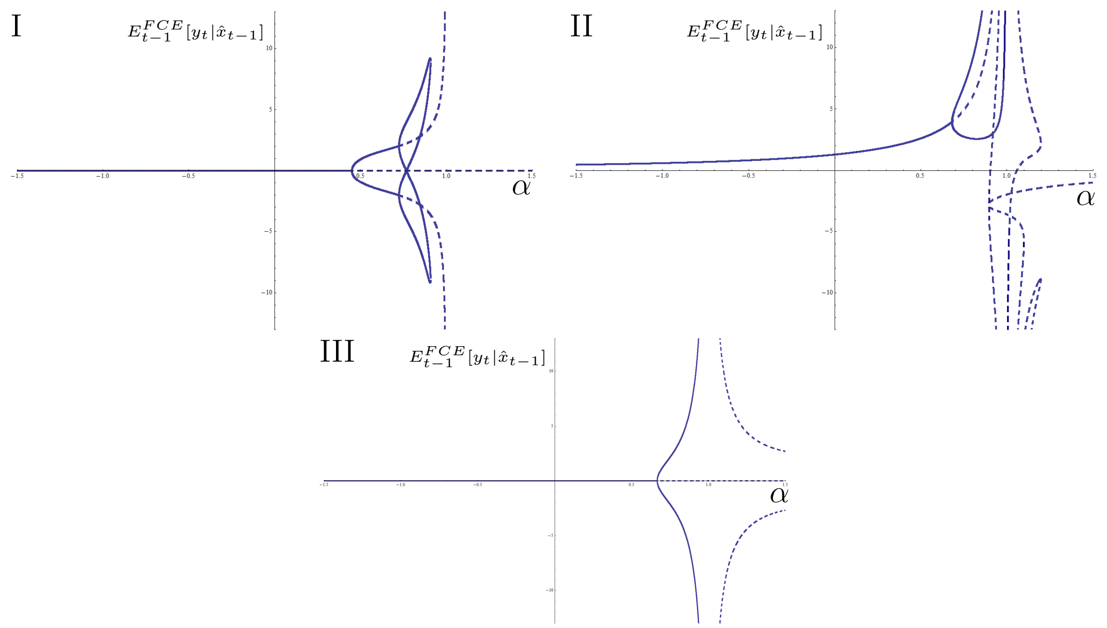

Figure 2: Pseudo-bifurcation diagrams of the associated differential equation (20). The first diagram (I) depicts the bifurcation for a model with a positive covariance between the exogenous shocks ($\sigma_{12} = .1$). The second (II) diagram depicts the bifurcation of a model with a positive intercept ($\mu = 1$). The third diagram (III) depicts the bifurcation of a model with the restriction that the combination weights sum to one. The parameter values are the same as those used in Figure 1. E-stability is indicated by the solid lines.

The T-map under this assumption and assumptions A1, A2, and A3 is given by

$$T\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha(a_2 + \gamma_1(a_1 - a_2)) \\ \alpha\gamma_1 b_1 + \zeta_1 \\ \alpha(a_2 + \gamma_1(a_1 - a_2)) \\ \alpha(1-\gamma_1)b_2 + \zeta_2 \\ \frac{(a_1-a_2)(a_1\alpha\gamma_1 + a_2(\alpha(1-\gamma_1)-1) + b_1\sigma_1^2(b_1\alpha\gamma_1+\zeta_1) + b_2\sigma_2^2(b_2(1+\alpha(\gamma_1-1))-\zeta_2)}{(a_1-a_2)^2 + b_1^2\sigma_1^2 + b_2^2\sigma_2^2} \end{pmatrix}. \quad (23)$$

The T-map again defines a system of polynomial equations with multiple solutions. However, the multiple equilibria in this case are different than in the unrestricted case because every equilibria shares a common intercept belief. This is due to the fact that $\alpha(\gamma_1 + \gamma_2) = 1$ cannot be satisfies in this case except trivially when $\alpha = 1$. Figure 2 shows that the stability results in this case are analogous to those in the unrestricted case. There exists a unique E-stable equilibrium for small or negative α and multiple E-stable equilibrium for some positive $0 < \bar{\alpha} < 1$.

The onset of multiple equilibria in the unrestricted and restricted cases also depends on the parameter ζ as well as α. The ζ vector, as shown in equation (17), is partly a function of the monetary policy rule parameters. Therefore, the number and stability of equilibria in the model is a function of policy. Figure 3 plots pseudo-bifurcation diagrams for the model with ζ_1 as the bifurcation parameter and α fixed at 0.9. The remainder of the parameter values are the same as in Figure 1. The diagrams illustrates that the number of equilibria and E-stability of the equilibria change as ζ is varied

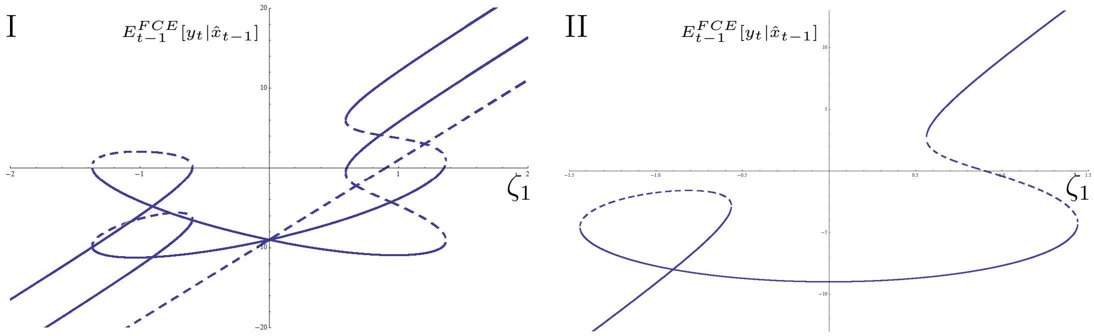

Figure 3: Pseudo-bifurcation diagrams of the associated differential equation (20). The first diagram (I) depicts the bifurcation for the unrestricted weights case. The second (II) diagram depicts the bifurcation for the case where the weights are restricted to sum to one. E-stability is indicated by the solid lines.

indicating that policy plays a role in determining whether there exists a unique stable equilibrium among the set of non-rational or non-Fundamental FCEs.

