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Abstract

Over the years, a large number of indexes of technological inefficiency have been specified, and a spate of papers has examined the properties satisfied by these indexes. This paper approaches the subject more synthetically, presenting generic results on classes of indexes and their properties. In particular, we consider a broad class of indexes containing almost all known indexes and a partition of this class into two subsets, *slacks-based indexes* and *path-based indexes*. Slacks-based indexes are expressed in terms of additive or multiplicative slacks for all inputs and outputs, and particular indexes are generated by specifying the form of aggregation over the coordinate-wise slacks. Path-based indexes are expressed in terms of a common contraction/expansion factor, and particular indexes are generated by specifying the form of the path to the frontier of the technology. Owing to an impossibility result in one of our earlier papers, we know that the set of all inefficiency indexes can be partitioned into three subsets: those that satisfy continuity (in quantities and technologies) and violate indication (equal to some specified value if and only if the quantity vector is efficient), those that satisfy indication and violate continuity, and those that satisfy neither. We prove two generic theorems establishing the equivalence of these two partitions: all slacks-based indexes satisfy indication and hence violate continuity, and all path-based indexes satisfy continuity and hence violate indication. We also discuss the few indexes that do not belong to either of these two sets. Our hope is that these results will help guide decisions about specification of the form of efficiency indexes used in empirical analysis.

Keywords: Efficiency Indexes, Inefficiency Indexes, Specification.

JEL Classification Codes: C6, D2.

1 Introductory Remarks.

Over the years, a large number of indexes of technological inefficiency (or, equivalently, technological efficiency) have been specified, and a spate of papers has examined the properties, or axioms, satisfied by these indexes. Russell and Schworm [19] carried out a systematic analysis, axiom by axiom and specification by specification. Their theorems suggest, however, a more synthetic structure and the possibility of generic results on classes of indexes and their properties. The purpose of the present paper is to present such results. In particular, we consider a broad class of indexes containing almost all known indexes and a partition of this class into two subsets, which we term "slacks-based indexes" and "path-based indexes". Slacks-based indexes are expressed in terms of additive or multiplicative slacks for all inputs and outputs, and particular indexes are generated by specifying the form of aggregation over the coordinate-wise slacks. Path-based indexes are expressed in terms of a path between a production vector and the boundary of the technology, and particular indexes are generated by specifying the form of the path.

Owing to an impossibility result of Russell and Schworm [19], we know that the set of all inefficiency indexes can be partitioned into three subsets: those that satisfy continuity (in quantities and technologies) and violate indication (equal to some specified value if and only if the quantity vector is efficient), those that satisfy indication and violate continuity, and those that satisfy neither indication nor continuity. Abstracting from the indexes satisfying neither axiom, we prove a generic theorem showing the equivalence of these two taxonomies—*i.e.*, showing that slacks-based indexes satisfy indication and hence violate continuity and path-based indexes satisfy continuity and hence violate indication. In our concluding remarks, we discuss briefly the few indexes that do not belong to either of these two sets.

2 Technology Sets and Efficiency.

The $\langle$input, output$\rangle$ production vector $\langle x, y\rangle \in \mathbf{R}_+^{n+m}$ is constrained to lie in a technology set $T \subset \mathbf{R}_+^{n+m}$. The output possibility set for input x is $P(x) = \{y \in \mathbf{R}_+^m \mid \langle x, y\rangle \in T\}$. Denote the origin of $\mathbf{R}_+^{n+m}$ by $0^{[n+m]} = \langle 0^{[n]}, 0^{[m]}\rangle$ and the (upper) frontier of T by $\partial(T)$. The inputs are indexed by $i = 1, \ldots, n$, and outputs are indexed by $j = 1 \ldots, m$.

