UNSW Business School
Working Paper

Never Stand Still Business School Economics

UNSW Business School Research Paper No. 2016 ECON 04

Double round-robin tournaments

Francesco De Sinopoli
Claudia Meroni
Carlos Pimienta

DOUBLE ROUND-ROBIN TOURNAMENTS

FRANCESCO DE SINOPOLI[†], CLAUDIA MERONI[‡], AND CARLOS PIMIENTA[§]

ABSTRACT. A tournament is a simultaneous n-player game that is built on a two-player game g. We generalize Arad and Rubinstein's model assuming that every player meets each of his opponents twice to play a (possibly) asymmetric game g in alternating roles (using sports terminology, once "at home" and once "away"). The winner of the tournament is the player who attains the highest total score, which is given by the sum of the payoffs that he gets in all the matches he plays. We explore the relationship between the equilibria of the tournament and the equilibria of the game g. We prove that limit points of equilibria of tournaments as the number of players goes to infinity are equilibria of g. Such a refinement criterion is satisfied by strict equilibria. Being able to analyze arbitrary two-player games allows us to study meaningful economic applications that are not symmetric, such as the ultimatum game.

1. INTRODUCTION

Consider the following two-player game:

	C	D
A	6,5	4,3
B	4,0	3,1

If two individuals were matched to play such a game with the objective to maximize their expected payoff, they would play the unique Nash equilibrium $\{A, C\}$. Now, think of a situation in which the two individuals are matched twice to play simultaneously the same game in alternating roles, with the objective to maximize the sum of the payoffs in the two matches. If both of them play A when in the role of the first player and C when in the role of the second player, they both get a payoff of 11. None of them has an incentive to deviate, since any other choice leads to a lower payoff, and $\{(A,C),(A,C)\}$ is the unique equilibrium of the game. Consider the same situation with the difference that now a prize is awarded to the individual with the highest total payoff, and

† DEPARTMENT OF ECONOMICS, UNIVERSITY OF VERONA, VERONA, ITALY.
‡ DEPARTMENT OF ECONOMICS, UNIVERSITY OF VERONA, VERONA, ITALY.
§ SCHOOL OF ECONOMICS, THE UNIVERSITY OF NEW SOUTH WALES, SYDNEY, AUSTRALIA.
Date: April 20, 2016.

the two individuals want to maximize their probability of winning the prize. In the event of a tie, each of them would be the winner with probability 1/2. In this case $\{(A,C),(A,C)\}$ is no longer an equilibrium. Indeed, if one of the two individuals is playing (A,C), the other one has an incentive to deviate, for instance, to (B,C), losing two points but lowering the payoff of his opponent by five points, becoming in this way the sole winner. It is easy to see that the unique (symmetric) equilibrium of the game is $\{(B,D),(B,D)\}$ and both players win with probability 1/2. Then, we can think of the same competitive situation extended to any number of individuals, where everyone meets all his opponents twice to play the same two-player game. Such a competition is called *double round-robin tournament*.

A double round-robin tournament is a simultaneous n-player game that is built on a two-player game g. Each player is matched with every other player twice and in every match the game g is played. In the two matches with the same opponent, a player plays once in the role of the first player and once in the role of the second player of g. Using sports terminology, we say that a player plays once "at home" and once "away". The winner of the tournament is the player with the highest score, where a player's score is the sum of the payoffs that he gets in all the matches he plays. This implies that players do not care about their absolute total score and maximize their probability of winning the tournament.

Arad and Rubinstein (2013) analyze tournaments of the *round-robin* type, where each player meets every other player once. In their analysis, the two-player game on which the tournament is built is a *symmetric* game. Moreover, the solution concept is that of a symmetric mixed strategy equilibrium, where a mixed strategy is executed only once and the player employs the resulting action in all his matches. Such a concept is appropriate to a situation in which individuals are drawn at random from a large population and are matched in pairs to play the same game anonymously. In this case, indeed, a mixed strategy can be interpreted as a distribution of actions in the population.

With our model, we aim to extend the analysis of tournaments to *asymmetric* games. Since any two individuals play the asymmetric game twice in alternating roles, the symmetry among players is restored. Hence, we also employ the solution concept of a symmetric Nash equilibrium. In the same spirit as Arad and Rubinstein (2013), we assume that each player employs always the same action when he is playing at home, as well as when he is playing away. Consider again the example at the beginning of this section. Under the assumptions of our model, the tournament with three players has a symmetric equilibrium in which every player chooses B at home and D away (the same as the two-individual case). For $n \geq 4$, instead, the symmetric Nash equilibrium of the tournament prescribes to choose A when playing at home and C when playing away, which are the Nash equilibrium strategies of the two-player game. This is not a coincidence, as we will see when analyzing the relationship between equilibria of the tournament and equilibria of the two-player game on which it is built. Intuitively, as n grows, deviating from the equilibrium strategies of the base game becomes less and less profitable. Indeed, the loss inflicted to each of the other players becomes negligible with respect to the higher payoffs that the other players obtain in the increasing number of matches among them.