3.3 Discussion and Extensions

The economic explanation for the pattern of a unique equilibria bifurcating into multiple stable equilibria as α increases lies in the positive feedback of expectations. When α is positive and close to one, all beliefs have a self-fulfilling quality. The data always moves in the same direction as the beliefs. The movement of the data towards the beliefs is subsequently reinforced by the optimal weights, which allows for some beliefs to become completely self-fulfilling.

The agents in this process are in some sense subject to a perverse form of the Lucas Critique. The reduced form correlations in the data that are used to calculate the optimal weights are a function of the forecasting strategy. However, unlike the Lucas Critique, where changes in policy alter the reduced form correlations and invalidate policy choices, the choice of optimal weights may reinforce the correlations. The reinforcement then prevents agents from detecting that their forecasts deviate from the fundamentals because their beliefs are consistent with past data.

There is nothing unique to the model studied in this paper with respect to the positive feedback necessary to generate the multiple equilibria and stability results. In fact, it is commonly found that the equilibria and E-stability results of learning models in simple settings carry over to richer, more realistic settings. Therefore, it is likely that any linear rational expectations model with a learnable REE will also have multiple, E-stable equilibria under forecast combination when there exists significant positive feedback.

4 Real-time Application and Comparison to DPS

One of the primary contributions of DPS and the learning and expectations literature is to show how expectation driven fluctuations can capture features of actual laboratory and real world economic data that are not explained by standard RE models. This section illustrates that optimal forecast combination with constant gain learning continues this tradition. The distinct dynamics observed replicate features of actual economic data that are not captured by either the RE or the DPS assumptions.

4.1 Constant Gain Learning

Optimal weight forecast combination is implemented by assuming that the agents estimate the parameters of their forecast rules and the optimal combination weights in real time using constant gain learning. Constant gain learning replaces the usual $\kappa_t = t^{-1}$ in the recursive least squares algorithm (18) with a constant $0 < \bar{\kappa} < 1$. A constant gain places more weight on the most recent observations. The stability of the model under constant gain learning for a sufficiently small gain is still governed by the E-stability conditions.[16] The main difference, however, is that the convergence is now in terms of a distribution of beliefs centered at the equilibrium. Small shocks in the model, therefore, may occasionally push expectations from the basin of attraction of one stable OW-FCE to another.[17]

Constant gain learning is argued by Sargent (2001), Orphanides and Williams (2005 and 2006), and Branch and Evans (2006b and 2007), to be an appropriate learning strategy when agents are concerned about structural breaks. In addition, the use of constant gain learning is found to be a route to interesting dynamics in Lucas-type monetary models in the case of model misspecification (Cho, Williams, and Sargent (2002), McGough (2006)) and Dynamic Predictor Selection (Branch and Evans (2007)) and is thus the natural assumption to employ.

4.2 Comparison to Dynamic Predictor Selection

This section compares forecast combination to dynamic predictor selection to show that there are important qualitative and theoretical differences between the two approaches. Dynamic predictor selection was first introduced by Brock and Hommes (1997) and posits that there exist heterogeneous agents who select among a menu of forecast rules by evaluating a fitness criteria. Based on the fitness criteria each agent chooses a single forecast rule to follow in each period. Heterogeneity is imposed by assuming that there is stochastic component to the agents' choices.

Branch and Evans (2007) apply the dynamic predictor selection framework to the model described by equation (17) by assuming that agent consider $\Pi_t = (\hat{\pi}_{1,t},\ \hat{\pi}_{2,t})'$ as a menu of forecast rules and that they use past mean squared forecast error (MSFE) as the fitness criteria. The agents tracks the MSFE of each model recursively using the

[16]See Evans and Honkapohja (2001) for a thorough treatment of this result.

[17]I do not simulate the decreasing gain case because the E-stability analysis fully characterizes the limiting dynamics. Given a set of initial conditions in a neighborhood of any of the E-stable OW-FCEs, the learning algorithm will converge asymptotically with probability one to the OW-FCE in that neighborhood.

following equation:

$$MSFE_{i,t} = MSFE_{i,t-1} + \lambda((\pi_t - \hat{\pi}_{i,t})^2 - MSFE_{i,t-1}) \tag{24}$$

for $i = 1, 2$, where λ is the gain parameter. The proportion of agents who choose each model based on this fitness criterion is determined by

$$n_{i,t} = \frac{exp(-\beta MSFE_{i,t-1})}{\sum_{i=1}^{k} exp(-\beta MSFE_{i,t-1})}, \tag{25}$$

where $n_{i,t}$ is the proportion of agents who choose the i^{th} model and β is a parameter that governs the relative speed at which agents abandon an underperforming forecast rule. The aggregate expectation of agents is a linear combination of the $\hat{\pi}_{i,t}$'s with the $n_{i,t}$'s as the weights

$$E^*_{t-1}\pi_t = \sum_{i=1}^{k} n_{i,t}\hat{\pi}_{i,t}. \tag{26}$$

The equilibrium concept in this framework is a Misspecification Equilibrium (ME), which was first introduced by Branch and Evans (2006). An ME is characterized by the same condition given by Remark 1, plus a fixed point in the proportion of agents that choose each rule given by equation (25). Branch and Evans (2007) show that there are three possible MEs in this model: two MEs where all agents choose the same forecast rules and a third where some fraction of agents chooses each rule. Analytic E-stability results are not available for these equilibria, but they report that, in simulations, the two homogeneous equilibria are stable under learning.

