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Sales Spotter: An Algorithm to Identify Sale Prices in Point-of-Sale Data

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Abstract: This paper develops an algorithm, called the sales spotter, which identifies the sale prices in the transaction price series provided in point-of-sale data. The goal of the sales spotter is to identify the maximum number of sale prices while minimizing the incorrect attribution of non-sale price reductions to sale prices. The spotter is developed and the values of its parameters are selected by analysing around 7.5 million flagged sales in a US supermarket scanner data. At the optimal values of the parameters, the spotter identifies 84% of authentic flagged sale weeks in the data.

Keywords: Promotional price, regular price, shelf price, sales filter, scanner data

JEL Classification Codes: E30, M37

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1 Introduction

The point-of-sale or scanner data sets, constructed from actual transactions recorded by scanning bar codes at the supermarket checkouts, provide detailed transaction information on prices and quantities of different varieties of items in a product category sold in different shopping outlets. These data sets, however, typically do not identify whether the transaction price corresponds to shelf price or promotional price. This may limit the use of these data sets because the distinction between the shelf and sale price may be important in many studies of pricing behaviour at the retail level.[1]

In recent years, a number of studies have used scanner data to investigate the persistence of retail prices, both with and without temporary price changes. These studies develop their own algorithms (henceforth called sales filters) in order to create a price series which reflects the most frequently occurring or representative price in a given period (e.g. Eichenbaum, Jaimovich, and Revelo, 2011; Chahrour, 2011; Hosken and Reiffen, 2004; Kehoe and Midrigan, 2008; Lloyd, Morgan, McCorriston, and Zgovu, 2011; Nakamura and Steinsson, 2008; Campbell and Eden, 2014). The details of the filters vary between the studies, which may make the series of regular prices generated from the filters different from each other, and reflect the ongoing debate on what price constitutes the most relevant price in studying 'sticky prices' in the context of macroeconomics (Chahrour, 2011). Kehoe and Midrigan, Eichenbaum et al. and Chahrour generally create a reference price series, reflecting the relatively persistent prices, from the modal prices and regard the other observed prices within a given window as temporary prices. While Eichenbaum et al. uses a fixed non-overlapping window, Chahrour shows that the use of a rolling window is better for filtering out the temporary price changes from the observed prices. The indicated temporary prices do not necessarily, and were not intended to, correspond to sale prices. For example, 21% of the temporary ('non-reference') prices in Eichenbaum et al. and 12% in Chahrour are actually found to be higher than the corresponding reference prices (i.e. price spikes) when they apply their filter to scanner data sets.

On the other hand, the sales filters in Hosken and Reiffen (2004), Lloyd et al. (2011), Nakamura and Steinsson (2008) and Campbell and Eden (2014) consider any temporary price movements as having occurred due to sales. In general, a price fall by more than a fixed percentage that continues for less than a fixed period of time before reverting to

[1] We use the terms, shelf and regular prices, and promotional and sale prices, interchangeably in this paper.

the original price or a new higher price is identified as a sale. For example, Campbell and Eden, Hosken and Reiffen and Lloyd et al. consider price falls of 10% or more, and with maximum durations of 2 weeks, 1 month and 12 weeks, respectively, as sales. These threshold numbers play a crucial role in the identification of sales. The evidence from the flagged sales of Dominick's Finer Foods data shows that a large number of sales have price dips of less than 10%, and that while a 2 week duration would miss many sales, 12 weeks would be too long and would wrongly identify non-sale price reductions as sale prices. Nakamura and Steinsson report that if the window of comparison period is increased from 1 month to 5 months and if only the V-shaped (or U-shaped) price changes are considered to be sales, the median frequency of price change decreases from 15.3 to 11.4 in the US during 1998-2005. Chahrour (2011) argues that the requirement of full price recovery in order to consider the end of a sale identifies fewer sales and, in turn, leads to lower estimates of price rigidity.

In scanner data which records transactions at weekly frequencies, an observed price is typically either a regular price or a sale price. This is because retail prices are usually set once a week at most, which is regarded by various authors as an important advantage of using scanner data sets for studying microeconomic adjustment of prices (see, for example, Chevalier, Kashyap, and Rossi, 2003; Dutta, Bergen, and Levy, 2002; Kehoe and Midrigan, 2008; Levy, Chen, Muller, Dutta, and Bergen, 2010). While most scanner data sets do not provide an indicator variable identifying whether the transaction price corresponds to regular or sale prices, an exception to this is the data on Dominick's Finer Foods, a retail chain in Chicago in the US.[2] Although, as reported by the data provider, Dominick's data only partially flags the sales, there are more than 7.5 million flagged sales in this data. We use the information from flagged sales in the Dominick's data to understand the key features of sales (such as, duration of sales, magnitude of price dips and percentage recovery of prices) and to develop an algorithm, the "sales spotter", along with its parameters to identify the sale prices in the data.

Furthermore, we have set the values of the parameters of the spotter after analysing the behaviour of the flagged sales in the Dominick's data. To give an example, for around 17% of the flagged sales in the Dominick's data, after the sale is over, the new regular prices are lower than the regular prices before the sale. We introduce a parameter which is related to the minimum percentage recovery from the initial dip to consider the end of a sale. While Kehoe and Midrigan (2008) and Lloyd et al. (2011) allow for lower recovery

[2] The data is available at http://research.chicagobooth.edu/marketing/databases/dominicks/index.aspx.

after the sale, Hosken and Reiffen (2004), Nakamura and Steinsson (2008), Campbell and Eden (2014) consider a price fall a sale only when the returned price is at least as high as the pre-sale regular price. In agreement with Chahrour (2011), we find that the restraint of full price recovery leads to missing a large number of authentic sales.

The task of identifying sales in point-of-sale data is not straightforward. While the majority of sale prices are of a regular type in the sense that price reduction is large, the duration of the reduced price is short and the price returns to its original price after the sale is over (U-shaped sales); there are also a large number of sales where the price dips are very small (around 15% of Dominick's flagged sales have price dips of 5% or less), the duration is long (around 5% have a duration of five weeks or more), and prices do not return to their original pre-sale prices (around one-third of the flagged sales return to a different price). The problem lies in identifying the latter types of sales, because in many cases these sales are indistinguishable from non-sale price changes. Further complications arise due to large number of missing prices, price variations within a single sale and frequent price spikes. For instance, if price spikes are not accounted for, a filter may identify a post-spike regular price as a sale price.

One might design an algorithm to capture the sales of very small price dip and long duration, but it would at the same time capture a large number of regular price reductions. The key to a good spotter is to draw a balance so that it maximizes the correct spotting of sales and minimizes the incorrect identification of non-sale price reductions. In order to achieve this, we incorporate four parameters in the design of the spotter, where one parameter addresses the missing prices and other three parameters address the heterogeneity in the key features of sales. We determine the values of the parameters through detailed observation of the data, and by setting up optimization rules consisting of an objective function and constraints. While the objective is to identify the maximum number of flagged sales weeks, the constraints ensure that the basic features of the identified sales are in line with those in the flagged sales.

Our approach to selecting the values of the parameters differs from the approach taken by the other filters where the values of the parameters are essentially set arbitrarily. Our study finds that the identification of sales is sensitive to the selected values of the parameters. The identification is particularly sensitive to the parameter related to the maximum duration of sales, suggesting that if the value of this parameter is set too high or low it may affect the construction of the regular price series, and consequently, our understanding of the pricing behaviour of firms.

The next section discusses some important features of the flagged sales in the Dominick's data. Section 3 describes the sales spotter and explains how the optimal values of the parameters of the sales spotter are obtained.[3] Section 4 draws conclusion.

2 Features of Flagged Sales in Dominick's Data

The Dominick's data contains information on prices paid by consumers and traded quantities of 29 products at weekly frequencies for around 100 stores of the chain. The period covered is nearly 400 weeks between September 1989 and May 1997 (although not all products are included from the beginning to the end). The data contains sales flag denoting three types of sales: (1) a discount on the regular price given regardless of the quantities purchased, which is referred to as a discount (D); (2) a discount on the regular price given if at least a certain minimum quantity is purchased, which is referred to as a bonus (B); and (3) a discount for coupon holders (C). The coupon transactions account for only 0.47% of all flagged sales and are not considered in our analysis. The product category, cigarettes, is excluded because the number of flagged sales in this category is negligible (only 21 flagged sales).[4]

In spite of the fact that not all sales are flagged, as reported by the data provider, a large number of sales — both in terms of the absolute number and the percentage of transactions — are flagged. There are 3,175,465 discounts and 4,391,937 bonuses, totalling over 7.5 million sales, flagged in the data (see Table 1). These sales cover 13.66 million weeks of transactions, accounting for 14.92% of total weekly transactions reported in the data. However, we detect some inconsistencies in the flagging of sales. These inconsistencies include the flagged sales where (1) the magnitude of a price reduction is non negative in the first period of the flagging of a sale (16.13% of flagged sales) and (2) the flagging of a sale not ending though the price has returned to at least the pre-sale regular price (2.93%).[5] These inconsistent flagged sales have been excluded from the calculations shown in Table 1 and Figure 1.

[3]The algorithm is written in the MATLAB program.

[4]The papers that use the Dominick's sales flag to study retail pricing behaviour include Chevalier et al. (2003) and Levy et al. (2010).

[5]Note that because sales are inconsistently flagged does not mean that all sale weeks within these sales are also inconsistently flagged. The inconsistency may happen when the flagging is shown mistakenly one week before or after the actual occurrence of sales. Our estimates show that the 10.07% of the flagged sales weeks are inconsistently flagged.