An alternative tournament model is the one in Laffond et al. (2000). As in our model, each player chooses one action and employs it in all his interactions. However, a player's payoff is given by the sum of the payoffs that he gets in all the (symmetric) games he plays, so players do care about their absolute total score.[1]

Note the difference between the tournament model that we adopt and the classic model of contests. In the classic contest model, players compete for a given prize by exerting an effort that increases their probability of winning (see, e.g., Green and Stokey (1983), Dixit (1987), Konrad (2009); among others). Each player's utility depends on his probability of winning, which is a function also of the other players' efforts, and on the cost of his own effort. In the tournament model, instead, the ranking of a player depends on the combination between his choice and the choices of all the other players, and actions are costless. A particular contest model is that of the elimination tournament, which consists of several rounds in which individuals play pair-wise matches (see, e.g., Rosen (1986), Konrad (2004), Groh et al. (2012)). Differently from our model, the winner of a match advances to the next round of the tournament, while the loser is eliminated from the competition.

Double round-robins are common in several sports competitions, especially in those with a large number of matches per season. Most professional association football leagues in the world are based on a double round-robin,[2] as are

[1] The same assumption is examined in the example at the beginning, when individuals have the objective to maximize the sum of the payoffs in the two matches. In that case, the unique equilibrium prescribes the Nash equilibrium strategies of the base game. With the appropriate modifications, this is consistent with the results in Laffond et al. (2000).

[2] Examples are the top european national leagues, like Spain's La Liga, England's Premier League, Germany's Bundesliga, Italy's Serie A. Moreover, double round-robins are used during the qualification phases of the FIFA World Cup and the respective continental leagues, and during the group phases of the UEFA Champions League and the Copa Libertadores de América.

most basketball leagues outside the United States.[3] In such competitions, the assumptions of our model fit for instance the case of the teams, which have to make several choices to comply with throughout the entire tournament (e.g. the players, the coach, the home field).

Besides this straightforward interpretation, we can think also of another interpretation of tournaments. Namely, if we focus on the Nash equilibria of a two-player game g, we can use tournaments as an equilibrium refinement. As a matter of fact, a mixed-strategy Nash equilibrium of g can be interpreted as a stable distribution of pure strategies in a large population, where individuals play g over time and maximize their expected payoff. One might ask whether a Nash equilibrium of g can also be interpreted as a stable distribution of actions in situations where many individuals are matched to play g in alternating roles and maximize their probability of winning. It turns out that only some of the equilibria of g are "stable" in this sense. Thus, tournaments provide a refinement criterion, which selects all the equilibria of g that are limit points of equilibria of the tournament built on g as the number of players goes to infinity.

Finally, as Arad and Rubinstein (2013) point out, the analysis of the relationship between the equilibria of the tournament and the equilibria of the base game can be useful for experimental design. Indeed, the tournament structure has been used in some experiments to study the agents' behavior in the game g. Such a design has been criticized because, in the case in which a prize is awarded to the participant with the highest score, individuals may have different incentives from those in the base game. In this work, we also aim to examine whether the tournament structure is appropriate for experiments based on asymmetric games. To preview, our analysis confirms the results obtained for symmetric games, as all the equilibria of the tournament based on a game g turn out to be a good approximation of equilibria of g.

We describe the model in the next Section. In Section 3, we analyze the interaction between any two players in the tournament. We examine the relationship between equilibria of the tournament and equilibria of the game on which it is built in Section 4, and we discuss it in some examples in Section 5.

2. The Model

A *double round-robin tournament* $D(g,n)$, simply referred to as *tournament* in the following, is a simultaneous n-player game built on a two-player game $g = (S_h, S_a, u_h, u_a)$. Each player plays g with every other player twice, *at home* and *away*. The sets S_h and S_a are the finite sets of *actions*, the former available to the player who is playing at home and the latter available to the player who

[3] The qualification stages of the Euroleague are an example.

is playing away. Every player is assumed to employ the same action $s_h \in S_h$ in all the matches he plays at home and the same action $s_a \in S_a$ in all the matches he plays away. The real-valued functions u_h and u_a are defined on $S_h \times S_a$. When the player who is at home plays s_h and the player who is away plays s_a, $u_h(s_h, s_a)$ and $u_a(s_h, s_a)$ are the payoffs they respectively obtain in the match. A player's *score* is the sum of the payoffs that he obtains in the $2(n-1)$ matches he participates in. The player with the highest score wins the tournament. In the case of a tie, the winner is chosen randomly among the set of top-scoring players. We assume that each player's objective is to maximize his probability of winning the tournament.