4.2.1 Theoretical Comparisons

The aggregate expectation under DPS may appear to be the same as under forecast combination, however, the weights under DPS reflect population proportions. This assumption rules out negative weights and weights larger than one, which can prevent agents from optimally responding to the observed misspecifications in the menu of forecast rules they possess.[18] The omission of potentially useful information observed

[18] To formally illustrate this argument, I present the derivation of optimal weights given by Timmermann (2006) who considers forecasting a mean zero process with two unbiased forecast rules. Ignoring time, the two forecast rules' errors can be written as $fe_1 = \pi - \hat{\pi}_1$ and $fe_2 = \pi - \hat{\pi}_2$, where $fe_1 \sim (0, \sigma_1^2)$, $fe_2 \sim (0, \sigma_2^2)$, $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$, and ρ_{12} is the correlation of the forecast errors. The combined forecast error is given by $fe_c = \gamma_1 fe_1 + (1 - \gamma_1)fe_2$, which implies

$$\sigma_c^2(\gamma_1) = \gamma_1^2\sigma_1^2 + (1 - \gamma_1)^2\sigma_2^2 + 2\gamma_1(1 - \gamma_1)\sigma_{12}. \tag{27}$$

Minimizing equation (27) with respect to γ_1 yields the optimal weight

$$\gamma_1^* = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}. \tag{28}$$

It is simple to show that $\sigma_c^2(\gamma_1^*) \leq \min(\sigma_1^2, \sigma_2^2)$ and that in general the optimal weights need not be convex if $\rho_{12} > \sigma_2/\sigma_1$.

in the data is also built into the selection assumption of DPS. The requirement that agents select a single rule means that agents disregard information about the future state of the economy simply due to small differences in the fitness criteria. Therefore, in some sense, the forecast combination assumption is a more individually rational response to model uncertainty than DPS.

Another distinction is that the optimal combination weights are completely data driven, whereas the weights in the DPS model are only partially data driven. The intensity of choice parameter β plays a large role in determining the weights and their transitions over time. This parameter is completely exogenous and has no real world analogue. There is no equivalent parameter under optimal weight forecast combination. The relative weights are chosen endogenously and in a way that is consistent with practices prescribed by the forecasting literature.

A final distinction between the two approaches is that in the endogenous weight case the equilibrium outcomes are largely disjoint. This is due to the fact that the ME concept requires Remark 1 to hold in equilibrium, while Remark 2, the violation of Remark 1, was integral in the existence of multiple equilibria in the OW-FCE case. Therefore, despite the similar aggregate structures, the models in many cases will produce distinct equilibrium predictions.

4.2.2 Real-time Learning Comparison

The main dynamic feature of DPS in the Lucas monetary model is endogenous time-varying volatility. The volatility is generated by the fact that there are two E-stable equilibria that each have distinct output and inflation volatilities in aggregate. The economy transitions occasionally from one stable equilibrium to another due to the random shocks in the model under constant gain learning, which causes the volatility of the aggregate time series to change over time.

Optimal weights forecast combination imparts a similar dynamic structure to the economy. There exist multiple E-stable equilibria that each imparts a different level of volatility to output and inflation. The model under constant gain learning will, like DPS, periodically transition from one stable equilibria to another and generate time-varying volatility. However, forecast combination exhibits an additional real-time feature that is not present under DPS. Under forecast combination there are also endogenous changes to the mean of output and inflation that occur due to off equilibrium paths.

To illustrate, compare, and contrast these dynamics I simulate the economy under two version of optimal weights forecast combination (OW-FC) and DPS. The two versions of OW-FC are the standard case considered in the paper and the restricted case, where the weights must sum to one. The differences in the simulations can thus be characterized by their respective restrictions of the combination weights. The OW-FC case has no restrictions, the restricted OW-FC case has weights restricted to one, and the DPS case has weights restricted to the unit simplex. The simulations are shown in Figure 4.

The first column of Figure 4 shows the time path of intercept beliefs and time path of inflation. The second column shows the time path of the endogenous weights and denotes the major transitions between equilibria with gray bars. The parameters for the simulations are given in the figure caption. The parameters values chosen satisfy assumptions A1, A2, and A3 and imply the existence of four stable OW-FCEs in the unrestricted OW-FC case and two stable equilibria in remaining cases. In addition, the shocks in all simulation are i.i.d. and the same sequence of shocks is used in each simulation. The use of the same shocks across the three simulations allows the dynamics generated by expectations to be easily discernible from those caused by the i.i.d. shocks themselves.

The graphs illustrate the following properties:

1. *Time-varying volatility:* Each simulation shows clear evidence of time-varying volatility. The magnitude of the volatility, however, differs between the cases. The OW-FC cases exhibit about twice the magnitude of fluctuations as in the DPS case. The source of the time-varying volatility is shown in the graphs of the combination weights. The changes in volatility coincide with the transitions between and fluctuation around stable equilibria.

2. *Breaks in trend inflation:* The unrestricted OW-FC case exhibit dramatic breaks to the trend of inflation over time with significant bouts of both deflation and inflation. The gray bars on the right-hand-side graph highlight each time a large break to trend inflation occurs. From the highlighted sections we can see that the changes to the trend occur when both weights are close to one. The breaks are thus occurring when beliefs are pushed into the neighborhood of the unstable Fundamental FCE. Both weights near one act to increase the positive feedback of the model. The increase in positive feedback pushes beliefs temporarily far away from any stable equilibrium of the model.

The equilibrium predictions of the three cases can for long periods of time remain very close. Figure 5 illustrates this by plotting the time path of the intercept beliefs for the three cases in a simulation with the same parameter values employed in Figure 4. The intercept beliefs drive the trend dynamics of inflation in the model. In this simulation, the intercept beliefs track each other closely for the first 600 periods before significant breaks occur to the OW-FC beliefs. The breaks once again occur in the standard OW-FC case when both weights are close to one. The breaks in the restricted case also occur due to a similar mechanism as the unrestricted case whenever the lone estimated weight is significantly larger than one in magnitude.

An optimal weight that is larger than one in magnitude reinforces any erroneous beliefs of a positive intercept that may exist due to random shocks. These beliefs are over time reflected in the agents' estimates of the intercepts through revisions in beliefs. The positive feedback of the model makes these beliefs temporarily self-fulfilling causing the dramatic spikes in inflation or deflation observed in the figures. The DPS weights by contrast are constrained to the unit simplex, which rules out these kinds of self-fulfilling off-equilibrium paths.