Table 1: Features of Flagged Sales in Dominick's Data

Products	Discounts				Bonuses				All Sales: Discounts & Bonuses			
	Total Weekly Transact. ('000)	Average Duration (in weeks)	Average Price Dip (%)†	Average Price Recovery (%)‡	Total Weekly Transact. ('000)	Average Duration (in weeks)	Average Price Dip (%)	Average Price Recovery (%)	Total Weekly Transact. ('000)	Average Duration (in weeks)	Average Price Dip (%)	Average Price Recovery (%)
Analgesics	31.61	1.10	22.36	99.50	91.09	1.90	14.06	99.63	122.70	1.70	16.20	99.60
Bath Soap	6.64	1.05	24.76	100.00	17.65	1.87	16.83	99.65	24.29	1.65	19.00	99.81
Beer	6.62	1.21	12.59	101.49	204.92	2.21	15.59	99.74	211.54	2.18	15.50	99.79
Bottled Juices	96.85	1.08	19.55	99.54	201.49	2.81	9.63	98.61	298.33	2.25	12.85	98.91
Cereals	73.16	1.11	32.14	98.28	98.19	1.97	13.30	99.13	171.35	1.61	21.35	98.77
Cheeses	163.93	1.11	18.67	99.10	327.32	2.40	10.35	99.08	491.25	1.97	13.13	99.09
Cookies	122.36	1.12	21.24	99.42	348.12	2.67	12.21	99.22	470.48	2.27	14.56	99.28
Crackers	53.00	1.08	22.04	99.63	128.01	2.90	9.37	98.55	181.01	2.36	13.08	98.87
Canned Soup	131.56	1.07	22.16	99.56	168.47	2.45	9.54	99.40	300.03	1.84	15.07	99.47
Dish Detergents	46.97	1.08	18.84	99.08	76.63	2.39	9.13	99.29	123.60	1.89	12.82	99.21
Front-end-candies	27.98	1.15	32.44	99.84	106.16	2.79	16.98	98.86	134.15	2.45	20.21	99.07
Frozen Dinners	107.12	1.07	24.56	99.10	89.19	2.44	13.42	99.82	196.30	1.69	19.49	99.43
Frozen Entrees	356.96	1.15	29.31	98.90	321.20	2.12	19.93	99.96	678.16	1.61	24.87	99.40
Frozen Juices	81.29	1.13	22.12	99.25	127.39	2.15	13.95	98.54	208.68	1.75	17.13	98.81
Fabric Softeners	39.37	1.06	16.68	99.35	76.85	2.83	9.09	98.75	116.22	2.23	11.66	98.95
Grooming Products	108.84	1.10	22.21	99.15	252.43	1.87	16.24	99.79	361.27	1.64	18.04	99.60
Laundry Detergents	68.66	1.11	22.16	98.70	116.65	2.55	10.95	99.24	185.30	2.02	15.10	99.04
Oatmeal	15.94	1.19	30.14	99.44	22.49	2.71	11.48	100.81	38.43	2.08	19.22	100.24
Paper Towels	29.35	1.13	13.82	99.89	46.62	2.44	9.75	97.88	75.97	1.94	11.32	98.66
Refrigerated Juices	74.03	1.14	24.14	97.92	142.10	2.29	12.30	99.53	216.13	1.90	16.36	98.98
Soft Drinks	1,155.10	1.21	23.14	98.29	655.28	1.83	22.57	99.84	1810.38	1.43	22.94	97.77
Shampoos	154.64	1.11	25.20	98.70	294.51	1.76	19.81	99.80	449.15	1.54	21.66	99.42
Snack Crackers	101.21	1.10	19.91	99.48	180.71	2.98	10.60	98.42	281.91	2.31	13.94	98.80
Soaps	28.48	1.07	16.27	100.00	79.64	2.67	7.81	100.12	108.12	2.25	10.04	100.09
Toothbrushes	61.42	1.08	24.86	98.98	99.13	1.94	20.98	98.67	160.56	1.61	22.46	98.79
Canned Tuna	13.75	1.13	22.42	98.57	108.80	2.99	7.61	99.04	122.55	2.78	9.27	98.99
Toothpastes	107.91	1.10	20.70	99.61	130.59	1.97	16.40	99.33	238.50	1.58	18.34	99.46
Bathroom Tissues	42.26	1.16	15.97	99.32	48.79	2.59	11.91	98.77	91.05	1.92	13.79	99.03
All Products*	3,175.47	1.15	23.25	98.82	4,391.94	2.28	15.04	98.91	7567.40	1.81	18.49	98.87

† Price dip is measured as the fall in the price in the first period of sale as a percentage of the last pre-sale regular price.
‡ Price recovery is the ratio of the first post-sale regular price to the last pre-sale regular price multiplied by 100.
* Total weekly transaction is obtained by summing the weekly transactions across products. Other figures are weighted averages where weights are the shares of the corresponding weekly transactions.

We study three features of the flagged sales: (1) the duration of sales in weeks, (2) the magnitude of price dip, which is measured as the fall in price in the first week of sales as the percentage of the preceding pre-sale regular price, and (3) the recovery of price after the sale is over, which is measured as the percentage of the pre-sale regular price. Table 1 shows that the average duration of sales is 1.15 weeks for discounts and 2.18 weeks for bonuses (medians are 1 week and 2 weeks, respectively). We find that 96.82% of discounts, 68.16% of bonuses and 80.10% of all sales lasted for 2 weeks or less. There is little variation in the duration of discounts across products. However, there are variations in the duration of bonuses, with the lowest duration being 1.76 weeks for shampoos and the highest being 2.99 weeks for canned tuna.[6]

While the duration is considerably lower for discounts compared to that for bonuses, the magnitude of price dip is greater for discounts than it is for bonuses. The average price fall is 23.25% for discounts and 18.49% for bonuses. However, there are large variations in the magnitude of price dip with around 31.78% of sales having price dips of less than 10% and 40.01% having price dips of more than 30% (see also the distribution in Figure 1). The recovery of prices after sales, on the other hand, are around the same, both for different products and types of sales. On average, the post-sale regular prices are around 99% of the pre-sale regular prices. The recovered price after the sale is 90-110% of the pre-sale regular price for 86.06% of sales, while it is exactly at 100% for 66.42% of sales. The price recovery is less than 100% for 17.01% of sales.[7]

3 Identifying Sale Prices

3.1 The Sales Spotter

Let p_t be the observed price in period t.[8] $p_t \in (r_t, s_t)$, where r_t and s_t denote the regular and sale prices in period t, respectively. The primary aim of the spotter is to identify whether p_t is a regular or a sale price. For this purpose, in a given period t, the spotter evaluates p_t in relation to its adjacent price observations using some specific rules. These

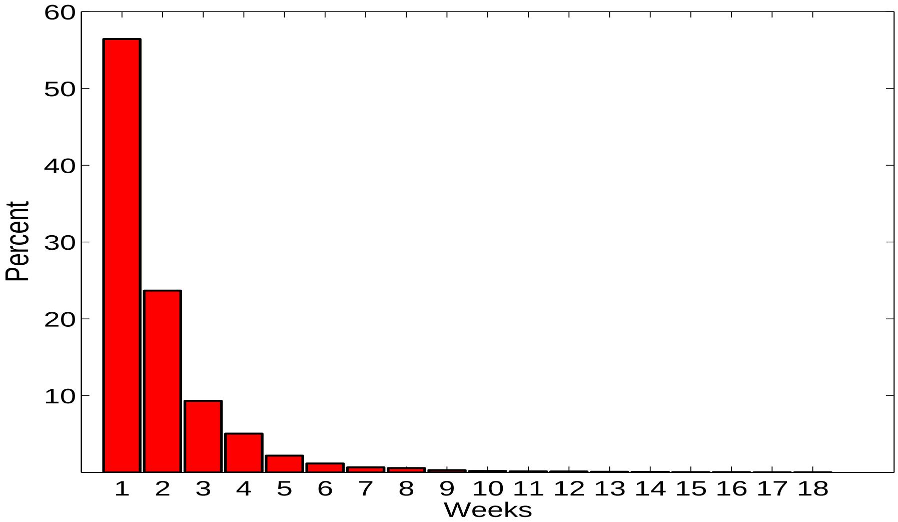
[6]See appendix table A-1 for the estimates of the standard deviations of the duration, price dip and recovery price of sales.

[7]In figure 1(b), 5 on the x-axis indicates the range $(0-5]$, 15 indicates $(10-15]$, 25 indicates $(20-25]$ and so on. Similarly, in figure 1(c), on the x-axis, $30 = (0-30]$, $50 = (30-50]$, $\ldots$, $110 = (90-110]$ and so on.

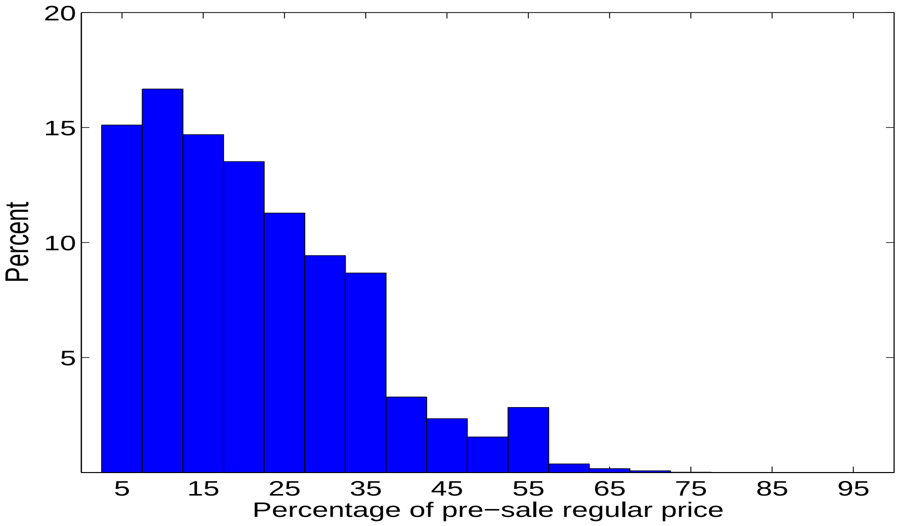
[8]In scanner data, the observed price corresponds to the transaction price of a distinct item sold in a distinct outlet in a given period (typically, in a week).

Figure 1: Distribution of Key Features of Sales

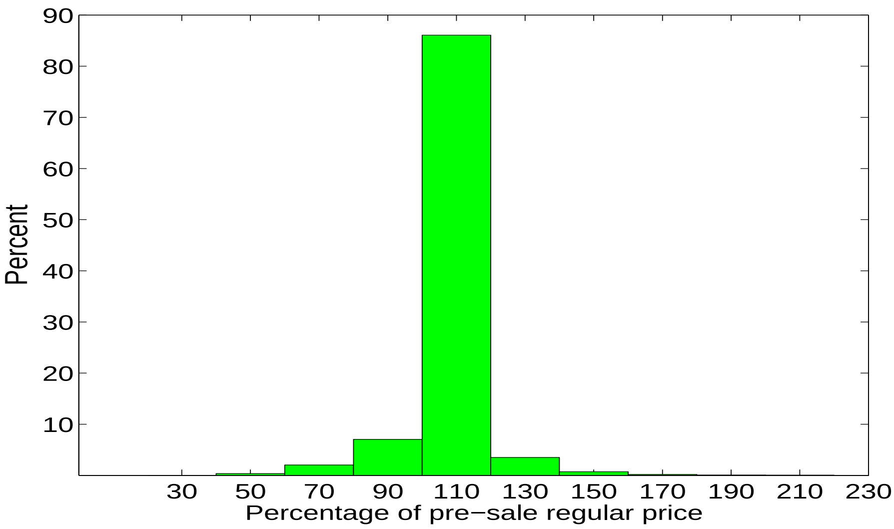
(a) Duration of sales



(b) Magnitude of price dip



(c) Price recovery of sales



rules are reflections of the basic features of sale price behaviour observed from scanner data. The spotter has 4 parameters which provide flexibility in the application of the rules to the data. The parameters are defined as follows:

1. M: the maximum number of periods the spotter is set to search backwards in time for an observed (non-missing) price;

2. K: the maximum number of periods for which a series of reduced prices are treated as sale prices;

3. E: minimum reduction of price as a proportion of the pre-sale regular price to be considered as a sale;

4. F: minimum recovery of the initial price dip as a proportion of the pre-sale regular price to consider the end of a sale.