We consider the collection of non-empty, closed technology sets, $\mathcal{T}$, that satisfy the following conditions:[1]

(i) $\langle x, y\rangle \in T$ and $\langle \bar{x}, -\bar{y}\rangle \geq \langle x, -y\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \in T$ (free disposability of inputs and outputs),

(ii) $y > 0^{[m]} \implies \langle 0^{[n]}, y\rangle \notin T$ (no free lunch), and

(iii) $P(x)$ is non-empty and bounded for all $x \in \mathbf{R}^n_+$.

Note that, owing to condition (i), $\langle x, y\rangle \in \partial(T)$ and $\langle x, -y\rangle \gg \langle \bar{x}, -\bar{y}\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \notin T$.

Many inefficiency indexes were originally defined on the particular subset of $\mathcal{T}$ generated by mathematical programming methods of constructing technology sets on a finite set of data points. This method, commonly referred to as Data Envelopment Analysis (DEA), generates convex polyhedral technologies (*i.e.*, intersections of finite numbers of half spaces).[2] Almost all of these indexes, however, can be applied to the more general class of technologies $\mathcal{T}$.

A production vector $\langle x, y\rangle \in T$ is technologically efficient (in the sense of Koopmans [13]) if $\langle x, -y\rangle > \langle \bar{x}, -\bar{y}\rangle$ implies $\langle \bar{x}, \bar{y}\rangle \notin T$. Denote the set of efficient production vectors in T by $\mathrm{Eff}(T)$.

3 Inefficiency Indexes: A Binary Taxonomy.

3.1 Technological inefficiency indexes.

Intuitively, a technological inefficiency index measures the "distance" from the production vector to a "reference point" on the frontier of T. Alternative inefficiency indexes are obtained by varying the selection of reference points and the measure of distance.

Formally, we define an inefficiency index as a mapping, $I : \Xi \to \mathbf{R}_+$, with image $I(x, y, T)$, where

$$\Xi = \big\{\langle x, y, T\rangle \in \mathbf{R}^{n+m}_{++} \times \mathcal{T} \mid \langle x, y\rangle \in T\big\}.$$

The effective range of I, denoted $R(I)$, is a closed subset of $\mathbf{R}_+$.[3,4] Denote the minimum point of the range by $\theta = \min R(I)$.

[1]Vector notation: $\bar{x} \geq x$ if $\bar{x}_i \geq x_i$ for all i; $\bar{x} > x$ if $\bar{x}_i \geq x_i$ for all i and $\bar{x} \neq x$; and $\bar{x} \gg x$ if $\bar{x}_i > x_i$ for all i.

[2]See Charnes, Cooper, Lewin, and Seiford [4].

[3]It is a straightforward matter of renormalization to convert an efficiency measure (typically mapping into the (0,1] interval) into an inefficiency measure.

[4]Note that, to focus on the salient issues at hand, we restrict the domain of I to positive

3.2 Slacks-based indexes.

Slacks-based indexes depend on the differences between the components of a given production vector $\langle x, y\rangle$ and the respective components of a frontier reference vector. The reference vector is determined as the solution to an optimization problem with an objective function formed by aggregating over the slacks.

Slacks-based indexes can be formulated in terms of either additive or multiplicative slacks: *viz.*, the additive or multiplicative difference between the production vector and the reference vector. We denote the additive slacks by $\langle s, t\rangle \in \mathbf{R}^n_+ \times \mathbf{R}^m_+$. Proportional input and output slacks are then given respectively by $\alpha_i = (x_i - s_i)/x_i \in (0, 1]$ for all i and $\beta_j = y_j/(y_j + t_j) \in (0, 1]$ for all j.[5] To facilitate easy comparison, we formulate all slacks-based indexes in terms of additive slacks (although some were originally formulated in terms of multiplicative slacks).

Formally, a slacks-based inefficiency index is defined by

$$I^s(x, y, T) = \max_{\langle s,t\rangle \in \mathbf{R}^{n+m}_+} \{\psi(s, t, x, y) \mid \langle x - s, y + t\rangle \in T\}, \tag{1}$$

where the function ψ is independent of units of measurement and increasing in the slack variables $\langle s, t\rangle$ and satisfies $\psi(0^{[n+m]}, x, y) = 0$ for all $\langle x, y\rangle \in T$. Members of this family of inefficiency indexes are generated by specifying the form of the function ψ.