A *pure strategy* s^i of player i is a mapping which assigns an action s_h to the matches he plays at home and an action s_a to the matches he plays away. The set of all pure strategies of each player is $S \equiv S_h \times S_a$. A *mixed strategy* σ^i of player i is an element of $\Sigma \equiv \Delta(S)$, the set of all probability distributions on S. A *strategy profile* $\sigma = (\sigma^1, \ldots, \sigma^n)$ is an element of Σ^n. Following Arad and Rubinstein (2013), in order to avoid non-existence problems we assume global randomization, that is, mixed strategies are executed only once and the player employs the resulting actions s_h and s_a in all the matches he plays at home and away respectively.

Given the structure of the problem, the tournament is a symmetric game (but the match g is usually not) and we focus on symmetric Nash equilibria.[4] Let $P(\sigma^i, \sigma)$ be the probability that player i wins the tournament when he plays the mixed strategy σ^i and his $(n-1)$ opponents play according to σ.

Definition 1. A strategy profile $\sigma = (\sigma^*, \ldots, \sigma^*)$ is a *symmetric Nash equilibrium* of the tournament $D(g,n)$ if

$$P(\sigma^*, \sigma) \geq P(\sigma', \sigma) \qquad \text{for all } \sigma' \in \Sigma.$$

The set Σ is nonempty, compact, and convex, and the function $P(\sigma^i, \sigma)$ is linear in σ^i and continuous in σ, making the associated best response correspondence convex-valued and upper semicontinuous. Thus, by standard fixed point theorems applied to finite symmetric games, a symmetric Nash equilibrium in mixed strategies always exists.

Notice that, if $\sigma = (\sigma^*, \ldots, \sigma^*)$ is a symmetric Nash equilibrium of the tournament, every action in the support of σ^* wins with probability $1/n$ when all the other $(n-1)$ players play σ^*. On the contrary, every action that is not in the support of σ^* wins the tournament with probability not greater than $1/n$.

[4]For the sake of notation, we will often denote a symmetric Nash equilibrium with the mixed strategy that every player chooses.

3. The Two-Player Interaction

A player in the tournament interacts twice with a given opponent, playing the match g against him once *at home* and once *away*. We now focus on such two-player interaction. This allows us to examine the relationship between mixed strategies of the tournament and strategy combinations of the game g.

To this end, given $g = (S_h, S_a, u_h, u_a)$, construct the two-player symmetric game $G = (S, u)$, where $u(s, s') = u_h(s_h, s'_a) + u_a(s'_h, s_a)$ is the utility of a player who plays s when his opponent plays s'. G summarizes the two matches that each player plays with any other player. Clearly, the double round-robin tournament built on top of g coincides with the round-robin tournament built on top of G. Note that each player's strategy set in the game G coincides with the set of mixed strategies of the tournament.

We define a *b-strategy* $b^i = (b^i_h, b^i_a)$ of player i as a pair of probability distributions, the first on S_h and the second on S_a. The set of all b-strategies of each player is $B \equiv \Delta(S_h) \times \Delta(S_a)$. Given a b-strategy b^i of player i, the corresponding *product mixed strategy* is the mixed strategy σ^i defined by $\sigma^i(s) = b^i(s_h) \cdot b^i(s_a)$ for every $s \in S$. Note that the set of b-strategies B is the set of strategy combinations of the game g. Thus, with slight abuse of notation, we will denote a strategy combination of g with b^i and a strategy of G with σ^i.

We can prove that mixed and b-strategies are related through an analogue of Kuhn's theorem (Kuhn, 1953).[5] As we will see in the next section, this result allows us to extend some general properties of round-robin tournaments to our tournaments.

Let two (mixed or b-) strategies of player i be *outcome-equivalent in the two-player interaction* of the tournament if, for every (mixed or b-) strategy of the opponent, they induce the same probability distributions on the payoffs that each player can get at home and away.

Proposition 1. *In the two-player interaction of the tournament, for any mixed strategy $\sigma^i \in \Sigma$ of player i there is an outcome-equivalent b-strategy $b^i \in B$, and vice versa.*

Proof. Consider the two matches that player i plays against player j, at home and away. Let $C(s_h)$ and $C(s_a)$ denote respectively the set of all pure strategies that choose s_h in the match at home and the set of all pure strategies that choose s_a in the match away.

For every mixed strategy σ^k of player k, $k = i, j$, consider the b-strategy b^k defined by $b^k(s_h) = \sum_{s \in C(s_h)} \sigma^k(s)$ and $b^k(s_a) = \sum_{s \in C(s_a)} \sigma^k(s)$. Fix a mixed strategy σ^j of player j. For any $s, s' \in S$, the weight that σ^i induces on player i's

[5] We cannot directly apply Kuhn's theorem given the lack of perfect recall.