The details of the spotter are as follows:

(1) Initial price: If p_t is the first observed price, then $p_t = r_t$, i.e. the initial price is always assumed to be a regular price. This is because it is not possible to compare the first observed price with the preceding adjacent prices, which are unknown.

(2) Let $m \in \{1, 2, \ldots, M\}$. If p_{t-m} $\forall m = 1, 2, \ldots, M$ are missing, then $p_t = r_t$. The spotter does not impute missing prices, instead, if it detects a missing price, it moves one period backward, up to the maximum of M periods.

(3) Let r_{t-m} be the regular price nearest to p_t (in terms of time period), where $m \leq M$. Also, let $E \in (0, 1)$. If $p_t \geq (1 - E)r_{t-m}$, then $p_t = r_t$. The parameter E is included in the spotter so that very small price decreases, such as those having taken place due to rounding of numbers or typographical errors, are not identified as a sale price.

(4) Now to price decreases, where $p_t < (1 - E)r_{t-m}$:

   (a) If $p_t = r_{t-2}$, i.e. $p_{t-1} = r_{t-1}$ is a price spike, then the spotter sets $p_t = r_t$. The spotter identifies one-period price spike and does not attribute the return of price from the spike to sales.

   (b) This is the case where $p_{t-1} = r_{t-1}$ is not a price spike. The price reductions that prevail for up to K periods are sale prices and price reductions that continue for more than K periods are identified as permanent reductions, with the reduced prices considered to be the new regular prices. In addition to $p_t < (1-E)r_{t-m}$, let $\{p_{t+1}, p_{t+2}, \ldots, p_{t+k}\} < (1-E)r_{t-m}$. The spotter distinguishes between two possible scenarios in (b)

(i) The first scenario is $p_{t+k+1} \geq r_{t-m}$, i.e. the price reduction continues for k periods and afterwards the price returns to a price at least as high as the pre-sale regular price. In this case, for all $k \leq K$, $[p_t \quad p_{t+1} \quad \ldots \quad p_{t+k} \quad p_{t+k+1}]' = [s_t \quad s_{t+1} \quad \ldots \quad s_{t+k} \quad r_{t+k+1}]'$ and $[r_t \quad r_{t+1} \quad \ldots \quad r_{t+k}]' = [r_{t-m} \quad r_{t-m} \quad \ldots \quad r_{t-m}]'$. And for all $k > K$, $p_t = r_t$ and, at this stage, the spotter does not identify $\{p_{t+1}, p_{t+2}, \ldots, p_{t+k}\}$, instead moves back to period $t+1$ and conducts the same operation from the beginning of 3. Suppose, $\{p_{t+1}, p_{t+2}, \ldots, p_{t+k}\} = r_t$, then $[p_{t+1} \quad \ldots \quad p_{t+k} \quad p_{t+k+1}]' = [r_{t+1} \quad \ldots \quad r_{t+k} \quad r_{t+k+1}]'$.

(ii) The second scenario is $p_{t+k+1} < r_{t-m}$, i.e. the returned price is less than the pre-sale regular price. At this point, the spotter further distinguishes between two possible cases depending on the extent to which p_{t+k+1} is lower than r_{t-m}. Let $F \in (0,1)$. The spotter finds whether p_{t+k+1} is more or less than F times the initial drop in price from r_{t-m}.
Case 1: p_{t+k+1} is close enough to the pre-sale regular price, r_{t-m}, to be considered the new regular price. More formally, if $(1-F)(r_{t-m} - p_t) \geq r_{t-m} - p_{t+k+1}$, for all $k \leq K$, $[p_t \quad p_{t+1} \quad \ldots \quad p_{t+k} \quad p_{t+k+1}]' = [s_t \quad s_{t+1} \quad \ldots \quad s_{t+k} \quad r_{t+k+1}]'$. And for all $k > K$, $p_t = r_t$ and the spotter moves back to period $t+1$ and conducts the same operation from the beginning of 3.
Case 2: In the case where p_{t+k+1} is less than F times the initial drop in price from r_{t-m}, the variation is too small to pull p_{t+k+1} out of the reduced prices and to consider it the new regular price. That is, if $(1-F)(r_{t-m} - p_t) < r_{t-m} - p_{t+k+1}$, for all $k \leq K$, the spotter at period t does not assign whether $\{p_t, p_{t+1}, \ldots, p_{t+k}\}$ are regular or sale prices. Instead, the spotter adds one period to k and conducts the same operation again from the beginning of point 4. This operation is continued for a maximum of K periods. And, as before, for all $k > K$, $p_t = r_t$ and the spotter moves back to period $t+1$ and conducts the same operation from the beginning of 3.

(5) End-of-sample price: Let p_t be the last observed price. The identification of a sale

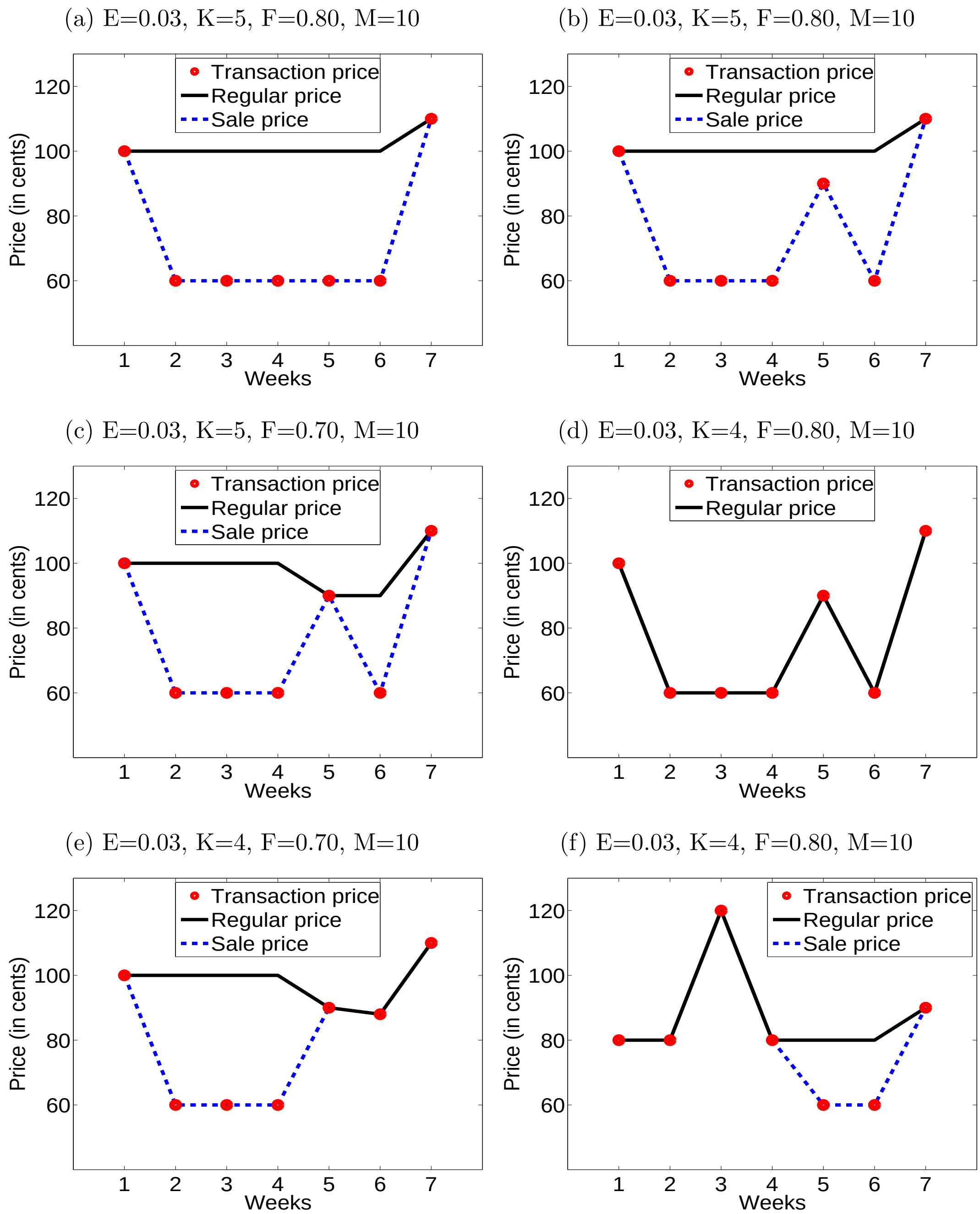
price is based on the preceding prices. If $p_t \geq (1-E)r_{t-m}$, then $p_t = r_t$ and if $p_t < (1-E)r_{t-m}$, then $p_t = s_t$.

Figure 2 shows examples of how the spotter works and what roles the parameters play in the identification of sale prices when there are no missing prices. These figures provide prices for 7 consecutive weeks. Since there are no missing prices, the parameter M does not play any role in identifying the sale prices in Figure 2. Let us begin with the simple case, shown in figure 2(a), where the parameters (E, K, F) are $(0.03, 5, 0.80)$ and the prices in weeks 1–7 are: 100, 60, 60, 60, 60, 60 and 110 cents. The price from week 1 to week 2 drops by 40%. Since this drop is larger than $(100 \times E)\%$, the spotter carries out further evaluation in order to find whether the reduced price is a sale or a regular price. The reduced price prevails from week 2 to 6 with no change in price in-between this period. Given that $K = 5$, this period of 5 weeks falls within the maximum allowable period for the reduced prices to be considered as sale prices. Hence the spotter identifies the prices in week 2–6 as sale prices.

The next 3 figures, 2(b), 2(c) and 2(d), demonstrate the role of parameter F in the determination of sales. The prices are the same as they were in figure 2(a) with only one difference; that the week 5 price is now 90 cents. While the parameter E remains the same at 0.03, the parameters K and F vary between the figures. In figure 2(b), given that the drop in price from week 1 to week 2 is 40 cents and $F = 0.80$, at least $(0.80 \times 40) = 32$ cents are required to be recovered to consider the increased price the new regular price. But the price recovery is 30 cents, and hence the sale continues in week 5. However, in figure 2(c), $F = 0.70$ and, therefore, $(0.70 \times 40) = 28$ cents of recovery is required to put an end to the sale, and hence 90 cents in week 5 is a post-sale regular price. Note that in figure 2(d), while the prices are the same as in 2(b) and 2(c), K is 4 weeks and $F = 0.80$. Since the price recovery in week 5 is not large enough for it to be a regular price (at least 32 cents of recovery is required) and the reduced prices prevail for more than 4 weeks, the reduced prices are not identified as sale prices.