The following indexes—converted to *in*efficiency indexes and/or normalized to satisfy $\psi(0^{[n+m]}, x, y) = \theta = 0$ where necessary—satisfy the definition (1) with the indicated specifications of ψ.[6]

- *Färe-Grosskopf-Lovell Index* (Färe, Grosskopf, and Lovell [9]):

$$\psi(s, t, x, y) = \left[\frac{1}{n+m}\left(\sum_i \frac{x_i - s_i}{x_i} + \sum_j \frac{y_j}{y_j + t_j}\right)\right]^{-1} - 1.$$

values of input and output quantities, thus avoiding some distracting boundary issues (see Levkoff, Russell, and Schworm [14] for an analysis of boundary problems).

[5] Of course, proportional slacks are conversely converted to additive slacks by $s_i = (1 - \alpha_i)x_i$ for all i and $t_j = (1 - \beta_j)y_j$ for all j.

[6] See Russell and Schworm [19] for an in-depth comparison of these indexes and those that follow below.

- *Quantity-Weighted Additive Index* (Charnes, Cooper, Golany, Seiford, and Stutz [3]):[7]
$$\psi(s,t,x,y) = \sum_i \frac{s_i}{x_i} + \sum_j \frac{t_j}{y_j}.$$
- *Weighted Additive Index* (Cooper and Pastor [8]):
$$\psi(s,t,x,y) = \sum_i u_i s_i + \sum_j v_j t_j,$$
where $u \gg 0$ and $v \gg 0$ are pre-specified weights.
- *Pastor-Ruiz-Sirvent Index* (Pastor, Ruiz, and Sirvent [16]):
$$\psi(s,t,x,y) = \frac{\frac{1}{n}\sum_i (x_i - s_i)/x_i}{\frac{1}{m}\sum_j (y_j + t_j)/y_j} - 1.$$
- *Measure of Efficiency Proportions* (Banker and Cooper [1]): equivalent to the Färe-Grosskopf-Lovell index.
- *Measure of Inefficiency Proportions* (Cooper, Park, and Pastor [7]): equivalent to the Quantity-Weighted Additive Index.
- *Slacks-Based Measure of Efficiency* (Tone [20]): equivalent to the Pastor-Ruiz-Sirvent Index.
- *Directional Slacks-Based Measure* (Fukuyama and Weber [12]): equivalent to the Weighted Additive Index.

3.3 Path-based indexes.

Path-based indexes require the specification of a path in input-output space that commences with any production vector $\langle x, y\rangle \in T$ and intersects the boundary of T exactly once. The path determines the reference point for any feasible production vector and measures the distance between the two points.

[7] Charnes, Cooper, Golany, Seiford, and Stutz also specified the "Additive DEA Model" (eschewing the weights), but that index is not independent of units of measurement and hence has been superceded by the Quantity-Weighted Additive Index.

To describe the path, we introduce a set Λ that is a closed (unit free) subset of $\mathbf{R}_+$ with $\mu = \min \Lambda$ and a map $\Omega : T \times \Lambda \to \mathbf{R}_+^{n+m}$. To denote the components of Ω, let $\Omega(x, y, \lambda) = \langle \Omega^x(x, y, \lambda), \Omega^y(x, y, \lambda) \rangle$, where $\Omega^x(x, y, \lambda) = \langle \Omega_1^x(x, y, \lambda), \ldots, \Omega_n^x(x, y, \lambda) \rangle$ and $\Omega^y(x, y, \lambda) = \langle \Omega_1^y(x, y, \lambda), \ldots, \Omega_m^y(x, y, \lambda) \rangle$.