utility $u^i(s,s') = u_h^i(s_h,s'_a)+u_a^i(s'_h,s_a)$ is equal to $\sigma^i(s)\sigma^j(s')$.[6] Thus, for any $s_h,s'_h \in S_h$ and $s_a,s'_a \in S_a$, σ^i induces a weight of $\sum_{s\in C(s_h)}\sum_{s'\in C(s'_a)}\sigma^i(s)\sigma^j(s')$ on $u_h^i(s_h,s'_a)$ and a weight of $\sum_{s\in C(s_a)}\sum_{s'\in C(s'_h)}\sigma^i(s)\sigma^j(s')$ on $u_a^i(s'_h,s_a)$. By construction, these weights are respectively equal to $b_h^i(s_h)b_a^j(s'_a)$ and to $b_a^i(s_a)b_h^j(s'_h)$, which are exactly the weights of $u_h^i(s_h,s'_a)$ and $u_a^i(s'_h,s_a)$ according to b^i, given b^j. In the same way, the equivalence holds also for the weights induced by σ^i and b^i on player j's utilities.

For the other way round, it is enough to consider for any b-strategy b^k of player k, $k=i,j$, the corresponding product mixed strategy σ^k. Given a (mixed or b-) strategy of player j, for any $s_h,s'_h \in S_h$ and $s_a,s'_a \in S_a$, the weights of $u_h^i(s_h,s'_a)$, $u_a^i(s'_h,s_a)$, $u_h^j(s'_h,s_a)$, and $u_a^j(s_h,s'_a)$ according to b^i and σ^i are clearly the same. □

Given the equivalence result stated in Proposition 1, the Nash equilibria of the game G could be equivalently defined in terms of b-strategies.

Let $b^i,b^j \in B$. We have

$$\begin{aligned}
u_h(s_h,b_a^j) &= \sum_{s_a\in S_a} b^j(s_a)u_h(s_h,s_a), \text{ and}\\
u_a(b_h^j,s_a) &= \sum_{s_h\in S_h} b^j(s_h)u_a(s_h,s_a).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
u_h(b_h^i,b_a^j) &= \sum_{s_h\in S_h}\sum_{s_a\in S_a} b^i(s_h)b^j(s_a)u_h(s_h,s_a),\\
u_a(b_h^j,b_a^i) &= \sum_{s_h\in S_h}\sum_{s_a\in S_a} b^i(s_a)b^j(s_h)u_a(s_h,s_a), \text{ and}\\
u(b^i,b^j) &= u_h(b_h^i,b_a^j)+u_a(b_h^j,b_a^i).
\end{aligned}$$

Clearly, in the game G a player has a profitable deviation from a given strategy when it is profitable for him to deviate either in the match at home, or in the match away, or in both. Therefore we have:

Proposition 2. *Let the strategies b^i and σ^i be outcome-equivalent in the two-player interaction. If $b^i=(b_h^i,b_a^i)$ is a Nash equilibrium of g, then (σ^i,σ^i) is a symmetric Nash equilibrium of G, and vice versa.*

Proof. First, note that $b^i=(b_h^i,b_a^i)$ is a Nash equilibrium of g if and only if the following conditions are satisfied:

$$\text{if } b^i(s_h)>0, \text{ then } u_h(s_h,b_a^i)=\max_{s'_h\in S_h} u_h(s'_h,b_a^i) \quad \forall s_h\in S_h, \tag{3.1}$$

[6]Note that, given σ^j, $\sigma^i(s)\sigma^j(s')$ is also the weight induced by σ^i on player j's utility $u^j(s',s) = u_h^j(s'_h,s_a)+u_a^j(s_h,s'_a)$.

$$\text{if } b^i(s_a) > 0, \text{ then } u_a(b^i_h, s_a) = \max_{s'_a \in S_a} u_a(b^i_h, s'_a) \quad \forall s_a \in S_a, \tag{3.2}$$

while (σ^i, σ^i) is a symmetric Nash equilibrium of G if and only if

$$\text{if } \sigma^i(s) > 0, \text{ then } u(s, \sigma^i) = \max_{s' \in S} u(s', \sigma^i) \quad \forall s \in S. \tag{3.3}$$

Because at least one action at home and one action away receive positive probability, conditions (3.1) and (3.2) can be rewritten as

$$\begin{aligned} &\text{if } b^i(s_h) \cdot b^i(s_a) > 0, \text{ then} \\ &u_h(s_h, b^i_a) + u_a(b^i_h, s_a) = \max_{s'_h \in S_h, s'_a \in S_a} u_h(s'_h, b^i_a) + u_a(b^i_h, s'_a) \quad \forall s_h \in S_h, s_a \in S_a, \end{aligned}$$

that is,

$$\text{if } b^i(s_h) \cdot b^i(s_a) > 0, \text{ then } u(s, b^i) = \max_{s' \in S} u(s', b^i) \quad \forall s = (s_h s_a) \in S.$$