Figure 2(e) differs from the previous three figures with respect to the price in week 6, which is now 88 cents. Hence, price drops by only 2 cents from week 5 to 6. This drop is less than the $E \times 90 = 2.7$ cents price drop required for the price to be considered a sale price. Hence, in contrast to that in figure 2(c), the week 6 price is a regular price. In figure 2(f), the price in week 3 is treated as a temporary spike and, therefore, the price reduction in week 4 is not treated as a sale. Hence, the price in week 4 is considered a regular price

Figure 2: Sales Spotter and its Parameters



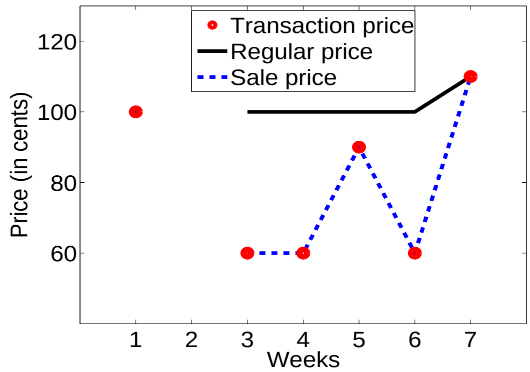
(a) E=0.03, K=5, F=0.80, M=10
(b) E=0.03, K=5, F=0.80, M=10
(c) E=0.03, K=5, F=0.70, M=10
(d) E=0.03, K=4, F=0.80, M=10
(e) E=0.03, K=4, F=0.70, M=10
(f) E=0.03, K=4, F=0.80, M=10

and then the subsequent price reductions are evaluated to find whether they are regular or sale prices. The examples in figures 2(a)–2(e) show that the selection of parameters are important in the identification of sales. The general idea would be to set the parameters K, F and E at a level so that the maximum number of sale prices are identified, while at the same time regular prices are not mis-identified as sale prices. However, these examples do not consider the presence of missing prices which are quite prevalent in scanner data sets and may introduce additional complication in the identification of sales.

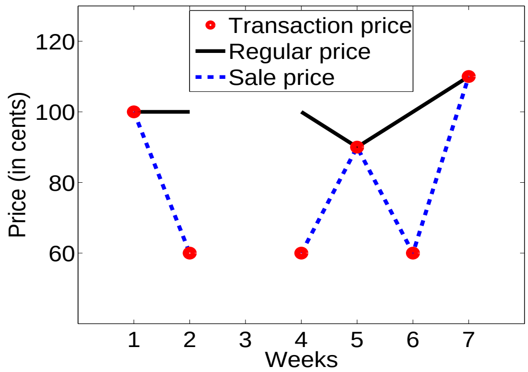
Figure 3 shows how missing prices impact on the identification of sale prices. The observed prices in these figures are the same as those in figures 2(b)–2(d) except that one of the prices is missing in each figure. Although the goal of the spotter is not to impute the missing prices, an implicit assumption about these prices are required in order to make the spotter work. In general, it is assumed that a missing price is the same as the price observed in the preceding week. For example, in figure 3(a), the price in week 2, which is between a regular and a reduced price, is missing. In week 3, the spotter attempts to compare the price for this week with the price in week 2. However, since the price is missing in week 2, the spotter searches backward up to $M = 10$ weeks for an observed price. In this case, a price is observed in week 1. The spotter assumes that the price in week 2 is a regular price and is the same as that in week 1, and counts the parameter K from week 3.

Figure 3: Missing Prices and the Sales Spotter

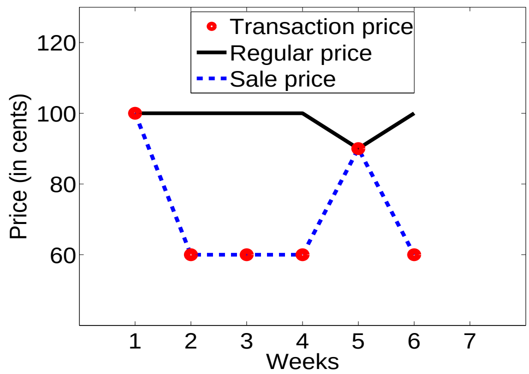
(a) E=0.03, K=4, F=0.80, M=10



(b) E=0.03, K=4, F=0.70, M=10



(c) E=0.03, K=4, F=0.70, M=10



In figure 3(b), the missing price is between two reduced prices, so the spotter assumes that the price in week 3 is also reduced and is the same as week 2. The sale prevails in weeks 2-4. In figure 3(c), the observed price in week 7, the last week in the sample, is missing. The price in week 6 is a sale price because the price drops by $(90 - 60) = 30$ cents between weeks 5 and 6, which is more than the $E \times 90 = 2.7$ cents required to be considered a sale price. The spotter assumes that the price in week 7 is the same as week 6 and identifies week 7 as a sales week.

3.2 Determining the Values of the Parameters of the Spotter

The parameter M is included due to the presence of frequent missing observations in scanner data. As expected, the number of flagged sales identified increases with M, though the increment is small and the rate of increase falls steadily with each subsequent increase in M. This holds for all 28 products. For the whole data set, an increase in M from 4 weeks to 13 weeks adds about 1% to the identification of the flagged sale weeks, then from 13 to 52 weeks adds less than 0.4%. Given such low increments and the possibility that a larger gap may identify non-sale price reductions as sales, the parameter M is set at 13 weeks, i.e. a quarter of a year. The parameter E is set so that any small or, rather, very small, price reductions documented in scanner data due to typographical error or rounding of numbers are not identified as sales. The flagged sales with an initial price dip of less or equal to 1%, 2%, 3%, 4% and 5% account for 0.98%, 3.76%, 7.45%, 11.17% and 15.11% of sales, respectively. This empirical observation leads us to choose $E = 0.02$ for all the products, implying that a price fall of more than 2% would undergo further evaluation for identification as a sale event. We have conducted sensitivity analysis with $E = 0.01, 0.04$ and $M = 52$ weeks. These are discussed later in the section.[9]

The values of two other parameters, K, the maximum duration of a sale, and F, the minimum price recovery to call the end of a sale, are chosen using an optimization rule. The objective of the optimization problem is to identify the maximum number of flagged sale weeks, subject to the constraints that the deviations between the flagged and identified sales in terms of average duration and price dips are equal or at the minimum. In other words, the constraints are set so that the spotter identifies sales whose main features are

[9]The studies that have considered only the price reductions of 10% or more as sale prices might have excluded a large number of sales from their analysis (e.g. Hosken and Reiffen, 2004; Lloyd et al., 2011; Campbell and Eden, 2014).

around the same as those of the flagged sales. The optimization problem is as follows:

$$max_{K,F} \quad \eta(K,F) = N_{I\cap f}(K,F)/N_f \tag{1}$$

subject to,

$$c_1 \;:\; mdur_I - mdur_f \leq 0 \tag{2}$$

$$c_2 \;:\; -mdip_I + mdip_f \leq 0 \tag{3}$$

$$c_3 \;:\; K - 18 \leq 0 \tag{4}$$

$$c_4 \;:\; -F + 0.25 \leq 0 \tag{5}$$

In in equation (1), N_f is the number of flagged sale weeks and $N_{I\cap f}(K,F)$ is the number of flagged sale weeks which are identified by the spotter. Hence, $\eta(K,F)$ denotes the proportion of flagged sale weeks which are identified by the spotter. In equations (2) and (3), $mdur_f$ and $mdur_I$ denote the mean durations, and $mdip_f$ and $mdip_I$ denote the mean of percentage price dips of the flagged and identified sales, respectively. Note that while $mdur_f$ and $mdip_f$ are fixed as obtained from the Dominick's data set, $mdur_I$ and $mdip_I$ change with the changes in the value of K and F.

In the objective function, a larger K is expected to identify more flagged sale weeks, although at a decreasing rate; $\delta\eta/\delta K = \eta'_K > 0$ and $\delta^2\eta/\delta K^2 = \eta''_K < 0$. However, a larger K would attribute some regular price falls to sale prices. This would increase the average duration and decrease the average price dip of the identified sales since, in general, the durations are higher and price falls are lower for regular price reductions. A larger F may increase the identification of the flagged sale weeks by allowing a sale to continue when there are small variations within the sale period, although it may attribute some post-sale regular prices to sale prices. If, however, F is too large (e.g. $F = 1$, implying at least 100% recovery), the effect may be the opposite because it may increase the duration which may become too large to call the price drop a sale. Hence, a priori, the direction of the effect of F on the identification of sales is less clear than that of K.

Turning our attention to the constraints now, c_1 restricts the mean duration of the identified sales to exceed that of the flagged sales as we increase K and F. c_2 restricts the mean price dip of the identified sales from falling below that of the flagged sales as we increase K and F. Hence, these two constraints hold two main features of sales to be around the same in the identified and flagged sales and, given that regular price falls are lower and last longer, provide a check to the false identification of non-sale price reductions as sale prices. c_3 restricts K to be no more than 18 weeks and c_4 requires that the price recovery is above 25% to consider the end of a sale.[10]

Figure 4 shows the combined objective and constraint functions of all 28 products. These values are obtained by taking the weighted average of the product-wise objective and constraint values, where weights are the proportion of flagged sales corresponding to each product. Figures 4(a), 4(d) and 4(g) show the actual values of the objective (η), c_1 and c_2 constraints of the "combined product", respectively, for feasible K and F. We fit linear, quadratic and cubic functions on the objective, c_1 and c_2 values and find that the cubic function provides a better fit for all three sets of values. The adjusted-R^2 of the cubic fit of the objective, c_1 and c_2 values are 98.07, 98.86 and 94.32 percent, respectively. This leads us to choose the cubic fits of all functions while solving the optimization problem. The figures show that the objective and constraint values are more sensitive to K than they are to F.[11]