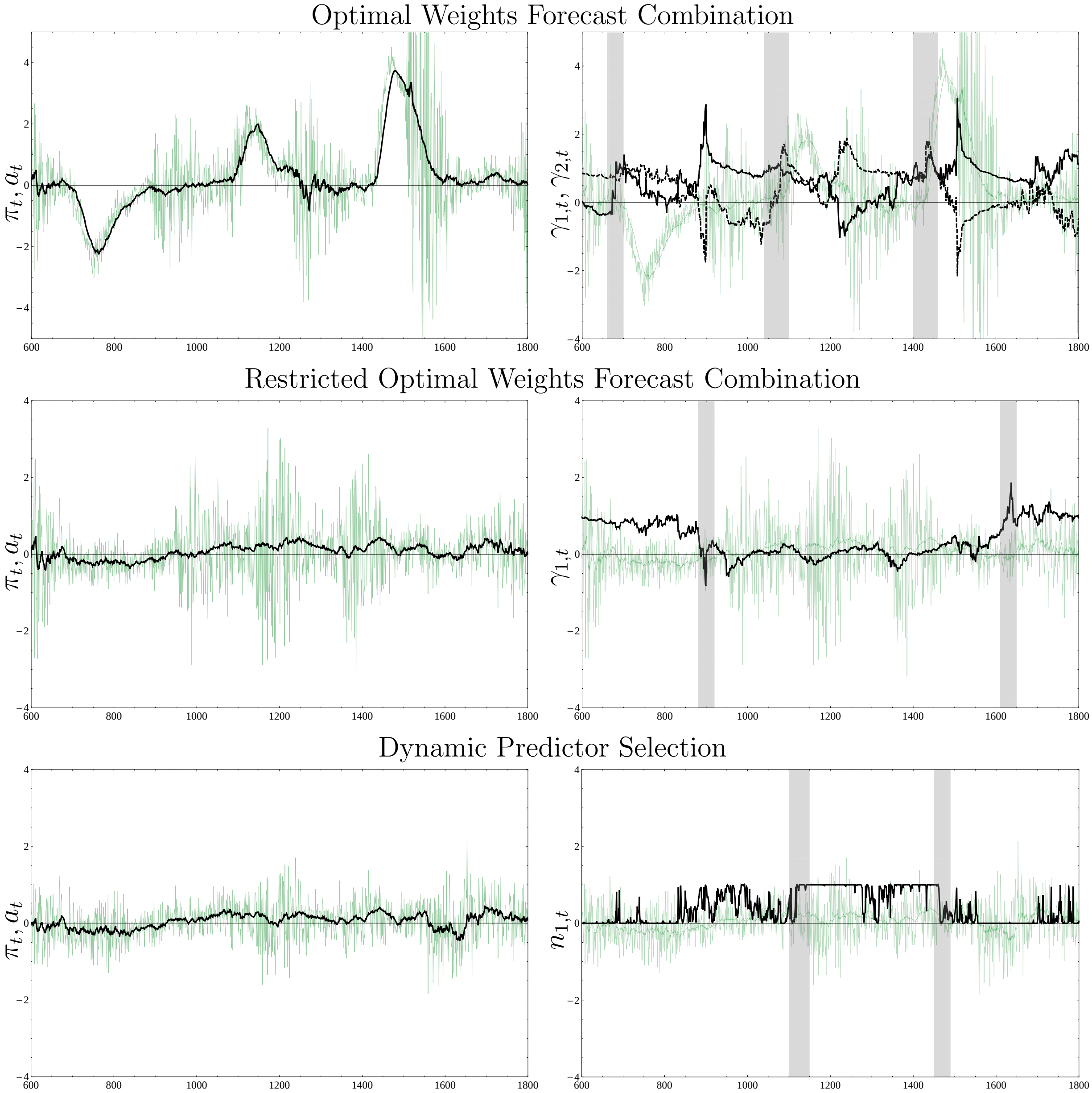

Figure 4: Simulated paths of inflation when agents use optimal weights forecast combination, restricted optimal weights forecast combination, and dynamics predictor selection. The parameter values for the simulations are $\alpha = 0.9$, $\zeta = (.5, .75)'$, $E[x_t x_t'] \sim N(0, 0.1I)$, $w_t \sim N(0, 0.25)$, and a gain of $\kappa = 0.08$. The remaining parameters for DPS are $\beta = 50$ and $\lambda = 0.35$. The first column shows the path of inflation and agents beliefs a_t. The second column shows the optimal combination weights and the DPS population weights. The gray bars indicate breaks to trend inflation in the OW-FC case and significant transitions between stable equilibria in the restricted OW-FC and DPS cases.

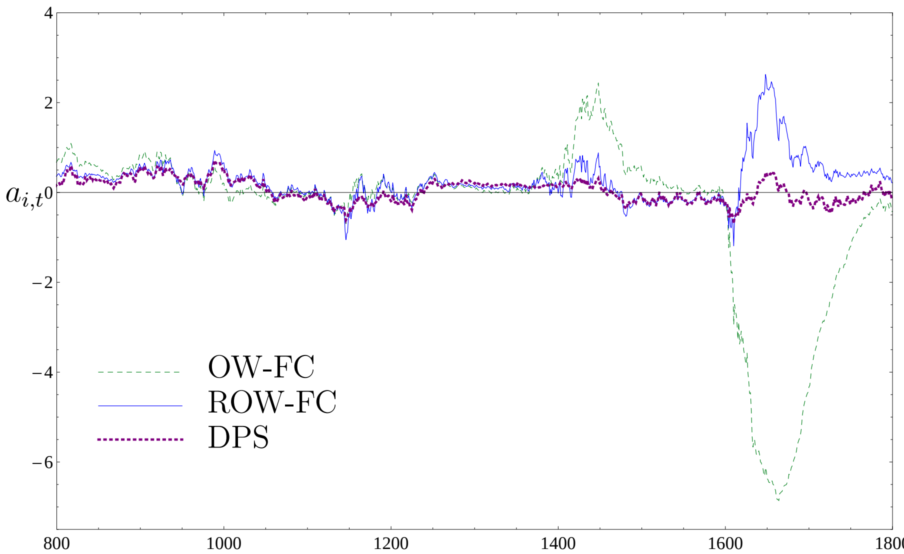

Figure 5: Simulated time path of the intercept beliefs $a_{i,t}$. The parameters used for the simulation are the same as in Figure 4.