Since b^i and σ^i are equivalent, $u(s, b^i) = u(s, \sigma^i)$ for every $s \in S$. Moreover, if $\sigma^i(s) > 0$ then $b^i(s_h) \cdot b^i(s_a) > 0$, while if $b^i(s_h) > 0$ then $\sigma^i(s_h s'_a) > 0$ for at least one $s'_a \in S_a$ and if $b^i(s_a) > 0$ then $\sigma^i(s'_h s_a) > 0$ for at least one $s'_h \in S_h$. The result readily follows. □

4. The Tournament

In this section, we analyze the relationship between equilibria of the game g and equilibria of the tournament built on g. The results obtained in the previous section suggest that two approaches can be used to analyze tournaments, one based on mixed strategies and the other based on b-strategies. First, we want to examine whether these two approaches are equivalent. To this end, we explore further how mixed and b-strategies are related in the tournament. Note that, given a b-strategy b^i, there may be more than one mixed strategy that is outcome-equivalent to b^i in the two-player interaction. Hence, we start by examining whether all the strategies that are equivalent in the two-player interaction are also equivalent in the tournament.

Two (mixed or b-) strategies of player i are *outcome-equivalent in the tournament* if, for every n and for every (mixed or b-) strategy of the other $(n-1)$ players, they induce the same probability of winning for each action. The following example shows that two strategies that are outcome-equivalent in the two-player interaction are not necessarily outcome-equivalent in the tournament.

Example 1. Consider the tournament with two players built on the following game g:

	C	D
A	2,0	0,2
B	0,1	1,0

We can easily construct the corresponding two-player symmetric game G:

	AC	AD	BC	BD
AC	2,2	0,4	3,0	1,2
AD	4,0	2,2	2,1	0,3
BC	0,3	1,2	1,1	2,0
BD	2,1	3,0	0,2	1,1

and the 4×4 matrix with the probabilities of winning for each action:

	AC	AD	BC	BD
AC	$\frac{1}{2},\frac{1}{2}$	0,1	1,0	0,1
AD	1,0	$\frac{1}{2},\frac{1}{2}$	1,0	0,1
BC	0,1	0,1	$\frac{1}{2},\frac{1}{2}$	1,0
BD	1,0	1,0	0,1	$\frac{1}{2},\frac{1}{2}$

Take the b-strategy $b^i = \left(\frac{1}{3}A + \frac{2}{3}B, \frac{1}{3}C + \frac{2}{3}D\right)$, which is the unique Nash equilibrium of g. The product mixed strategy $\sigma^i = \left(\frac{1}{9}AC + \frac{2}{9}AD + \frac{2}{9}BC + \frac{4}{9}BD\right)$ and the strategy $\tilde{\sigma}^i = \left(\frac{1}{3}AC + \frac{2}{3}BD\right)$ are both outcome-equivalent to b^i in the two-player interaction, and they are both symmetric Nash equilibria of G. However, they are not outcome-equivalent in the tournament. Indeed, note for instance that σ^i wins against AC with probability $\frac{13}{18}$, while $\tilde{\sigma}^i$ wins against the same action with probability $\frac{5}{6}$.

Henceforth, we refer to strategies that are outcome-equivalent in the two-player interaction simply as *equivalent* strategies. The previous example implies that a choice between the two approaches, based on mixed strategies and on b-strategies, has to be made. In particular, note that the set of b-strategies is a strict subset of the set of mixed strategies, thus one may wonder whether a full analysis can be carried out focusing just on this lower-dimensional set. However, the same example shows that such a limitation would lead to the problem of non-existence of Nash equilibria.[7]

[7]In particular, let $P(b^i, b)$ be the probability that player i wins the tournament when he plays b^i and his opponents play according to b. Note that the set B is nonempty, compact, and convex. However, the function $P(b^i, b)$ is continuous in b but it is not quasi-concave in b^i, so standard arguments cannot be used to state that the best response correspondence is convex-valued. Actually, the non-convexity of such a correspondence is the reason why Nash equilibria in b-strategies may not exist.

To see that the tournament in Example 1 does not have symmetric Nash equilibria in b-strategies, note first that if a b-strategy is a symmetric Nash equilibrium when the strategy set of each player is B, then the corresponding product mixed strategy is a symmetric Nash equilibrium when the strategy set of each player is Σ.[8] Then, notice that there is no symmetric Nash equilibrium neither in pure strategies nor in completely-mixed strategies (as AC is weakly dominated by AD). It follows that there cannot be neither a symmetric Nash equilibrium in pure b-strategies nor in completely-mixed b-strategies. Finally, since the tournament does not have symmetric Nash equilibria with support $\{AC, AD\}$, there exists no symmetric Nash equilibrium in b-strategies where players play A at home with probability one. Analogously, no other partially-mixed symmetric Nash equilibrium in b-strategies exists, as there are no symmetric Nash equilibria with support $\{BC, BD\}$, $\{AC, BC\}$, or $\{AD, BD\}$.[9]

Since Nash equilibria in b-strategies may not exist, our analysis will be based on mixed strategies, as outlined in Section 2. Now, we focus on the relationship between equilibria of the tournament and equilibria of the game on which it is built.