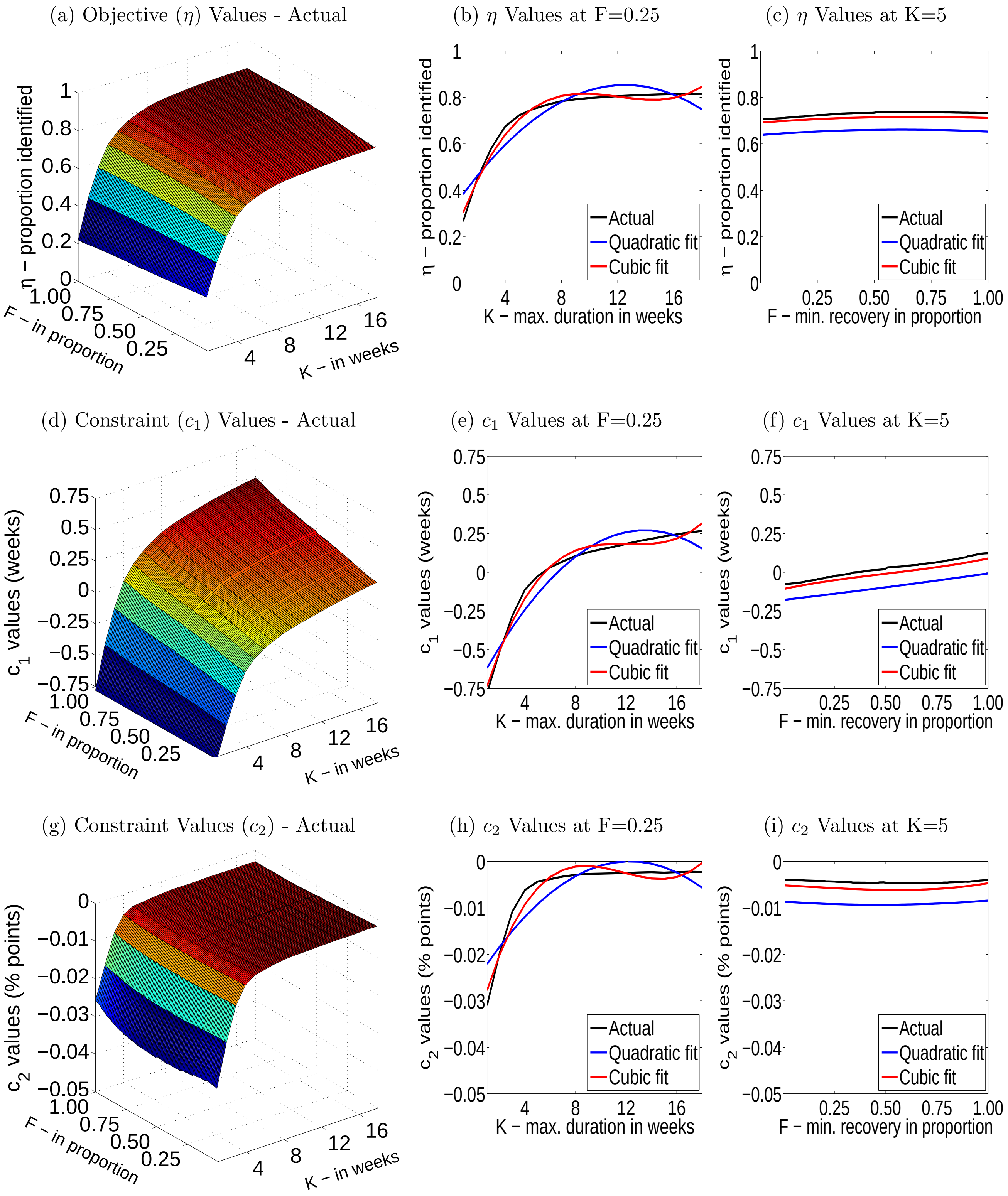
Figures 4(b), 4(e) and 4(h) show the actual and fitted values of the objective, c_1 and c_2 functions at $F = 0.25$, respectively. These figures show that the objective and constraint values grow rapidly for up to $K = 5$ weeks, then the rate of increase slows down or nearly stalls. For example, with regard to the objective function, the identified flagged sale weeks are 58.0, 72.3, 76.9 and 79.2 percent of the total flagged sales weeks for $K = 3, 5, 7$ and 9, respectively. Figures 4(c), 4(g) and 4(i) show the actual and fitted values of the objective, c_1 and c_2 functions at $K = 6$, respectively. These figures show that while the effect of parameter F on the objective and c_2 values are small, its effect on c_1 values is non-trivial and positive. The positive effect implies that a larger F tends to identify price reductions of a larger duration, some of which may be non-sale price falls, as sale prices. The figures indicate that the values related to the duration of sales — the parameter K and constraint c_1 — would play an important role in the identification of sale prices.

Table 2 shows the optimal values of K and F, the corresponding objective value and indicators showing which constraints are active in the solution to the optimization problem. The optimal K and F for the combined product are 6 weeks and 0.25, respectively. The corresponding objective value is 0.75. The active constraints in the solution are c_1 and c_2, where the former restrains the mean duration of the identified sales to be the same

[10]Only 0.33% and 0.01% of sales have duration of more than 18 weeks and price recovery of less than 25%, respectively.

[11]Given that the objective and constraint functions are smooth and bounded in a fixed domain, a solution to the optimization problem exists.

Figure 4: Objective and Constraint Values Aggregated over All Products



of the flagged sales and the latter sets the requirement of at least 25% price recovery for declaring the end of a sale. Looking at the product-wise results, the optimal K ranges from 3 to 9 weeks, with K at $5-6$ weeks for 16 out of 28 products. The optimal F is less than 0.50 for 23 products, of which F takes the minimum of 0.25 for 19 products. The largest optimal F is found to be for analgesics, 0.67. While constraint c_1 is binding for 22 products, constraint c_2, ensuring that the magnitude of price dips are around the same in the flagged and identified sales, is active for 13 products. Both c_1 and c_2 are found to be active for 7 products.[12]

The average objective value at the optimal solutions of all products is 0.738, implying that around 73.8% of the flagged sale weeks are identified by the spotter. We find that around 10.1% of flagged sale weeks are inconsistently flagged because they correspond to sales whose price in the first or last period of sale is larger than the regular price preceding the sale. Since these flagged sales are not intended to be identified, the spotter identifies around 84.0% of the authentic flagged sale weeks.

Table 3 shows the sensitivity of the optimal values with respect to changes in the constraints and the values of the parameters M and E. With respect to c_1, when we let the average duration of the identified sales to exceed that of the flagged sales by one standard deviation of the duration of the flagged sales, the optimal K, F, and the corresponding objective values of the combined product become 8, 0.25 and 0.784, respectively. Hence, when compared with the base case, this perturbation in c_1 increases the objective value by 3.4 percentage points. If we let the constraint c_2 exceed by two standard deviations of the duration of the flagged sales, then the objective value increases to 0.822. The results show that the identification of sales is sensitive to c_1, because c_1 restricts K from becoming too large. If c_1 is removed from the optimization problem, the optimal K and F, and the objective value become 10, 1.0 and 0.828, respectively.

We conduct similar sensitivity analysis with respect to constraint c_2, where we perturb the constraint by adding one and two standard deviations of the size of the price dips of the flagged sales and, finally, by removing the constraint from the optimization problem. Since c_2 is not binding in the combined product, any perturbation to c_2 would not impact the optimal solution of the combined product. However, the perturbation in c_2 affects

[12]Kehoe and Midrigan (2008) chose 5 weeks and Chahrour (2011) chose 6 weeks for the size of window within which the current price is compared with the forward prices, while Campbell and Eden (2014) consider only 2 weeks and Lloyd et al. (2011) consider price reductions up to 12 weeks as sales. Our finding appears to be supportive of the selection of window size of Chahrour, and Kehoe and Midrigan, although their filters have a different objective and use a different mechanism.

Table 2: Optimal Values of Parameters by Products

Products	Optimal Values†			Active Constraints‡			
	K	F	η	c_1	c_2	c_3	c_4
Analgesics	3	0.67	0.682	0	1	0	0
Bath Soap	4	0.57	0.820	0	1	0	0
Beer	5	0.25	0.811	1	0	0	1
Bottled Juices	6	0.25	0.706	1	0	0	1
Cereals	5	0.45	0.711	1	1	0	0
Cheeses	5	0.25	0.717	1	1	0	1
Cookies	6	0.25	0.778	1	0	0	1
Crackers	7	0.25	0.792	1	0	0	1
Canned Soup	5	0.25	0.699	1	1	0	1
Dish Detergents	5	0.25	0.674	1	0	0	1
Front-end-candies	5	0.47	0.713	0	1	0	0
Frozen Dinners	8	0.25	0.776	1	0	0	1
Frozen Entrees	5	0.25	0.763	1	0	0	1
Frozen Juices	5	0.25	0.715	1	0	0	1
Fabric Softeners	7	0.25	0.654	1	0	0	1
Grooming Products	4	0.51	0.786	0	1	0	0
Laundry Detergents	6	0.25	0.657	1	0	0	1
Oatmeal	9	0.49	0.656	1	0	0	1
Paper Towels	6	0.25	0.630	1	0	0	1
Refrigerated Juices	6	0.25	0.712	1	0	0	1
Soft Drinks	6	0.25	0.746	1	0	0	1
Shampoos	5	0.25	0.773	1	1	0	1
Snack Crackers	7	0.34	0.763	1	1	0	0
Soaps	7	0.25	0.726	1	1	0	0
Toothbrushes	4	0.62	0.758	0	1	0	0
Canned Tuna	6	0.25	0.654	1	0	0	1
Toothpastes	3	0.52	0.720	0	1	0	0
Bathroom Tissues	8	0.25	0.655	1	1	0	1
Average*	6	0.30	0.738	–	–	–	–
Combined**	6	0.25	0.750	1	0	0	1

† For all products, $M = 13$ and $E = 0.02$. Here K is set to the closet integer corresponding to the optimal value and η is the corresponding actual objective value (see appendix table A-3 for the optimal values that includes decimals and the fitted objective values.)

‡ 1 means active constraint and 0 means inactive constraint.

* Weighted average of the optimal values of all products where weights are the proportion of flagged sales.

** Uses the weighted average of objective and constraint functions in the optimization problem where weights are the proportion of flagged sales.

Table 3: Sensitivity Analysis

Constraints & Parameters	All Products K	All Products F	All Products η	Combined Product K	Combined Product F	Combined Product η
Base case[†]						
Combined Prod.	6	0.30	0.738	6	0.25	0.750
Constraint c_1[‡]						
$+ (1 \times \sigma_{dur_f})$	7	0.46	0.783	8	0.25	0.784
$+ (2 \times \sigma_{dur_f})$	8	0.65	0.807	10	0.63	0.822
$= \{\emptyset\}$**	8	0.79	0.809	10	0.69	0.828
Constraint c_2*						
$+ (1 \times \sigma_{mag_f})$	6	0.29	0.753	6	0.25	0.750
$+ (2 \times \sigma_{mag_f})$	6	0.28	0.752	6	0.25	0.750
$= \{\emptyset\}$**	6	0.25	0.750	6	0.25	0.750
Parameter F						
≥ 0.00	6	0.14	0.740	6	0.00	0.731
≥ 0.50	5	0.51	0.736	5	0.50	0.734
≥ 0.75	5	0.75	0.735	5	0.75	0.735
≥ 1.00	4	1.00	0.666	4	1.00	0.667
Parameter E						
$= 0.01$	5	0.32	0.741	5	0.25	0.737
$= 0.04$	6	0.30	0.700	6	0.25	0.696
Parameter M						
$= 52$ weeks	6	0.31	0.759	6	0.25	0.754

[†] In the base case, $E = 0.02$ and $M = 13$ weeks. Here K is set to the closet integer corresponding to the optimal value and η is the corresponding actual objective value.
[‡] σ_{dur_f} denotes the standard deviation of the duration of the flagged sales.
* σ_{mag_f} denotes the standard deviation of the price dip of flagged sales.
** $c_1 = \{\emptyset\}$ and $c_2 = \{\emptyset\}$ mean that the respective constraint is removed.

the optimal solution for 7 products. The average impact of the one- and two-standard deviation perturbations on these 7 products is the increase in the objective values from 0.682 to 0.701 and 0.723, respectively. The removal of c_2 increases the objective value to 0.741. However, as shown in Table 3, the average impact of any perturbation in c_2 on all 28 products is very small. This finding is supportive of our earlier observation that while c_2 — at least for some products — is an important constraint, c_1 plays a more dominant role in the determination of sales.