4.2.3 Actual Inflation Comparison

This section compares simulations of the model under OW-FC and DPS to the GDP Deflator measure of U.S. inflation. The comparison to inflation is meant to establish why the dynamics generated under optimal weight forecast combination are economically relevant by qualitatively matching model implied dynamics to actual dynamics. The exercise, however, is not meant to be a formal calibration exercise.

Branch and Evans (2007) show that DPS in the Lucas monetary model can generate time-varying volatility in simulated inflation that is qualitatively similar to the observed volatility in actual U.S. inflation. Optimal weight forecast combination can replicate this finding as well as generate an additional important feature of U.S. inflation. The empirical features of interest in this exercise are those highlighted in Stock and Watson (2007). Stock and Watson note that U.S. inflation is well described by a parsimonious unobserved components stochastic volatility model (UC-SV) that allows for changing volatility in both permanent and transient shocks. They model the inflation process as

$$
\begin{aligned}
\pi_t &= \tau_t + \eta_t, \text{ where } \eta_t = \sigma_{\eta,t}\zeta_{\eta,t} \\
\tau_t &= \tau_{t-1} + \epsilon_t, \text{ where } \epsilon_t = \sigma_{\epsilon,t}\zeta_{\epsilon,t} \\
\ln\sigma^2_{\eta,t} &= \ln\sigma^2_{\eta,t-1} + \psi_{\eta,t} \\
\ln\sigma^2_{\epsilon,t} &= \ln\sigma^2_{\epsilon,t-1} + \psi_{\epsilon,t},
\end{aligned}
$$

where $\zeta_t = (\zeta_{\eta,t}, \zeta_{\epsilon,t})$ is i.i.d. $N(0, I_2)$, $\psi_t = (\psi_{\eta,t}, \psi_{\epsilon,t})$ is i.i.d. $N(0, 0.2I_2)$, and ζ_t and ψ_t are independently distributed. They note that there is significant time variation in both the permanent and transient shocks to inflation over time. The OW-FC concept can capture both of these features in the simple Lucas-type monetary model with i.i.d. shocks.

Figure 6 shows the UC-SV estimates for actual inflation and simulated inflation. The parameter values for the simulations are the same as in Figure 4, except with a change to the gain parameter. I follow Branch and Evans (2006b) and set the gain to $\bar{\kappa} = 0.0115$ to calibrate the rate at which agents respond to new information to be consistent with actual forecasting behavior at a monthly frequency.[19] This allows the speed at which agents respond to shocks to be plausible for the comparison. The economy is simulated for 795 periods. The 795 periods are then aggregated by simple averaging to 265 periods to mimic the quarterly observations of the GDP Deflator.

The OW-FC simulation is chosen to include an endogenous break to trend inflation.[20] The endogenous break to trend inflation generates shocks to the permanent component of inflation that are qualitatively similar to what is observed in actual inflation. DPS, by contrast, cannot generate similar large breaks to the trend inflation, which prevent it from replicating the large swings in volatility observed in the permanent component of inflation. Both OW-FC and DPS, however, are capable of replicating the time-varying volatility of the transient component of inflation.

[19]Branch and Evans (2006b) find that $\bar{\kappa} = 0.0345$ is the most likely gain to explain the quarterly forecasts observed in the SPF. I have scaled the number to make it consistent with monthly observations.

[20]The simulations each discard the first 15,000 initial periods. The simulated path presented is a 795 period subsample taken from a larger simulation after the initial 15,000 periods to illustrate common possible dynamics.