Recall that the double round-robin tournament built on g coincides with the round-robin tournament built on the corresponding game G. Arad and Rubinstein (2013) prove two main results about the relationship between the equilibria of a round-robin tournament and the equilibria of the symmetric base game. Thus, we can apply directly their results to the relationship between the equilibria of a tournament $D(g, n)$ and the equilibria of the corresponding game G. As a consequence of Proposition 1, we can then extend such results to the equilibria of the game g.

The following proposition is an analogue of Proposition 1 in Arad and Rubinstein (2013), which can be directly extended to our framework as a consequence of the results presented in Section 3. Thus, we can state it without proof.

Proposition 3. *Let σ^i be the limit point of a subsequence of symmetric Nash equilibria of $D(g, n)$ as $n \to \infty$. Then σ^i is a (symmetric) Nash equilibrium of G, and the equivalent b-strategy $b^i = (b^i_h, b^i_a)$ is a Nash equilibrium of g.*

Proposition 3 tells us that the set of limit points of the equilibria of a tournament as $n \to \infty$ is a subset of the Nash equilibrium set of the base game. Proposition 4 shows that this subset is strict.

[8]This follows from the fact that every pure strategy is a b-strategy.

[9]As a matter of fact, the unique (symmetric) Nash equilibrium strategy of the tournament is $\sigma^i = \frac{1}{3}AD + \frac{1}{3}BC + \frac{1}{3}BD$.

Proposition 4. *Let $b^i = (b^i_h, b^i_a)$ be a Nash equilibrium of g and let $\Sigma^i(b^i)$ be the set of all equivalent mixed strategies. The set $\Sigma^i(b^i)$ does not necessarily contain a limit point of a sequence of symmetric Nash equilibria of $D(g,n)$ as $n \to \infty$.*

Proof. See the ultimatum game presented in Example 3, where UAA is not a limit point of equilibria of the tournament.[10] □

Note that Propositions 3 and 4 suggest a refinement criterion, since only some equilibria of g have the nice property of being approachable by equilibria of large tournaments. The next proposition shows that the strongest equilibrium refinement proposed in the literature, the one of strict equilibrium, has such a nice property.

Proposition 5. *Let $b^i = (b^i_h, b^i_a)$ be a strict Nash equilibrium of g, and let s be the only equivalent strategy. Then, there exists an integer $\bar{n}$ such that s is a symmetric Nash equilibrium of the tournament $D(g,n)$ for every $n \geq \bar{n}$.*

Proof. Recall that a Nash equilibrium is strict if every deviation implies a loss. Let $k = \min_{s' \neq s} u(s,s) - u(s',s)$, i.e., the minimum loss that a player incurs in any two-player interaction if he deviates, which is strictly positive. Moreover, let $\bar{k} = \max_{s' \neq s} u(s,s) - u(s,s')$, i.e., the maximum loss (or the minimum gain) of the opponent that such deviation entails. Clearly, a player does not have an incentive to deviate from s if $(n-1)k \geq \bar{k}$, that is, if $n \geq \frac{\bar{k}+k}{k}$. Defining $\bar{n} = \left\lceil \frac{\bar{k}+k}{k} \right\rceil$, the result follows. □

From this standpoint, even the relationship between mixed and b-strategies should be studied as n goes to infinity. Let b^i be a Nash equilibrium of g. One may conjecture that if a mixed strategy that is equivalent to b^i is the limit point of a sequence of equilibria of the tournament as $n \to \infty$, then all the equivalent mixed strategies are. The following example shows that this is not the case.

Example 2. Consider the tournament based on the following game g:

	C	D
A	$0,0$	$1,-1$
B	$1,-1$	$0,0$

whose corresponding two-player symmetric game G is:

[10] Alternatively, one can consider the double round-robin tournament built on the "degenerate" game used to prove Proposition 2 in Arad and Rubinstein (2013). Note that such proposition cannot be directly extended to our framework, given our constraints on G.

	AC	BD	AD	BC
AC	$0,0$	$0,0$	$1,-1$	$-1,1$
BD	$0,0$	$0,0$	$-1,1$	$1,-1$
AD	$-1,1$	$1,-1$	$0,0$	$0,0$
BC	$1,-1$	$-1,1$	$0,0$	$0,0$