With regard to constraint c_4, if the minimum price recovery is reduced to 0, the objective value remains around the same as the base case scenario. This is expected because there are only a few sales which have price recovery of less than 25% of the pre-sale regular price. On the other hand, if only the sales with the full price recovery is considered, the objective value for the combined product is reduced to 0.667. Hence, if the spotter had identified only U-shaped sales, it would have missed around 8.3% of sale weeks. The results from the perturbation of c_4 indicate that although there are many sales whose recovered price are lower than the pre-sale regular price, they are probably not far below the pre-sale regular price.[13]

We have also conducted sensitivity analysis to changes in M and E. Although the presence of frequent missing prices makes the inclusion of M in the spotter necessary, the change in its value has a negligible effect on the identification of flagged sales. For example, if M is increased from 13 weeks (the base case) to 52 weeks, the identification of sales weeks increases by only 0.4% points. However, the effect of the change in E on the identification of sales can be significant. This is expected because a large number of sales have very small price dips. We find that if E is increased from 0.02 to 0.04, the spotter misses identifying around 5.4% of the flagged sales weeks.

The flagged sales which are not identified by the spotter include: (1) sale prices which are the initial prices (2.06%); (2) sales prices which have price dips of less than $(100 \times E)$% of the regular price (3.07%); (3) sales whose price recovery is less than $(100 \times F)$% of the regular price (0.01%); and (4) sale prices with durations larger than optimal K (2.83%). Note that the longer duration sales correspond to around 8.02% of flagged sale weeks. Hence, the sales which have durations longer than the optimal duration account for the largest percentage of sale weeks not identified by the spotter.[14]

We summarize our results with regard to determining the value of the parameters as follows: (1) The parameter for the maximum duration of sales plays the most important role in identifying the sale prices. Our evidence from 28 products shows that the optimal value for this parameter ranges from 3 to 9 weeks. The average is 6 weeks, the value that we suggest using if one would prefer to use one value for all products. Since the identification of sales is sensitive to the value of this parameter, sensitivity analysis may be conducted in studies of pricing behaviour where the distinction between the regular and sale prices is important (such as, measuring price stickiness and inflation). The values in the range of 4 – 8 weeks might be a reasonable choice in undertaking the sensitivity analysis, noting that a lower value would miss identifying some sales and a higher value would add some sales at the expense of attributing some non-sale price reductions to sales. (2) Any mechanism to

[13]It is common for the sales filters to consider only the fully recovered price falls as sale prices (e.g Hosken and Reiffen, 2004; Nakamura and Steinsson, 2008; Campbell and Eden, 2014).

[14]The sum of the percentage of unidentified sales would not match exactly with the unidentified sale weeks implied by the objective functions. This is because of the distinction between the percentage of sales and the percentage of sale weeks, overlaps of sales in the above 4 categories and other unknown reasons.

identify sales should be cognizant of the fact that for a large number of sales the recovered price after the sale is lower than the pre-sale regular price. Our analysis indicates that if we set the parameter for the minimum price recovery requirement at 0.25, the spotter maximizes the identification of sales. However, our sensitivity analysis shows that the value of this parameter exerts a significant influence on the identification only when it gets close to 1. This leaves us with some room for flexibility in choosing the value for this parameter, and leads us to suggest that a one-half recovery should be considered large enough to end a sale. (3) A large number of sales have very small price dips and, therefore, the identification of sales is sensitive to the minimum percentage reduction required in price to consider a price reduction a sale. We set this value at 2% based on observing the Dominick's data. Our analysis leads us to suggest that a 5% price drop should be considered large enough to be considered as a sale.

4 Conclusion

Since many scanner data sets do not provide an indicator for whether the transaction took place during a promotion or not, for many studies on retail pricing behaviour, an algorithm is required to be developed to identify sale prices from all transaction prices. This task is not straightforward because there are wide variations in sale prices in terms of the duration of sales, magnitude of price dips and recovered price after the sale. Further complications arise from a large number of missing prices and frequent price spikes. Some features of sale and non-sale price reductions overlap, making their distinctions difficult.

We have developed an algorithm, the sales spotter, coded in the MATLAB programme, in order to identify sale prices in point-of-sale data. The spotter depends on 4 parameters reflecting the main features of sale prices and providing flexibility in the way spotter identifies sales prices from the transaction price series. While one parameter is related to how the spotter handles the missing prices in the data, the other parameters are related to the maximum duration of sales, the magnitude of price dips, and the recovery of price after the sale is over. We set the values of the parameters by analyzing the basic features of around 7.5 million flagged sales in the data, with the goal of maximizing the spotting of sales while preventing the identification of non-sale price changes as sale prices.

We find that among the four parameters, the identification of sales is most sensitive to the parameter related to the maximum duration of sales, K. We find that in our base case scenario, the optimal K is 6 weeks and the spotter identifies around 84% of the authentic sale weeks. The remaining sale weeks were not identified mainly because they correspond to sales which have large durations and very small price dips.

Our sensitivity analysis shows that if we remove the constraint from the optimization problem which prevents the value of K from becoming too large, then the optimal K becomes 10 weeks, leading to the identification of around 93% of authentic sale weeks. We find that if the spotter had only identified sales whose recovery price is at least as high as the pre-sale regular price, then it would have missed spotting around 8.3% of authentic sale weeks. We further find that if the spotter had considered price dips of 4% or more, instead of 2% or more, it would have missed identifying another 5.4% of authentic sale weeks.

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Appendix

Table A-1: Standard Deviation of Some Key Features of the Flagged Sales in Dominick's Data

Products†	Duration (in weeks)			Price Dip(%)			Price Recovery(%)		
	Discounts	Bonuses	All	Discounts	Bonuses	All	Discounts	Bonuses	All
Analgesics	0.23	0.60	0.59	11.52	9.53	11.13	5.85	6.71	6.37
Bath Soap	0.24	0.86	0.76	9.88	10.51	10.75	8.89	8.59	8.72
Beer	0.37	0.73	0.78	10.20	6.29	7.35	13.16	5.48	7.68
Bottled Juices	0.19	1.12	1.15	10.09	7.41	9.74	5.75	6.41	6.14
Cereals	0.21	0.77	0.69	13.31	10.43	14.54	8.86	7.84	8.31
Cheeses	0.31	0.78	0.86	8.82	6.71	9.02	5.80	5.62	5.75
Cookies	0.21	1.06	1.12	10.30	7.20	8.94	6.29	6.71	6.59
Crackers	0.27	1.03	1.15	8.53	10.40	9.35	5.65	26.91	22.26
Canned Soup	0.16	0.79	0.83	7.28	6.32	9.91	3.40	6.17	5.14
Dish Detergents	0.13	0.97	0.95	9.54	6.25	9.64	4.74	6.80	6.13
Front-end-candies	0.23	0.93	0.99	14.56	16.66	15.09	7.81	26.06	22.72
Frozen Dinners	0.14	0.99	0.96	9.33	9.95	11.74	6.98	8.85	8.07
Frozen Entrees	0.31	0.76	0.74	10.66	11.45	12.24	6.44	11.08	9.37
Frozen Juices	0.18	0.72	0.74	9.81	7.32	9.63	6.31	5.22	5.74
Fabric Softeners	0.18	1.56	1.48	9.73	5.97	8.87	5.67	7.09	6.62
Grooming Products	0.23	0.71	0.66	11.61	11.09	11.57	8.50	11.07	10.11
Laundry Detergents	0.21	1.20	1.14	11.88	8.26	10.97	7.84	7.16	7.44
Oatmeal	0.19	0.97	0.99	10.85	8.91	12.36	13.00	8.22	10.73
Paper Towels	0.22	0.87	0.84	7.48	6.67	7.30	5.73	7.10	6.51
Refrigerated Juices	0.21	0.70	0.74	11.11	9.81	11.65	8.39	9.16	8.90
Soft Drinks	0.34	1.05	0.86	8.04	13.86	12.66	21.80	29.69	24.26
Shampoos	0.27	0.70	0.64	13.32	9.96	11.75	12.42	8.95	10.49
Snack Crackers	0.21	1.31	1.35	11.28	7.65	10.42	7.59	7.06	7.27
Soaps	0.22	1.08	1.12	9.44	5.94	8.79	3.92	5.15	4.73
Toothbrushes	0.29	0.60	0.61	12.73	11.76	12.36	10.60	11.01	10.83
Canned Tuna	0.18	1.11	1.19	11.81	3.88	8.03	6.14	3.29	4.05
Toothpastes	0.26	0.66	0.63	10.00	9.85	10.35	7.43	8.65	8.09
Bathroom Tissues	0.19	1.21	1.10	7.51	12.46	11.35	4.17	4.07	4.13
Average*	0.26	0.92	0.88	9.97	10.03	11.67	10.63	13.56	12.03

† See Table 1 for the measures of averages of these features of sales.
* This is the weighted average of all products where weights are the proportion of flagged sales of the respective products to all flagged sales in the data.

Table A-2: Adjusted R-squares of the Polynomial Fits to the Objective and Constraint Functions

Products	Objective function			Constraint: c_1 function			Constraint: c_2 function		
	Linear	Quad.	Cubic	Linear	Quad.	Cubic	Linear	Quad.	Cubic
Analgesics	43.84	77.03	93.27	79.88	89.04	96.82	53.69	81.21	94.82
Bath Soap	43.50	76.97	92.58	67.20	84.09	94.52	47.52	69.11	86.32
Beer	44.79	75.96	92.00	58.05	79.55	92.01	44.66	78.76	93.33
Bottled Juices	62.82	93.86	99.48	66.45	93.92	99.55	42.70	77.59	93.77
Cereals	55.84	86.39	97.30	87.47	93.62	97.62	53.28	80.22	93.15
Cheeses	56.75	87.77	93.97	67.17	91.08	98.39	43.93	78.82	94.39
Cookies	56.98	89.39	98.22	60.90	89.65	98.37	42.54	77.22	94.28
Crackers	63.43	93.64	99.48	64.82	93.07	99.41	43.02	76.91	92.77
Canned Soup	59.69	91.66	99.13	71.03	92.49	99.00	47.29	80.55	95.53
Dish Detergents	67.01	93.00	98.99	79.72	95.30	99.30	48.70	79.30	94.36
Front-end-candies	66.10	93.38	99.03	67.17	92.94	99.18	47.28	81.09	95.83
Frozen Dinners	55.84	87.68	98.01	67.55	91.43	98.61	43.31	77.03	93.52
Frozen Entrees	49.30	84.20	96.70	63.33	89.57	98.25	34.09	70.33	90.07
Frozen Juices	56.72	89.42	98.38	74.88	94.21	99.03	41.62	76.94	93.52
Fabric Softeners	77.49	96.82	99.26	83.28	96.66	99.22	51.20	83.66	95.69
Grooming Products	38.43	72.30	90.33	73.53	87.24	96.01	30.04	63.36	85.54
Laundry Detergents	66.43	94.58	99.63	77.17	96.05	99.69	47.83	81.32	95.55
Oatmeal	70.31	94.02	98.96	84.34	96.00	99.04	60.68	85.25	95.73
Paper Towels	74.87	94.51	99.05	85.16	96.75	99.18	40.21	74.62	93.34
Refrigerated Juices	57.86	87.92	97.85	75.07	92.87	98.65	37.91	71.37	89.99
Soft Drinks	62.42	88.01	96.55	77.52	94.60	98.73	26.55	65.49	87.07
Shampoos	39.37	70.85	89.31	76.32	87.17	95.51	53.14	78.07	91.56
Snack Crackers	68.55	94.85	99.20	74.04	96.25	99.46	49.16	83.40	96.98
Soaps	65.12	93.99	99.58	71.77	94.20	99.45	45.49	79.47	94.59
Toothbrushes	50.30	81.22	94.81	72.59	90.54	97.49	52.56	85.89	97.05
Canned Tuna	71.31	97.32	99.51	73.48	96.52	99.77	46.63	81.33	95.79
Toothpastes	44.56	77.63	93.32	74.54	89.06	97.19	49.09	77.77	94.08
Bathroom Tissues	71.32	93.18	98.57	78.76	95.55	98.98	44.16	78.53	94.48
Average*	58.60	88.81	98.07	72.36	92.87	98.86	44.46	78.25	94.32

* This is the weighted average of all products where weights are the proportion of flagged sales of the respective products to all flagged sales in the data.