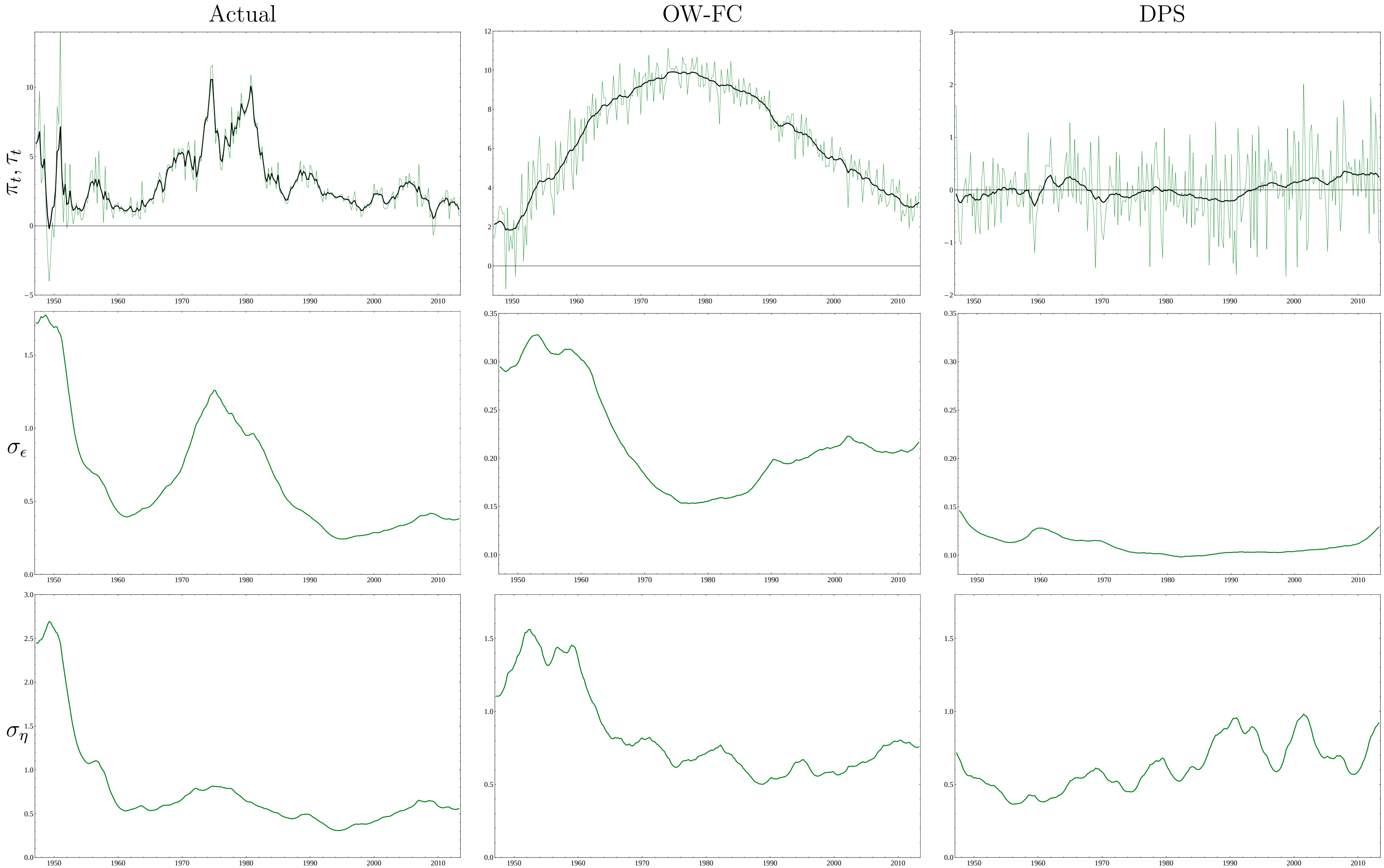

Figure 6: Actual and simulated paths of inflation and the UC-SV model estimates of the stochastic trend τ_t, the standard deviation of the permanent shock σ_ϵ, and the standard deviation of the transient shock σ_η for inflation. Actual inflation is the U.S. GDP Deflator measure from 1947Q4 to 2013Q2. The parameter values of the simulation are given in the text.

5 Conclusion

This paper fills a gap in the literature on expectations and bounded rationality by exploring the equilibrium and dynamic consequences of homogeneous agents who employ forecast combination techniques, rather than model selection, to form expectations due to model uncertainty. The consequence of heterogeneous agents who face the model uncertainty problem, but engage in model selection, has been well explored in the literature. This paper establishes that despite the similarities of the two approaches that the forecast combination approach yields distinct equilibrium and dynamic predictions.

The Forecast Combination Equilibrium concept permits multiple equilibria when agents are assumed to employ an optimal weights combination strategy. The number and stability of the equilibria depend on the menu of forecast rules considered as well as the policy parameters of the model. The concept is capable of generating endogenous time-varying volatility and endogenous breaks to the trend of inflation when agents estimate beliefs and optimal weights using constant gain learning in a Lucas-type monetary model that only experiences i.i.d. random shocks. The dynamics replicate key qualitative feature of U.S. inflation data illustrating the potential of the concept to explain a range of economic phenomena.

6 Appendix

Selector Matrix Example: The selector matrices given by the u_i's are $m_i \times n+1$ matrices that can be thought of as identity matrices, which have the rows that do not correspond to included exogenous variables deleted. As an example, consider the case where $z_{t-1} = (1, x_{1,t-1}, x_{2,t-1}, x_{3,t-1})'$ and the misspecified model is given by

$$\hat{y}_{1,t} = a_1 + b_1 x_{1,t-1} + b_3 x_{3,t-1}.$$

The model can be written as $\hat{y}_{1,t} = \phi_1' u_1 z_{t-1}$, where $\phi_i = (a_1, b_1, b_3)'$ and

$$u_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Derivation of T-map and associated differential equation for the OW-FCE: The derivation follows Chapter 13 of Evans and Honkapohja (2001) and can be computed by calculating

$$ER^{-1}\mathbf{z_{t-1}}(\mathbf{y_t} - \mathbf{z_{t-1}}\Theta_{t-1}).$$

The nesting of the equilibrium conditions (8) and (11) can be seen by multiplying the $\mathbf{z_{t-1}}$ through the brackets to yield

$$ER^{-1}\begin{pmatrix} u_1 z_{t-1} z'_{t-1}\Omega + \alpha\sum_{i=1}^{2}\gamma_i u_1 z_{t-1} z'_{t-1} u'_i\phi_i - u_1 z_{t-1} z'_{t-1} u'_1\phi_1 \\ u_2 z_{t-1} z'_{t-1}\Omega + \alpha\sum_{i=1}^{2}\gamma_i u_2 z_{t-1} z'_{t-1} u'_i\phi_i - u_2 z_{t-1} z'_{t-1} u'_2\phi_2 \\ Y_t z'_{t-1}\Omega + \alpha Y_t\Gamma' Y_t - Y_t\Gamma' Y_t \end{pmatrix}$$

and then pushing through the expectations operator, rearranging terms, and factoring out $\mathbf{z_{t-1}z'_{t-1}}$ to arrive at

$$R^{-1}E\mathbf{z_{t-1}z'_{t-1}}\begin{pmatrix} (u_1\Sigma_z u'_1)^{-1}(u_1\Sigma_z\Omega + \alpha\gamma_2 u_1\Sigma_z u'_2\phi_2) + (\alpha\gamma_1 - 1)\phi_1 \\ (u_2\Sigma_z u'_2)^{-1}(u_2\Sigma_z\Omega + \alpha\gamma_1 u_2\Sigma_z u'_1\phi_1) + (\alpha\gamma_2 - 1)\phi_2 \\ (\phi'_1 u_1\Sigma_z u'_1\phi_1)^{-1}(\phi'_1 u_1\Sigma_z(\Omega + (\alpha-1)\gamma_2 u_2\phi_2)) + (\alpha-1)\gamma_1 \\ (\phi'_2 u_2\Sigma_z u'_2\phi_2)^{-1}(\phi'_2 u_2\Sigma_z(\Omega + (\alpha-1)\gamma_1 u_1\phi_1)) + (\alpha-1)\gamma_2 \end{pmatrix}.$$