We require that Ω be continuous and satisfy the following conditions for all $\langle x, y \rangle \in T$:

(a) $\langle x, y \rangle = \Omega(x, y, \mu)$;

(b) Ω_i^x is decreasing in λ and increasing in x_i for $i = 1, \ldots, n$ and Ω_j^y is increasing in λ and y_j for $j = 1, \ldots, m$; and

(c) $\lim_{\lambda \to \infty} \Omega_j(x, y, \lambda) = \infty$ for some output j.

For any feasible point $\langle x, y \rangle$, the path containing it is defined by the set $\{\Omega(x, y, \lambda) \mid \lambda \in \Lambda\}$. Along the path, inputs are contracted and outputs are expanded as λ increases.

Path-based indexes are defined by

$$I^p(x, y, T) = \max\{\lambda \in \Lambda \mid \Omega(x, y, \lambda) \in T\}. \tag{2}$$

Note that $\Omega(x, y, I^p(x, y, T))$ is the point of intersection between the path $\{\Omega(x, y, \lambda) \mid \lambda \in \Lambda\}$ and the boundary of the technology set. Therefore, $\Omega(x, y, I^p(x, y, T)) \in \partial(T)$ is the *reference vector* for any feasible point $\langle x, y \rangle$.

The following indexes satisfy the definition (2) with the indicated specifications of Λ and Ω .

- *The Hyperbolic Index* (Färe, Grosskopf, and Lovell [1985]):

  $$\Lambda = [1, +\infty) \quad \text{and} \quad \Omega(x, y, \lambda) = \langle x/\lambda, \lambda y \rangle.$$

- *The Directional-Distance Index*[8] (Luenberger [15] and Chung, Färe, and Grosskopf [5]):

  $$\Lambda = [0, \infty) \quad \text{and} \quad \Omega(x, y, \lambda) = \langle x - \lambda g_x,\ y + \lambda g_y \rangle,$$

  where $g = \langle g_x, g_y \rangle \in \mathbf{R}_{++}^{n+m}$.

[8]The directional distance index is adapted from the shortage function of Luenberger [15] to the measurement of efficiency by Chung, Färe, and Grosskopf [5]. Both restrict g only to the non-negative orthant, but restriction to the positive orthant improves the properties of the index (see Russell and Schworm [19]).

- *The Briec [2] Index*[9]:

$$\Lambda = [0, 1] \quad \text{and} \quad \Omega(x, y, \lambda) = \langle (1 - \lambda)x, (1 + \lambda)y \rangle.$$

4 A Generic Theorem.

Among the axioms, or properties, of efficiency and inefficiency indexes that have been posited in the literature, two are salient for the above taxonomy:

Indication of efficiency (I): For all $\langle x, y, T \rangle \in \Xi$, $I(x, y, T) = \theta$ if and only if $\langle x, y \rangle \in \text{Eff}(T)$.

Joint continuity (C): I is continuous in $\langle x, y, T \rangle$.[10]

The intrinsic appeal of these axioms is self-evident. Condition (I), introduced in the context of input-oriented indexes by Färe and Lovell [1978], requires that an inefficiency index distinguish between inefficient and efficient production vectors. Specifically, the indication condition requires that the index is equal to its minimal value if and only if the production vector is efficient for the given technology.

In formulating condition (C), Russell [17, page 256] argued that continuity is a compelling property, "for it provides assurance that 'small' errors of measurement (of, *e.g.*, input or output quantities) result only in 'small' errors of efficiency measurement." If the technology is constructed (with error) from data on production vectors, the argument for continuity in the technology is perhaps even more compelling.

These two properties take on more importance when one considers the following result:

R-S Impossibility Result (Russell and Schworm [19, Theorem 1]): There does not exist an inefficiency index satisfying (I) and (C) on $\mathcal{T}$.

Thus, the set of all inefficiency indexes can be partitioned into three sets: (i) those that satisfy the indication property, (ii) those that satisfy continuity, and (iii) those that satisfy neither. For reasons we discuss briefly in Section

[9]Briec [2] derives this index from the directional-distance function by using the definition of I_{DD} with the direction $g = \langle x, y \rangle$.