Table A-3: Optimal Values of the Parameters by Products

Products	Optimal Values†			Active Constraints‡			
	K	F	f	c_1	c_2	c_3	c_4
Analgesics	3.31	0.672	0.641	0	1	0	0
Bath Soap	3.79	0.569	0.738	0	1	0	0
Beer	5.01	0.250	0.811	1	0	0	1
Bottled Juices	5.53	0.250	0.664	1	0	0	1
Cereals	4.47	0.453	0.658	1	1	0	0
Cheeses	4.80	0.250	0.672	1	1	0	1
Cookies	5.93	0.250	0.768	1	0	0	1
Crackers	6.70	0.250	0.797	1	0	0	1
Canned Soup	4.77	0.250	0.665	1	1	0	1
Dish Detergents	4.72	0.250	0.633	1	0	0	1
Front-end-candies	4.91	0.467	0.657	0	1	0	0
Frozen Dinners	7.67	0.25	0.792	1	0	0	1
Frozen Entrees	4.92	0.250	0.751	1	0	0	1
Frozen Juices	4.67	0.250	0.677	1	0	0	1
Fabric Softeners	6.80	0.250	0.647	1	0	0	1
Grooming Products	3.55	0.510	0.695	0	1	0	0
Laundry Detergents	5.63	0.250	0.629	1	0	0	1
Oatmeal	8.91	0.487	0.665	1	0	0	1
Paper Towels	6.36	0.250	0.645	1	0	0	1
Refrigerated Juices	5.84	0.250	0.699	1	0	0	1
Soft Drinks	6.14	0.250	0.762	1	0	0	1
Shampoos	5.29	0.250	0.779	1	1	0	1
Snack Crackers	6.75	0.344	0.764	1	1	0	0
Soaps	6.72	0.251	0.717	1	1	0	0
Toothbrushes	3.65	0.616	0.678	0	1	0	0
Canned Tuna	6.00	0.250	0.632	1	0	0	1
Toothpastes	3.15	0.521	0.662	0	1	0	0
Bathroom Tissues	8.44	0.250	0.686	1	1	0	1
Average*	5.46	0.298	0.724	–	–	–	–
Combined**	5.55	0.250	0.735	1	0	0	1

† For all products, $M = 13$ and $E = 0.02$. f corresponds to the fitted (cubic) objective value.
‡ 1 implies active and 0 implies inactive constraint.
* Weighted average of the optimal values of all products where weights are the proportion of flagged sales in the products to all flagged sales.
** Uses the objective and constraint values which are the weighted average of the respective values of all products, and weights are the proportion of flagged sales for each product.

Table A-4: Un-identified Flagged Sales as a Percentage of Total Flagged Sales

Products*	Inconsis -tent Flagging	Initial Prices as Sales	Duration $K >$ Optimal	Price Dip $< E\%$	Price Recovery $< F\%$
Analgesics	9.45	3.61	7.11	4.49	0.00
Bath Soap	4.71	6.38	4.38	0.69	0.00
Beer	5.10	3.39	6.62	0.69	0.00
Bottled Juices	12.90	1.92	3.71	6.71	0.00
Cereals	11.04	2.25	2.04	7.25	0.00
Cheeses	11.32	1.13	2.69	7.84	0.00
Cookies	7.40	2.10	2.66	2.48	0.00
Crackers	9.16	2.24	4.16	2.98	0.00
Canned Soup	11.99	1.10	2.73	4.46	0.00
Dish Detergents	11.18	2.09	3.64	6.15	0.00
Front-end-candies	13.65	2.08	4.67	1.46	0.02
Frozen Dinners	8.56	2.90	0.60	2.69	0.01
Frozen Entrees	10.23	1.44	1.41	1.51	0.01
Frozen Juices	12.08	0.70	1.77	4.03	0.01
Fabric Softeners	11.52	2.31	4.10	6.38	0.07
Grooming Products	5.29	4.29	4.40	1.25	0.01
Laundry Detergents	12.84	3.09	3.28	6.48	0.00
Oatmeal	11.72	0.92	0.79	10.73	0.00
Paper Towels	15.17	2.02	3.45	2.96	0.00
Refrigerated Juices	10.75	1.02	1.26	5.67	0.01
Soft Drinks	10.05	1.46	1.49	0.88	0.03
Shampoos	6.68	5.00	3.15	0.54	0.01
Snack Crackers	9.78	1.71	3.57	2.86	0.00
Soaps	7.89	3.07	2.43	11.79	0.00
Toothbrushes	8.32	3.07	5.14	1.04	0.02
Canned Tuna	13.38	1.22	8.38	7.12	0.00
Toothpastes	7.87	2.92	4.91	1.75	0.00
Bathroom Tissues	13.35	1.19	1.11	3.80	0.00
All Products**	10.07	2.06	2.83	3.07	0.01

* Inconsistent flagging figures refer to the percentage of sale weeks and others refer to the percentage of sales.
** This is the weighted average of all products where weights are the proportion of flagged sales of the respective products to all flagged sales in the data.

Table A-5: Sensitivity of Optimal Values to Changes in Constraint c_1

Products†	$c_1 + (1 \times \sigma_{dur_f})^{\ddagger}$			$c_1 + (2 \times \sigma_{dur_f})^{\ddagger}$			$c_1 = \{\emptyset\}^*$		
	K	F	f	K	F	f	K	F	f
Analgesics	3.31	0.672	0.641	3.31	0.672	0.641	3.31	0.672	0.641
Bath Soap	3.79	0.569	0.738	3.79	0.569	0.738	3.79	0.569	0.738
Beer	5.50	0.500	0.820	5.96	0.700	0.845	10.52	0.250	0.893
Bottled Juices	6.67	0.286	0.726	6.90	0.689	0.748	6.90	0.689	0.748
Cereals	4.50	0.531	0.660	4.50	0.531	0.660	4.50	0.531	0.660
Cheeses	5.53	0.499	0.730	5.81	0.805	0.751	5.81	0.805	0.751
Cookies	6.86	0.250	0.812	9.54	0.250	0.859	10.06	1.000	0.875
Crackers	7.38	0.773	0.846	7.38	0.773	0.846	7.38	0.773	0.846
Canned Soup	4.94	0.675	0.685	4.94	0.675	0.685	4.94	0.675	0.685
Dish Detergents	5.61	0.250	0.688	6.95	0.250	0.746	11.28	1.000	0.833
Front-end-candies	4.91	0.467	0.657	4.91	0.467	0.657	4.91	0.467	0.657
Frozen Dinners	9.36	0.722	0.847	10.01	1.00	0.868	10.01	1.000	0.868
Frozen Entrees	6.44	0.250	0.812	8.64	0.483	0.860	9.32	0.942	0.876
Frozen Juices	8.03	0.421	0.816	8.04	0.559	0.821	8.04	0.559	0.821
Fabric Softeners	8.04	0.250	0.691	8.40	0.536	0.709	8.40	0.536	0.709
Grooming Products	3.55	0.510	0.695	3.55	0.510	0.695	3.55	0.510	0.695
Laundry Detergents	6.23	0.250	0.658	6.98	0.250	0.686	7.63	0.481	0.712
Oatmeal	10.76	0.893	0.711	12.86	1.00	0.727	12.86	1.000	0.727
Paper Towels	11.19	0.419	0.737	13.00	0.797	0.749	13.00	0.797	0.749
Refrigerated Juices	9.67	0.718	0.788	10.20	1.00	0.800	10.20	1.000	0.799
Soft Drinks	9.29	0.453	0.822	10.26	0.837	0.849	10.75	1.000	0.852
Shampoos	5.62	0.311	0.794	5.77	0.429	0.805	5.81	0.540	0.810
Snack Crackers	6.94	0.839	0.789	6.94	0.839	0.789	6.94	0.839	0.789
Soaps	6.77	0.526	0.731	6.77	0.526	0.731	6.77	0.526	0.731
Toothbrushes	3.65	0.616	0.678	3.65	0.616	0.678	3.65	0.616	0.678
Canned Tuna	7.17	0.250	0.702	9.47	0.250	0.785	13.00	1.000	0.795
Toothpastes	3.15	0.521	0.662	3.14	0.521	0.662	3.14	0.521	0.662
Bathroom Tissues	12.26	0.943	0.740	13.00	1.000	0.748	13.00	1.000	0.748
Average**	6.93	0.464	0.767	7.70	0.646	0.788	8.16	0.791	0.797
Combined**	7.66	0.250	0.801	9.50	0.627	0.841	10.08	0.691	0.830

† For all products, $M = 13$ and $E = 0.02$. f corresponds to the fitted (cubic) objective value.
‡ σ_{dur_f} denotes the standard deviation of the duration of the flagged sales.
* $c_1 = \{\emptyset\}$ means that constraint c_1 has been removed from the optimization problem.
** See footnotes of Table A-3.