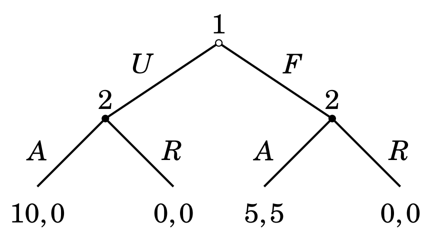
The game g has a unique Nash equilibrium, $b^i = \left(\frac{1}{2}A + \frac{1}{2}B, \frac{1}{2}C + \frac{1}{2}D\right)$. Proposition 3 implies that at least one of the mixed strategies that are equivalent to b^i (and, hence, symmetric equilibria of G) must be a limit point of a subsequence of symmetric Nash equilibria of $D(g,n)$ as $n \to \infty$. Now, take the equivalent mixed strategy $\sigma^i = \left(\frac{1}{2}AC + \frac{1}{2}BD\right)$. For any n sufficiently large, if the other $(n-1)$ players play according to σ^i, strategy $\frac{1}{2}AC + \frac{1}{2}BD$ wins the tournament with probability $1/n$, while strategies AD and BC win with probability close to 1/2. It follows that σ^i is never an equilibrium of $D(g,n)$ for large n, and it is not even a limit point of equilibria of the tournament as $n \to \infty$. Indeed, since the probability of winning is continuous in the strategies of the opponents, the above argument implies that, for every $\tilde{\sigma}^i \in \Sigma$ and ϵ small enough, $(1-\epsilon)\sigma^i + \epsilon\tilde{\sigma}^i$ is not an equilibrium of the tournament with sufficiently large n.

5. Examples

We present now some examples, where the relationship between equilibria of the base game and equilibria of the tournament is discussed. In particular, we compare the limit points of equilibria of tournaments with the standard refinements in the literature.

The first example shows that a dominated Nash equilibrium of g can be a symmetric equilibrium of $D(g,n)$ for every n. Moreover, it shows that a stable set of g in the sense of Kohlberg and Mertens (1986) does not necessarily contain an equilibrium of the tournament $D(g,n)$ for any n, and not even a limit point.

Example 3 (Ultimatum game)**.** Consider the following ultimatum game, where Player 1 can offer a fair (F) or unfair (U) proposal about how to split 10 dollars, and Player 2 can either accept (A) or reject (R) it:

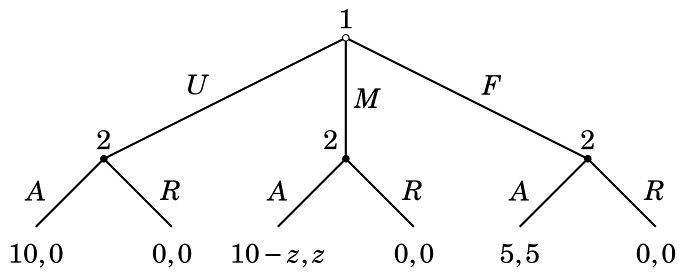


Let the match g be the corresponding normal form game:

	AA	AR	RA	RR
U	$10,0$	$10,0$	$0,0$	$0,0$
F	$5,5$	$0,0$	$5,5$	$0,0$

Note that the mixed strategy FRA is a symmetric equilibrium of the tournament for every n, and the equivalent b-strategy (F,RA) is an undominated Nash equilibrium of g. Also URR is a symmetric equilibrium of the tournament for every n, but the equivalent b-strategy (U,RR) is a dominated Nash equilibrium of g. On the contrary, note that the b-strategy (U,AA) is a strictly perfect equilibrium of g, therefore a stable set as defined by Kohlberg and Mertens (1986). The equivalent strategy UAA, however, is not a symmetric equilibrium of the tournament for any n, and neither is a close-by strategy. Indeed, in the matches in which he moves second, each player has the incentive to deviate and reject the unfair offer, in order to inflict a loss of 10 to his opponents. Hence, if everybody conforms to UAA, a player can increase his probability of winning from $1/n$ to 1 by deviating to URA, and therefore UAA is not even a limit point of equilibria.

Example 4 (Modified ultimatum game)**.** Consider now a modified version of the ultimatum game, in which Player 1 can make also an intermediate (M) offer that gives an amount z to Player 2, with $0 < z < 5$:



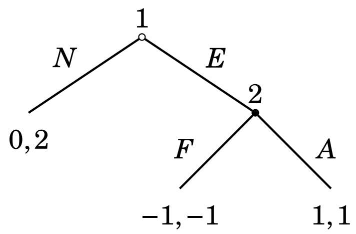
Let the match $g(z)$ be the corresponding normal form game:

	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
U	$10,0$	$10,0$	$10,0$	$10,0$	$0,0$	$0,0$	$0,0$	$0,0$
M	$10-z,z$	$10-z,z$	$0,0$	$0,0$	$10-z,z$	$10-z,z$	$0,0$	$0,0$
F	$5,5$	$0,0$	$5,5$	$0,0$	$5,5$	$0,0$	$5,5$	$0,0$

As before, to offer the fair proposal when moving first and to accept only it when moving second is a symmetric Nash equilibrium of the tournament for every n and every z. Now, however, to accept also an unfair proposal can be part of an equilibrium strategy of the tournament. Indeed, for each z, $MRAA$ is a symmetric Nash equilibrium of the tournament $D(g(z),n)$ if and only if $n \geq n_z = \lceil \frac{10}{z} \rceil$. Note that $\lim_{z\to 0} n_z = \infty$.