[10]As in Russell [17] and Russell and Schworm [19], we adopt the topology of closed convergence on $\mathcal{T}$.

5, we have little interest in those satisfying neither. Instead, we are interested in characterizing a binary partition of indexes—extant or yet to be formulated—into those satisfying indication and those satisfying continuity. To this end, we show that all indexes with the structure (1) satisfy the indication property but violate continuity, whereas all indexes with the structure (2) satisfy continuity but violate the indication property.

Theorem.

(a) I^s violates (C) and satisfies (I).

(b) I^p violates (I) and satisfies (C).

Proof. See the Appendix.

5 Discussion.

Axioms other than indication and continuity have been explored in the literature, but none seems as dispositive as these two. Invariance with respect to units of measurement has been built into the definitions of slacks-based and path-based indexes (1) and (2), and all indexes in the literature satisfy this fundamental condition.[11]

Similarly, slacks-based and path-based indexes both satisfy (weak) monotonicity: $\langle x, y\rangle \in T$, $\langle x', y'\rangle \in T$, and $\langle x', -y'\rangle \geq \langle x, -y\rangle$ imply $I(x', y', T) \geq I(x, y, T)$. This is evident from the definitions of the two types of indexes. If $\langle s^o, t^o\rangle$ solves the optimization problem in (1) at $\langle x, y, T\rangle$ and $\langle x', -y'\rangle \geq \langle x, -y\rangle$, then (by free disposability) $\langle x' - s^o, y' + t^o\rangle \in T$; consequently $\langle s^o, t^o\rangle$ is a feasible solution at $\langle x', y', T\rangle$, implying that $I^s(x', y', T) \geq I^s(x, y, T)$. Similarly, if λ^o solves the optimization problem in (2) at $\langle x, y, T\rangle$, then free disposability and $\langle x', -y'\rangle \geq \langle x, -y\rangle$ implies $\Omega(x', y', \lambda^o) \in T$ so that $I^p(x', y', T) \geq I^p(x, y, T)$.[12]

Many efficiency studies, including those employing data envelopment analysis, restrict the collection of allowable technologies to convex polyhedral sets.

[11] Other than the "Additive Model" of Charnes, Cooper, Golany, Seiford, and Stutz [3], which has been superseded by the Weighted Additive Index formulated in the same paper for the explicit purpose of establishing unit invariance.

[12] It is not possible to show that either class of indexes satisfies strict monotonicity: $\langle x, y\rangle \in T$, $\langle x', y'\rangle \in T$, and $\langle x', -y'\rangle \geq \langle x, -y\rangle$ imply $I(x', y', T) > I(x, y, T)$. One particular specification, the Weighted Additive Index, does satisfy this condition, but it has the disadvantage of requiring the use of arbitrary weights to correct for the dependence of the "Additive Model" on unit changes.

It can be shown that, on this restricted class of technologies, slacks-based inefficiency indexes are continuous in input and output quantities.[13] Nevertheless, part (a) of the Theorem (and, a fortiori, part (b)) remains valid on this restricted class of technologies. Thus, the indication-continuity trade-off between slacks-based and path-based technologies remains.

Researchers have specified inefficiency indexes that do not belong to either the slacks-based or path-based family. The most interesting is the Weighted Holder Distance Function (Briec [2]), which is based explicitly on a mathematical distance function. This index has the same properties as the path-based indexes but to our knowledge has yet to be applied in practice.[14] Other indexes that do not fit into our taxonomy have distinctively inferior properties. For example, the multi-stage indexes (*e.g.*, Coelli [6]), sequentially combining path-based and slacks-based properties, appear to violate (weak) monotonicity (see the discussion of the Zieschang [21] index in Russell and Schworm [18]). Indexes that concomitantly combine slacks-based and path-based properties (*e.g.*, the generalized hyperbolic measure of Färe, Grosskopf, and Lovell [9]) appear to combine the worst features of both types of indexes, violating both continuity and identification.