Table A-6: Sensitivity of Optimal Values to Changes in Constraint c_2

Products†	$c_2 + (1 \times \sigma_{mag_f})^\ddagger$			$c_2 + (2 \times \sigma_{mag_f})^\ddagger$			$c_2 = \{\emptyset\}^*$		
	K	F	f	K	F	f	K	F	f
Analgesics	3.54	0.681	0.666	3.76	0.571	0.688	3.99	0.250	0.701
Bath Soap	4.81	0.337	0.807	4.95	0.250	0.813	4.95	0.250	0.813
Beer	5.01	0.250	0.811	5.01	0.250	0.811	5.01	0.250	0.811
Bottled Juices	5.53	0.250	0.664	5.53	0.250	0.664	5.53	0.250	0.664
Cereals	4.56	0.348	0.662	4.64	0.279	0.665	4.67	0.250	0.665
Cheeses	4.80	0.250	0.672	4.80	0.250	0.672	4.80	0.250	0.672
Cookies	5.93	0.250	0.768	5.93	0.250	0.768	5.93	0.250	0.768
Crackers	6.70	0.250	0.797	6.70	0.250	0.797	6.70	0.250	0.797
Canned Soup	4.77	0.250	0.665	4.77	0.250	0.665	4.77	0.250	0.665
Dish Detergents	4.72	0.250	0.633	4.72	0.250	0.633	4.72	0.250	0.633
Front-end-candies	5.03	0.338	0.665	5.10	0.250	0.669	5.10	0.250	0.669
Frozen Dinners	7.67	0.250	0.792	7.67	0.250	0.792	7.67	0.250	0.792
Frozen Entrees	4.92	0.250	0.751	4.92	0.250	0.751	4.92	0.250	0.751
Frozen Juices	4.67	0.250	0.677	4.67	0.250	0.677	4.67	0.250	0.677
Fabric Softeners	6.80	0.250	0.647	6.80	0.250	0.647	6.80	0.250	0.647
Grooming Products	3.81	0.524	0.717	4.09	0.536	0.739	5.20	0.250	0.801
Laundry Detergents	5.63	0.250	0.629	5.63	0.250	0.629	5.63	0.250	0.629
Oatmeal	8.91	0.487	0.665	8.91	0.487	0.665	8.91	0.487	0.665
Paper Towels	6.36	0.250	0.645	6.36	0.250	0.645	6.36	0.250	0.645
Refrigerated Juices	5.84	0.250	0.699	5.84	0.250	0.699	5.84	0.250	0.699
Soft Drinks	6.14	0.250	0.762	6.14	0.250	0.762	6.14	0.250	0.762
Shampoos	5.29	0.250	0.779	5.29	0.250	0.779	5.29	0.250	0.779
Snack Crackers	7.01	0.250	0.771	7.01	0.250	0.771	7.01	0.250	0.771
Soaps	6.72	0.251	0.717	6.72	0.251	0.717	6.72	0.251	0.717
Toothbrushes	4.15	0.635	0.719	4.50	0.354	0.740	4.66	0.250	0.747
Canned Tuna	6.00	0.250	0.632	6.00	0.250	0.632	6.00	0.250	0.632
Toothpastes	3.48	0.538	0.693	3.84	0.556	0.724	4.47	0.250	0.759
Bathroom Tissues	8.44	0.250	0.686	8.44	0.250	0.686	8.44	0.250	0.686
Average**	5.52	0.291	0.726	5.55	0.281	0.729	5.63	0.251	0.733
Combined**	5.55	0.250	0.735	5.55	0.250	0.735	5.55	0.250	0.735

† For all products, $M = 13$ and $E = 0.02$. f corresponds to the fitted (cubic) objective value.
‡ σ_{mag_f} denotes the standard deviation of the duration of the flagged sales.
* $c_2 = \{\emptyset\}$ means that constraint c_2 has been removed from the optimization problem.
** See footnotes of Table A-3.

Table A-7: Sensitivity of Optimal Values to Changes in Constraint c_4

Products†	$F \geq 0$‡			$F \geq 0.50$‡			$F \geq 0.75$‡			$F \geq 1$‡		
	K	F	f	K	F	f	K	F	η	K	F	f
Analgesics	3.29	0.000	0.626	3.31	0.676	0.641	3.30	0.750	0.640	3.11	1.000	0.612
Bath Soap	3.80	0.569	0.738	3.80	0.569	0.738	3.72	0.750	0.732	3.40	1.000	0.689
Beer	5.61	0.000	0.810	4.75	0.500	0.763	4.74	0.750	0.761	4.72	1.000	0.745
Bottled Juices	6.22	0.000	0.687	5.18	0.500	0.647	4.93	0.750	0.628	4.63	1.000	0.600
Cereals	4.47	0.453	0.689	4.44	0.500	0.656	4.07	0.750	0.630	3.21	1.000	0.562
Cheeses	4.94	0.173	0.678	4.50	0.500	0.657	4.26	0.750	0.640	3.90	1.000	0.601
Cookies	6.67	0.000	0.785	5.53	0.500	0.753	5.19	0.750	0.733	4.82	1.000	0.706
Crackers	7.09	0.118	0.806	6.22	0.500	0.781	5.81	0.750	0.759	5.27	1.000	0.724
Canned Soup	4.79	0.217	0.667	4.57	0.500	0.656	4.33	0.750	0.638	3.91	1.000	0.600
Dish Detergents	5.11	0.000	0.642	4.52	0.500	0.623	4.33	0.750	0.606	3.99	1.000	0.578
Front-end-candies	4.91	0.467	0.657	4.91	0.500	0.657	4.81	0.750	0.647	4.55	1.000	0.620
Frozen Dinners	8.46	0.194	0.793	6.19	0.500	0.777	5.29	0.750	0.750	4.43	1.000	0.698
Frozen Entrees	5.66	0.000	0.800	4.55	0.500	0.737	4.24	0.750	0.717	3.85	1.000	0.678
Frozen Juices	5.39	0.000	0.698	4.39	0.500	0.662	4.14	0.750	0.641	3.74	1.000	0.602
Fabric Softeners	7.50	0.000	0.661	6.47	0.500	0.638	6.23	0.750	0.626	5.89	1.000	0.608
Grooming Products	3.55	0.510	0.695	3.55	0.510	0.695	3.39	0.750	0.675	2.64	1.000	0.580
Laundry Detergents	6.26	0.000	0.642	5.32	0.500	0.615	5.08	0.750	0.597	4.75	1.000	0.570
Oatmeal	8.91	0.487	0.667	8.84	0.500	0.667	7.55	0.750	0.658	5.90	1.000	0.619
Paper Towels	7.75	0.000	0.663	5.91	0.500	0.632	5.52	0.750	0.610	4.76	1.000	0.560
Refrigerated Juices	7.51	0.000	0.720	5.27	0.500	0.677	4.76	0.750	0.642	4.03	1.000	0.577
Soft Drinks	8.47	0.065	0.776	4.97	0.500	0.735	4.29	0.750	0.702	3.79	1.000	0.660
Shampoos	5.51	0.162	0.782	4.68	0.500	0.762	4.27	0.750	0.737	2.52	1.000	0.568
Snack Crackers	6.75	0.344	0.764	6.40	0.500	0.753	5.94	0.750	0.731	5.45	1.000	0.697
Soaps	6.72	0.251	0.717	6.17	0.500	0.702	5.84	0.750	0.683	5.54	1.000	0.657
Toothbrushes	3.65	0.616	0.678	3.65	0.616	0.678	3.61	0.750	0.672	3.15	1.000	0.619
Canned Tuna	6.32	0.00	0.649	5.77	0.500	0.618	5.58	0.750	0.602	5.35	1.000	0.580
Toothpastes	3.15	0.521	0.662	3.15	0.521	0.662	2.96	0.750	0.639	2.11	1.000	0.522
Bathroom Tissues	9.29	0.165	0.687	7.44	0.500	0.677	6.55	0.750	0.652	5.07	1.000	0.590
Average*	6.33	0.142	0.735	4.93	0.506	0.706	4.54	0.750	0.685	3.98	1.000	0.633
Combined*	6.40	0.000	0.750	5.08	0.500	0.719	4.72	0.750	0.698	4.33	1.000	0.663

† For all products, $M = 13$ and $E = 0.02$. f corresponds to the fitted (cubic) objective value.

‡ In the base case, $c_4 : F \geq 0.25$.

* See footnotes of Table A-3.

Table A-8: Sensitivity of Optimal Values to Changes in the Parameters E & M

Products†	$E = 0.01$			$E = 0.04$			$M = 52$		
	K	F	f	K	F	f	K	F	f
Analgesics	3.34	0.660	0.665	3.27	0.685	0.570	3.29	0.658	0.647
Bath Soap	3.60	0.556	0.723	4.62	0.595	0.785	4.12	0.527	0.778
Beer	4.98	0.250	0.794	5.28	0.250	0.813	4.81	0.636	0.775
Bottled Juices	5.49	0.250	0.694	5.66	0.250	0.578	5.53	0.250	0.665
Cereals	4.48	0.417	0.692	4.77	0.445	0.599	4.56	0.373	0.668
Cheeses	4.55	0.400	0.697	4.85	0.250	0.548	4.80	0.250	0.673
Cookies	5.73	0.250	0.765	6.21	0.250	0.722	5.93	0.250	0.771
Crackers	6.33	0.284	0.786	6.79	0.250	0.718	6.73	0.250	0.805
Canned Soup	4.67	0.267	0.683	4.88	0.316	0.556	4.77	0.250	0.666
Dish Detergents	4.76	0.250	0.665	4.84	0.250	0.538	4.72	0.250	0.636
Front-end-candies	4.35	0.398	0.611	5.03	0.250	0.646	4.93	0.467	0.661
Frozen Dinners	6.92	0.250	0.794	8.84	0.264	0.719	7.73	0.250	0.785
Frozen Entrees	4.84	0.250	0.752	5.12	0.250	0.737	4.93	0.250	0.754
Frozen Juices	4.64	0.250	0.687	4.68	0.250	0.607	4.68	0.250	0.678
Fabric Softeners	6.52	0.250	0.661	7.47	0.250	0.559	6.80	0.250	0.653
Grooming Products	3.56	0.498	0.700	3.94	0.557	0.709	3.51	0.516	0.700
Laundry Detergents	5.44	0.250	0.658	5.98	0.250	0.555	5.63	0.250	0.633
Oatmeal	7.44	0.392	0.731	9.55	0.401	0.650	8.87	0.485	0.667
Paper Towels	6.26	0.250	0.658	6.59	0.250	0.588	6.35	0.250	0.648
Refrigerated Juices	5.57	0.250	0.711	6.32	0.250	0.639	5.85	0.250	0.701
Soft Drinks	6.06	0.250	0.762	6.30	0.250	0.756	6.11	0.250	0.765
Shampoos	5.26	0.250	0.780	5.49	0.250	0.775	5.04	0.345	0.786
Snack Crackers	6.14	0.522	0.746	6.70	0.466	0.718	6.76	0.339	0.766
Soaps	5.64	0.514	0.718	7.22	0.250	0.586	6.77	0.250	0.723
Toothbrushes	3.55	0.605	0.675	3.71	0.615	0.655	3.70	0.605	0.690
Canned Tuna	5.94	0.250	0.654	6.09	0.250	0.509	6.03	0.250	0.638
Toothpastes	3.17	0.510	0.671	3.14	0.531	0.641	3.19	0.517	0.672
Bathroom Tissues	7.39	0.330	0.699	10.03	0.250	0.634	8.44	0.250	0.687
Average*	5.29	0.316	0.728	5.69	0.304	0.681	5.45	0.311	0.725
Combined*	5.43	0.250	0.743	5.74	0.250	0.691	5.55	0.250	0.739

† In the base case, $E = 0.02$ and $M = 13$ weeks. f corresponds to the fitted (cubic) objective value.
* See footnotes of Table A-3.