The associated differential equation of the system can then be written as

$$\begin{aligned}\dot{\Theta} &= R^{-1}E\mathbf{zz}'(T(\Theta) - \Theta) \\ \dot{R} &= E\mathbf{zz}' - R,\end{aligned}$$

where since $R^{-1}E\mathbf{zz}' = I$ at a fixed point, the appropriate equation to analyze a given equilibrium is

$$\dot{\Theta} = T(\Theta) - \Theta.$$

Proposition 1: To prove existence first note that by construction $u_i u'_i = I$, where I is an $m_i \times m_i$ identity matrix, and consider the following two lemmas:

Lemma 2: If Σ_z is diagonal (A2), then $u_i\Sigma_z^{-1}u'_i u_i\Sigma_z u'_i = I$

Proof: By assumption Σ_z is a diagonal matrix and by construction $u'_i u_i = \Psi$ is a square diagonal matrix with only ones and zeros on the diagonal. Matrix multiplication of diagonal matrices implies that $\Sigma_z^{-1}u'_i u_i\Sigma_z$ can be computed as

$$\begin{aligned} & diag(\sigma_{z,1}^{-1}, \sigma_{z,2}^{-1}, ..., \sigma_{z,k}^{-1}) * diag(\psi_1, \psi_2, ..., \psi_k) * diag(\sigma_{z,1}, \sigma_{z,2}, ..., \sigma_{z,k}) \\ &= diag(\sigma_{z,1}^{-1}\psi_1\sigma_{z,1}, \sigma_{z,2}^{-1}\psi_2\sigma_{z,2}, ..., \sigma_{z,k}^{-1}\psi_k\sigma_{z,k}) \\ &= diag(\psi_1, \psi_2, ..., \psi_k).\end{aligned}$$

Therefore, $u_i\Sigma_z^{-1}u'_i u_i\Sigma_z u'_i = u_i\Psi u'_i = u_i u'_i u_i u'_i = I$. $\square$

Lemma 3: If Σ_z is diagonal (A2), then $(u_1\Sigma_z u'_i)^{-1}u_i\Sigma_z = u_i$

Proof: Suppose that $(u_1\Sigma_z u_i')^{-1}u_i\Sigma_z = A$, such that $A \neq u_i$, then

$$\begin{aligned}
u_i\Sigma_z &= u_i\Sigma_z u_i' A \\
u_i &= u_i\Sigma_z u_i' A\Sigma_z^{-1} \\
u_i u_i' &= u_i\Sigma_z u_i' A\Sigma_z^{-1} u_i' \\
I &= u_i\Sigma_z u_i' A\Sigma_z^{-1} u_i' \quad \text{by construction} \\
u_i\Sigma_z^{-1}u_i' &= u_i\Sigma_z^{-1}u_i' u_i\Sigma_z u_i A\Sigma_z^{-1}u_i' \\
u_i\Sigma_z^{-1}u_i' &= A\Sigma_z^{-1}u_i' \quad \text{by Lemma 2} \\
vec(u_i\Sigma_z^{-1}u_i') &= vec(A\Sigma_z^{-1}u_i') \\
vec(\Sigma_z^{-1})(u_i \otimes u_i) &= vec(\Sigma_z^{-1})(u_i \otimes A) \\
vec(\Sigma_z^{-1})(u_i \otimes (u_i - A)) &= 0
\end{aligned}$$

Thus, either $u_i = 0$, or $A = u_i$ and a contradiction is established. □

Now consider the equation for the equilibrium parameter beliefs given by

$$\phi_i = [(1-\alpha\gamma_i)u_i\Sigma_z u_i']^{-1}(u_i\Sigma_z\Omega + \alpha\sum_{j\neq i}\gamma_j u_i\Sigma_z u_j'\phi_j)$$

using Lemmas 2 and 3 this can be simplified to

$$\phi_i = (1-\alpha\gamma_i)^{-1}(u_i\Omega + \alpha\sum_{j\neq i}\gamma_j A_{i,j}\phi_j),$$

where $u_i u_j' = A_{i,j}$. Now assuming A1, A3, and $\alpha \neq \frac{1}{\omega}$, it follows that $A_{i,j}\phi_j = 0$ for all j in the above sum. To see this, recall that u_i and u_j may only share the first row in common by A3. Therefore, everywhere $A_{i,j}$ has a non-zero value corresponds to the intercept parameter belief. However, due to A1 ($\mu = 0$) and $\alpha \neq \frac{1}{\omega}$, the equilibrium parameter belief must also be zero. Thus, $(1-\alpha)^{-1}u_i\Omega$ and $\Gamma = (1, ..., 1)'$ are the unique EW-FCE.

Finally, to very that the EW-FCE is equivalent to the REE, substitute in the equilibrium beliefs, weights, and A3 into the ALM

$$\begin{aligned}
y_t &= (\Omega' + \alpha\sum_{i=1}^{k}(1-\alpha)^{-1}\Omega' u_i' u_i)z_{t-1} + w_t \\
y_t &= (\Omega' + \alpha(1-\alpha)^{-1}\Omega' \begin{bmatrix} \omega & \\ & I \end{bmatrix})z_{t-1} + w_t.
\end{aligned}$$