Both slacks-based and path-based indexes can be restricted to a subspace of $\mathbf{R}_{+}^{n+m}$ (to generate, *e.g.*, input-oriented indexes and output-oriented indexes). Subspace adaptations of the path-based efficiency indexes retain continuity. Subspace specializations of slacks-based indexes satisfy indication if the indication axiom is relaxed to encompass only subspace efficiency. Therefore, with suitable modifications of the axioms, the theorem remains true for inefficiency indexes defined on subspaces.

To summarize, the axiomatic analysis of inefficiency measurement seems to boil down to the choice between the indication and continuity properties. The incompatibility of these two properties poses a fundamental quandary for the researcher. The trade-off between the slacks-based and the path-based indexes evident in the Theorem reflects the trade-off between these two properties. If the researcher values indication more than continuity, employ or specify a slacks-based index; if the researcher values continuity more than indication, employ or specify a path-based index.

We close by paying tribute to the formulations in two landmark papers that

[13] The proof follows closely the continuity proof of selected slacks-based indexes in Russell and Schworm [19, pp. 155–156].

[14] The definition of path-based indexes (2) could be modified to encompass this index—after all, the Holder distance identifies the shortest path from a production vector to the boundary of the technology—but the redefinition would complicate the analysis (and would be a bit contrived).

crystallize the fundamental dichotomy, and resultant specification quandary, of inefficiency indexes and axioms. The Farrell [11] (input based) index is the model for path-based indexes (satisfying continuity but violating indication), and the Färe-Lovell [10] (input based) index is the model for slacks-based indexes (satisfying indication but violating continuity). All subsequently specified (in)efficiency indexes are variations on these two themes.[15]

Appendix: Proof of the Theorem.

Proof of (a) Slacks-Based Indexes Violate Continuity and Satisfy Indication.

It suffices to prove that I^s satisfies (I), since the R-S Impossibility Result then implies that this class of indexes violates (C).

Suppose that $\langle x, y\rangle \in T$, with $\langle x, y\rangle \gg 0^{[n+m]}$, is not efficient so that there exists a production vector $\langle x', y'\rangle \in T$ satisfying $\langle x', y'\rangle \gg 0^{[n+m]}$ and $\langle x', -y'\rangle < \langle x, -y\rangle$. If $\langle s', t'\rangle$ is the solution to

$$\max_{\langle s,t\rangle \in \mathbf{R}_+^{n+m}} \Big\{ \psi(x', y', s, t)) \mid \langle x' - s, y' + t\rangle \in T \Big\},$$

then $\langle x'-s', y'+t'\rangle$ is a boundary point of T. As $\langle x'-s', y'+t'\rangle \leq \langle x', -y'\rangle < \langle x, -y\rangle$, there exists an $\langle s^o, t^o\rangle \in \mathbf{R}_+^{n+m}$ such that $\langle s^o, t^o\rangle > \langle s', t'\rangle$ and $\langle x - s^o, y + y^o\rangle = \langle x' - s', y' + t'\rangle \in T$. As ψ is increasing in $\langle s, t\rangle$, it must be that $I^s(x, y, T) > I^s(x', y', T) \geq 0$.

Next suppose that $I^s(x, y, T) > 0$. Then $\langle s, t\rangle > 0^{[n+m]}$, so that there exists a point $\langle x', y'\rangle \in T$ satisfying $\langle x', y'\rangle \gg 0^{[n+m]}$ and $\langle x', -y'\rangle < \langle x, -y\rangle$. Therefore, (x, y) is inefficient.

Consequently, I^s satisfies indication (I) and therefore violates (C) . $\diamondsuit$

Proof of (b) Path-Based Indexes Satisfy Continuity and Violate Indication.

It suffices to prove that I^p satisfies (C), since the R-S Impossibility Result then implies that this class of indexes violates (I).