Lastly, we present a further example in which dominated equilibria of g are limit points of symmetric equilibria of the tournament as $n \to \infty$.

Example 5 (Entry game)**.** Consider the following "entry game", in which Firm 1 has to decide whether or not to enter the market, and Firm 2 has to decide how to compete, either aggressively (Fight) or not (Accomodate):



Let the match g be the corresponding normal form game:

	F	A
N	$0,2$	$0,2$
E	$-1,-1$	$1,1$

The game g has an undominated Nash equilibrium, (E,A). The equivalent strategy EA is a symmetric Nash equilibrium of the tournament $D(g,n)$ for every $n \geq 3$. Moreover, g has a continuum of dominated Nash equilibria, $\left\{(N, \alpha F + (1-\alpha)A) : \frac{1}{2} \leq \alpha \leq 1\right\}$. Note that NF is a symmetric equilibrium of the tournament $D(g,n)$ for $n \geq 4$. To see whether the other mixed strategies in the continuum $\left\{(\alpha NF + (1-\alpha)NA) : \frac{1}{2} \leq \alpha \leq 1\right\}$ are symmetric equilibria of the tournament for large values of n, note first that the only profitable deviation to consider is playing E instead of N when in the role of the first player. Then, for a given $\alpha \in \left[\frac{1}{2}, 1\right)$, consider a player who plays $(\alpha EF + (1-\alpha)EA)$ while all the other players are playing $(\alpha NF + (1-\alpha)NA)$. Let x be the number of players that play NF in equilibrium. The player who plays E attains a score of $3n-3-2x$, the players that play NA get $2n-3$, while the players that play NF get $2n-5$ (so they never win the tournament). Of course, when all the players play the same strategy, they all win with probability $1/n$. Thus, playing E when in the role of the first player is a profitable deviation if and only if

$$\mathbb{P}\left(x = \frac{n}{2}\right)\frac{1}{n-x} + \mathbb{P}\left(x < \frac{n}{2}\right) > \frac{1}{n}, \tag{5.1}$$

where $\mathbb{P}\left(x = \frac{n}{2}\right) = \binom{n-1}{n/2}\alpha^{n/2}(1-\alpha)^{n-1-n/2}$ is positive only if n is an even number, and $\mathbb{P}\left(x < \frac{n}{2}\right) = \sum_{k=0,\dots,m-1}\binom{n-1}{k}\alpha^k(1-\alpha)^{n-1-k}$, with $m = \left\lceil \frac{n}{2} \right\rceil$.

When $\alpha = \frac{1}{2}$, E is always a profitable deviation, so $\left(\frac{1}{2}NF + \frac{1}{2}NA\right)$ is never a symmetric Nash equilibrium of the tournament for any n. For a fixed $n > 4$, the lhs of (5.1) is decreasing in α and equals the rhs at a value $\alpha_n^* \in \left(\frac{1}{2}, 1\right)$. It follows that, for every $n > 4$, $\left\{(\alpha NF + (1-\alpha)NA) : \alpha_n^* \le \alpha \le 1\right\}$ is a continuum of Nash equilibria of the tournament $D(g, n)$. In particular, α_n^* is decreasing in n and approaches $\frac{1}{2}$ as n goes to infinity.We can thus conclude that all the strategies in the continuum, included $\left(\frac{1}{2}NF + \frac{1}{2}NA\right)$, are limit points of equilibria of the tournament as $n \to \infty$.

References

Arad, A. and Rubinstein, A. Strategic tournaments. *American Economic Journal: Microeconomics*, 5(4):31–54, 2013.

Dixit, A. K. Strategic behavior in contests. *American Economic Review*, 77(5):891–98, 1987.

Green, J. R. and Stokey, N. L. A comparison of tournaments and contracts. *Journal of Political Economy*, 91(3):349–64, 1983.

Groh, C., Moldovanu, B., Sela, A., and Sunde, U. Optimal seedings in elimination tournaments. *Economic Theory*, 49(1):59–80, 2012.

Kohlberg, E. and Mertens, J.-F. On the strategic stability of equilibria. *Econometrica*, 54(5):1003–37, 1986.

Konrad, K. A. Bidding in hierarchies. *European Economic Review*, 48(6):1301–1308, 2004.

Konrad, K. A. *Strategy and Dynamics in Contests*. New York: Oxford University Press, 2009.

Kuhn, H. W. Extensive games and the problem of information. In Kuhn, H. and Tucker, A., editors, *Contributions to the Theory of Games*, volume 2, pages 193–216. Princeton: Princeton University Press, 1953.

Laffond, G., Laslier, J.-F., and Breton, M. L. K-player additive extension of two-player games with an application to the Borda electoral competition game. *Theory and Decision*, 48(2):129–137, 2000.

Rosen, S. Prizes and incentives in elimination tournaments. *American Economic Review*, 76(4):701–15, 1986.