Now using A1 such that $\Omega = (0, \zeta')'$, it follows that the above expression simplifies to

$$y_t = (1-\alpha)^{-1}\Omega' z_{t-1} + w_t.$$

To show that proposed FCE is unique it must be the case that

$$\Delta(\Gamma^*) = I - \alpha \sum_{i=1}^{k} \Sigma_z u_i' (u_i \Sigma_z u_i')^{-1} u_i'$$

is invertible. Applying Lemma 3 it follows that

$$\begin{aligned}
\Delta(\Gamma^*)\Sigma_z &= (I - \alpha \sum_{i=1}^{k} \Sigma_z u_i' (u_i \Sigma_z u_i')^{-1} u_i')\Sigma_z \\
\Delta(\Gamma^*)\Sigma_z &= \Sigma_z - \alpha \Sigma_z \sum_{i=1}^{k} u_i' u_i \\
\Delta(\Gamma^*)\Sigma_z &= \Sigma_z - \alpha \Sigma_z \begin{bmatrix} \omega & \\ & I_n \end{bmatrix} \\
\Delta(\Gamma^*) &= (I - \alpha A),
\end{aligned}$$

which is invertible as long as $\alpha \neq 1$ and $\alpha\omega \neq 1$. □

Proposition 2: From the proof of Proposition 1 it follows that

$$\phi_i = (1 - \alpha\gamma_i)^{-1} u_i \Omega$$

and that the beliefs are equivalent to the REE in aggregate. Substituting these beliefs into the optimal weights condition yields

$$\gamma_i (1 - \alpha)(1 - \alpha\gamma_i)^{-1} \Psi = \Xi + (\alpha - 1) \sum_{j \neq i} \Lambda_j,$$

where $\Psi = \Omega' u_i' u_i \Sigma_z u_i' u_i \Omega$, $\Xi = \Omega' u_i' u_i \Sigma_z \Omega$, and $\Lambda_j = (1 - \alpha\gamma_j)^{-1} \gamma_j \Omega' u_i' u_i \Sigma_z u_j' u_j \Omega$. Assuming A1 and $\alpha \neq \frac{1}{\sum_{i=1}^{k} \gamma_i}$, it follows that intercept belief are zero, which implies $\Lambda_j = 0$ for all $j \neq i$. Now noting that $\Xi\Psi^{-1} = 1$, it follows that

$$\gamma_i (1 - \alpha)(1 - \alpha\gamma_i)^{-1} = 1,$$

which implies the optimal weight is $\gamma_i = 1$. □

Lemma 1: The methods presented follow Wiggins (1990). A bifurcation may be characterized by deriving an approximation to the center manifold of the dynamic system. The dynamic behavior of the system on the center manifold determines the dynamics in the larger system. To demonstrate the derivation of the center manifold, consider the following dynamic system

$$\dot{x} = Dx \qquad x \in \mathbb{R}^n.$$

The system has n eigenvalues such that $s + c + u = n$, where s is the number of eigenvalues with negative real parts, c is the number of eigenvalues with zero real parts, and u is the number eigenvalues with positive real parts. Suppose that $u = 0$, then the system can be written as

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \epsilon), \\ \dot{y} &= By + g(x, y, \epsilon), \qquad (x, y, \epsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}, \\ \dot{\epsilon} &= 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} f(0,0,0) = 0, &\qquad Df(0,0,0) = 0, \\ g(0,0,0) = 0 &\qquad Dg(0,0,0) = 0, \end{aligned}$$

A and B are diagonal matrices with the corresponding eigenvalues on the diagonal, and $\epsilon \in \mathbb{R}$ is the bifurcation parameter. Suppose that the system has a fixed point at $(0, 0, 0)$. The center manifold is defined locally as

$$W^c_{loc}(0) = \{(x, y, \epsilon) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R} \mid y = h(x, \epsilon), |x| < \delta, |\epsilon| < \delta, h(0, 0) = 0, Dh(0, 0) = 0\}.$$

The graph of $h(x, \epsilon)$ is invariant under the dynamics generated by the system, which gives the following condition:

$$\dot{y} = D_x h(x, \epsilon)\dot{x} + D_\epsilon h(x, \epsilon)\dot{\epsilon} = Bh(x, \epsilon) + g(x, h(x, \epsilon), \epsilon). \tag{30}$$

The equation can be used to approximate $h(x, \epsilon)$ to form $f(x, h(x, \epsilon), \epsilon)$, which gives the dynamics of the system on the center manifold and allows for a bifurcation to be identified. The conditions for the existence of a supercritical pitchfork bifurcation at (0,0,0) are

$$\begin{array}{lll} f(0,0,0) = 0 & \frac{\partial f}{\partial x}(0,0,0) = 0 & \frac{\partial f}{\partial \epsilon}(0,0,0) = 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0,0) = 0 & \frac{\partial^2 f}{\partial x \partial \epsilon}(0,0,0) \neq 0 & \frac{\partial^3 f}{\partial x^3}(0,0,0) < 0. \end{array}$$

To apply the center manifold reduction technique to the T-map given by equation (21) and (20) it must be put into the normal form of (29). This is done by translating the fixed point $a_i = 0, b_i = \zeta_i/(1 - \alpha)$, and $\gamma_i = 1$ for $i = 1, 2$ to lie at the origin and finding a linear transformation V to put the eigenvalues of the system into the matrices A and B. Let the translated fixed point at the origin be represented by Θ^*. Then calculating $(DT - I)|_{\Theta^*}$, the Jacobian of the T-map at the fixed point, let V be a linear transformation such that

$$V^{-1}(DT - I|_{\Theta^*})V = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is a zero matrix and B is a diagonal matrix with the stable eigenvalues on the diagonal. The T-map in normal form is thus

$$\begin{aligned} \dot{u} &= Au + f(u, v, \epsilon) \\ \dot{v} &= Bv + g(u, v, \epsilon) \\ \dot{\epsilon} &= 0 \end{aligned}$$

where f and g are the terms of order two and higher. Now that it is in normal form, the center manifold can be approximated using a Taylor expansion by taking derivatives of equation (30). The second order approximation of center manifold is

$$f(u, h(u, \epsilon), \epsilon) \approx 2u\epsilon - \frac{\left(\frac{1}{12} - \frac{i}{6}\right) u^3}{\sigma_2^2 \zeta_2^2} - \frac{\left(\frac{1}{6} - \frac{i}{3}\right) u^3 \epsilon}{\sigma_2^2 \zeta_2^2}.$$

The above equations satisfy the conditions for the existence of a supercritical pitchfork bifurcation. □

Proposition 4: The result follows directly from Lemma 1. The existence of a supercritical pitchfork bifurcation of the Fundamental OW-FCE steady state at $\alpha = \frac{1}{2}$ implies that steady state is stable under learning for $\alpha < \frac{1}{2}$ and that two new E-stable equilibrium have come into existence.

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