[15] In fact, the hyperbolic (in)efficiency index is a straightforward generalization of the Farrell index (which takes a radial path to the boundary of the input-requirement set), and the Färe-Grosskopf-Lovell index is a straightforward generalization of the Färe-Lovell index.

Consider a sequence $\{x^\nu, y^\nu, T^\nu\}_{\nu=1}^\infty$ converging (in the topology of closed convergence) to $\langle x^o, y^o, T^o\rangle$. We need to show that $I^p(x^\nu, y^\nu, T^\nu)$ converges to $I(x^o, y^o, T^o)$.

Before addressing directly the issue of continuity of I^p, we formally establish two (intuitively obvious) facts.

(i) The intersection of the curve $\{\Omega(x, y, \lambda) \mid \lambda \in \Lambda\}$ and the frontier $\partial(T)$ is a singleton for each $\langle x, y, T\rangle \in \Xi$ (and hence for each element of the sequence $\{x^\nu, y^\nu, T^\nu\}_{\nu=1}^\infty$ and its limit $\langle x^o, y^o, T^o\rangle$).

Suppose not: for some $\langle x, y, T\rangle \in \Xi$, $\{\Omega(x, y, \lambda) \mid \lambda \in \Lambda\} \cap \partial(T)$ contains two points, $\langle x', y'\rangle$ and $\langle \hat{x}, \hat{y}\rangle$. Since Ω_i^x is decreasing in λ for all i and Ω_j^y is increasing in λ for all j, either $\langle x', -y'\rangle \gg \langle \hat{x}, -\hat{y}\rangle$ or $\langle \hat{x}, -\hat{y}\rangle \gg \langle x', -y'\rangle$. As these are frontier points, each inequality violates the free disposability assumption (FD), which proves the result.

(ii) Since Ω is continuous, the sequence of paths $\{\Omega(x^\nu, y^\nu, \lambda) \mid \lambda \in \Lambda\}_{\nu=1}^\infty$ converges to the path $\{\Omega(x^o, y^o, \lambda) \mid \lambda \in \Lambda\}$ (in the topology of closed convergence).

Consider a sequence, $\{\Omega(x^\nu, y^\nu, \lambda^\nu)\}_{\nu=1}^\infty$, converging to $\Omega(x^o, y^o, \lambda')$, where (by continuity of Ω) $\lambda' = \lim_{\nu\to\infty} \lambda^\nu$. As Λ is a closed set, $\lambda' \in \Lambda$, establishing that $\Omega(x^o, y^o, \lambda') \in \{\Omega(x^o, y^o, \lambda) \mid \lambda \in \Lambda\}$.

Now consider a point $\Omega(x^o, y^o, \lambda')$ with $\lambda' \in \Lambda$ and construct the sequence $\{\Omega(x^\nu, y^\nu, \lambda')\}_{\nu=1}^\infty$. Clearly, $\{\Omega(x^\nu, y^\nu, \lambda')\} \in \{\Omega(x^\nu, y^\nu, \lambda) \mid \lambda \in \Lambda\}$ for all ν and $\lim_{\nu\to\infty} \Omega(x^\nu, y^\nu, \lambda') = \Omega(x^o, y^o, \lambda')$, establishing convergence.

(iii) Continuity of I^p.

Assume I^p is not continuous. Then there exists a sequence $\{\langle x^\nu, y^\nu, T^\nu\rangle\}_{\nu=1}^\infty$ converging to $\langle x^o, y^o, T^o\rangle$ for which the sequence $\{I^p(x^\nu, y^\nu, T^\nu)\}_{\nu=1}^\infty$ does not converge to $I^p(x^o, y^o, T^o)$.

Let $\lambda^\nu = I^p(x^\nu, y^\nu, T^\nu)$ for $\nu = 1\ldots,\infty$ and $\lambda^o = I^p(x^o, y^o, T^o)$ so that the sequence $\{\lambda^\nu\}_{\nu_k=1}^\infty$ does not converge to λ^o. Therefore, there exists a subsequence $\{\lambda^{\nu_k}\}_{\nu_k=1}^\infty$ such that

$$\text{either} \quad \lambda^{\nu_k} > \lambda^o \quad \text{or} \quad \lambda^{\nu_k} < \lambda^o$$

for all elements of the subsequence.

By the definition of path-based indexes (2), $\Omega(x, y, I^p(x, y, T))$ is the intersection of the paths $\{\Omega(x, y, T) \mid \lambda \in \Lambda\}$ and $\partial(T)$. Moreover, (i) above implies that this intersection consists of a single point. Finally, condition (ii) above implies that

$$\{\Omega(x^\nu, y^\nu, \lambda^\nu)\}_{\nu=1}^\infty \to \Omega(x^o, y^o, \lambda^\nu)$$

for all $\lambda^\nu = 1, \ldots, \infty$ and

$$\{\Omega(x^\nu, y^\nu, \lambda^o)\}_{\nu=1}^{\infty} \to \Omega(x^o, y^o, \lambda^o).$$

Since Ω is continuous and $\lambda^\nu \not\to \lambda^o$, the sequence $\{\Omega(x^o, y^o, \lambda^\nu)\}_{\nu=1}^{\infty}$ does not converge to $\Omega(x^o, y^o, \lambda^o)$.

To ease the notation, denote $\Omega(x^\nu, y^\nu, I^p(x^\nu, y^\nu, T^\nu))$ by $\langle \tilde{x}^\nu, \tilde{y}^\nu \rangle$ for all ν and $\Omega(x^o, y^o, I^p(x^o, y^o, T^o))$ by $\langle \tilde{x}^o, \tilde{y}^o \rangle$.

The condition (c) on Ω implies that

$$\text{either} \quad \langle \tilde{x}^{\nu_k}, -\tilde{y}^{\nu_k} \rangle \gg \langle \tilde{x}^o, -\tilde{y}^o \rangle \quad \text{or} \quad \langle \tilde{x}^{\nu_k}, -\tilde{y}^{\nu_k} \rangle \ll \langle \tilde{x}^o, -\tilde{y}^o \rangle$$

for all elements in the subsequence $\{\nu_k\}_{k=1}^{\infty}$.

As $T^\nu \to T^o$ and $\{\Omega(x^\nu, y^\nu, \lambda) \mid \lambda \in \Lambda\}_{\nu=1}^{\infty}$ converges to $\{\Omega(x^o, y^o, \lambda) \mid \lambda \in \Lambda\}$, there exists a sequence $\{\bar{x}^\nu, \bar{y}^\nu\}_{\nu=1}^{\infty} \subset \partial(T^\nu)$ converging to $\langle \tilde{x}^o, \tilde{y}^o \rangle$. Hence, there exists a subsequence of $\{\nu_k\}$ denoted $\{\nu_{k_j}\}$ such that

$$\text{either} \quad \langle \tilde{x}^{\nu_{k_j}}, -\tilde{y}^{\nu_{k_j}} \rangle \gg \langle \bar{x}^{\nu_{k_j}}, -\bar{y}^{\nu_{k_j}} \rangle \quad \text{or} \quad \langle \tilde{x}^{\nu_{k_j}}, -\tilde{y}^{\nu_{k_j}} \rangle \ll \langle \bar{x}^{\nu_{k_j}}, -\bar{y}^{\nu_{k_j}} \rangle$$

for all elements of the subsequence.

As $\langle \tilde{x}^\nu, \tilde{y}^\nu \rangle$ and $\langle \bar{x}^\nu, \bar{y}^\nu \rangle$ are contained in the frontier of T^ν, either of these strict inequalities violates free disposability (FD).

This contradiction implies that $I^p(x^\nu, y^\nu, T^\nu)$ converges to $I^p(x^o, y^o, T^o)$ so that I^p satisfies continuity (C), in which case it must violate indication (I). $\diamondsuit$

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