Business School / School of Economics

UNSW Business School Working Paper

UNSW Business School Research Paper No. 2017 ECON 09

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March 31, 2017

Abstract: This paper investigates a problem of optimal growth with resource exhaustibility and pollution externality, based on a unified framework that explicitly considers augmentable man-made capital, exhaustible resource reserves, and accumulative environmental pollutants as three stock variables for optimal control analysis. Characterizations of the social optimum show that for any given man-made capital and resource reserves, resource extraction flows generated in optimal growth with both resource exhaustibility and pollution externality are smaller than those with only resource exhaustibility, and taking account of pollution externality resulting from resource extraction reduces the growth rate of consumption if man-made capital and natural resources are complements in final goods production. Existence, uniqueness and comparative statics of the steady state are analyzed. Conditions for transitional dynamics stability of optimal growth with resource exhaustibility and pollution externality are established. Expositions are made on whether allocations in a market equilibrium are consistent with the social optimum outcomes.

Key words: Sustainability; Economic Growth; Exhaustible Resources; Pollution Externality; Environmental Damage; Optimal Control Problems.

JEL Classification: Q54; Q43; Q32; O13; O44; C61

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*Jin acknowledges supports from the University of New South Wales Vice-Chancellor's Research Fellowship (RG152486). Woodland gratefully acknowledges the financial support provided by the Australian Research Council through its Discovery Grants program for this research (DP140101187). Views expressed here should not be attributed to the University of New South Wales or to the Australian Research Council.

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1 Introduction

One of the most important challenges facing our society is the endeavor to sustain economic prosperity without depleting the Earth's exhaustible natural resources and without jeopardizing the vulnerable environment (World Bank, 2011). A sensible way to solve this complex problem requires taking a holistic approach to target the intricate connection between economic growth, natural resource, and the environment.

First, there is a clear case that growth and natural resources are closely linked given that natural resources (such as minerals and fossil fuels) provide intermediate inputs to produce final goods. An accessible supply of resource is productive to the economy and a decline in supply due to resource scarcity will constrain production and impose shadow costs on the economy (e.g., Hamilton, 2009; Dargay and Gately, 2010; Reynolds and Baek, 2012; Lutz et al., 2012). Second, extraction and utilization of natural resources, particularly the fossil fuels that still dominate world energy markets, is tightly connected to anthropogenic emissions of environmental pollutants. Extraction of polluting resources gives rise to emissions of conventional air pollutants at local levels (e.g., particulate, sulfur oxides, and nitrogen oxides), and greenhouse gas emissions from burning fossil-based resource have been identified as the dominant contributor to climate change at the global level (IPCC, 2007; World Bank, 2011; IEA, 2014). Third, there is no disagreement that a polluted environment (e.g., particulate, acid rain and rising ozone levels) generates negative externality effects that are exacting heavy losses in social welfare (Chay and Greenstone, 2003; Chen et al., 2013; Zivin and Neidell, 2012). In particular, the climate change effects such as heat waves, floods, droughts and storm surges induced by greenhouse gas emissions bring about a greater prevalence of pollution damages and disutility (IPCC, 2007; Weitzman, 2007).

In this context, we think it is significant to bring together economic growth, natural resource and the environment into a unified framework and investigate their underlying dynamic relationship in a sustainable growth path. By doing so, this study will help enhance our understanding of economic mechanisms relating to growth sustainability, resource exhaustibility and pollution externality, and improve the effectiveness of policy responses to the grand challenge facing our society. Therefore, the central aim of this research is to examine optimal growth with both resource exhaustibility and pollution externality, based on a unified framework with an explicit account of dynamic interactions between augmentable man-made capital, exhaustible resource reserves and accumulative environmental pollutants.

1.1 Related Literature

To explore this issue, we think there is a clear case for taking the theory of natural resource and environmental economics as a point of departure. Accordingly, we restrict ourselves to a brief review of two strands in resource and environmental economics literature on which our own analysis draws.

First, it is natural to think of the planet as a source of supplying energy, minerals and other natural resources, and this interpretation of the nature's function to mankind led to a large and still growing literature on growth and resource exhaustibility.[1] In particular, this strand of literature is enriched with the theory of optimal extraction of exhaustible resources finding its origin in the pioneering work of Hotelling (1931) and afterward the seminal Dasgupta-Heal-Solow-Stiglitz (DHSS thereafter) models, which features model representations of an economy by two capital stocks - an augmentable man-made capital and an exhaustible resource reserve (Dasgupta and Heal, 1979; Solow, 1974; Stiglitz, 1974). Existing works along the line of DHSS models allow the economists to fruitfully address various interesting issues, for example, (1) relations between capital accumulation, resource depletion, and the Hartwick's rule (e.g., Hartwick, 1977; Dixit et al., 1980; Withagen and Asheim, 1998; Asheim et al., 2012; Mitra et al., 2013); (2) effects of non-renewable resources taxation on sustainable growth path (e.g., Schou, 2002; Groth and Schou, 2007); and (3) relations between knowledge accumulation, endogenous technology change, and exhaustible natural resources (e.g., Barbier, 1999; Tsur and Zemel, 2003; Grimaud and Rouge, 2003, 2005; Bretschger and Smulders, 2012).

While the DHHS models lay a solid foundation for analyzing the relations between optimal growth and resource scarcity, there is an apparent need to extend the DHHS framework by incorporating an explicit account of the environment that is closely linked to natural resources. Specifically, given the fact that extraction and utilization of polluting resources such as fossil-based energy resources brings about anthropogenic emissions of environmental pollutants and damages, investigations of issues related to growth and resource extraction are expected to gain new insights if pollution externality resulting from resource extraction is incorporated into traditional growth models with only resource exhaustibility.

[1]See Krautkraemer (1998) for a survey.

Second, the connection between growth and the environment has received attention because of a large body of empirical works studying the relation between economic development and environmental degradation, known as the Environmental Kuznets Curve (e.g., Grossman and Krueger, 1995; Selden and Song, 1994; Stern and Common, 2001; Dasputa et al., 2002). A rapidly expanding theoretical literature attempts to rationalize the empirical relations by examining growth, pollution abatement and environmental quality, see, for example, Keeler et al. (1974); Forster (1973); Gruver (1976); Tahvonen and Withagen (1996); Selden and Song (1995); Mohtadi (1996); Stokey (1998); John and Pecchenino (1994), and more recently Bartz and Kelly (2008); Rubio et al. (2009).[2]

Basically, the models specified in these works describe the economy by two stocks - an augmentable man-made capital and a stock of environmental quality or pollutants. As the factor affecting dynamics of the environmental stocks, pollution emissions or the rate of environmental degradation is modeled as a flow variable that depends on goods production and consumption or capital accumulation, without establishing the connection between pollution emissions and natural resources. However, given the fact that emissions of environmental pollutants are basically driven by extraction and utilization of natural resources (e.g., emissions profiles of carbon pollutants are shaped by extraction and combustion of fossil-based energy resources), a finer modelling framework should feature a representation of natural resources as the direct determinant of pollution emissions. This argument coincides with a strand of recent environmental economics literature known as the Green Paradox, saying that the supply-side profiles of fossil-based resources (such as resource exhaustibility, extraction costs and emissions intensity) are underlying fundamental determinants of fossil-induced environmental consequences (Sinn, 2012; Grafton et al., 2012; Smulders et al., 2012; van der Ploeg and Withagen, 2012a, 2014). From this point of view, there is also a particular need to establish the connection between natural resources and the environment for addressing environmental and pollution control problems.

For the above-mentioned reasons, this work aims to create a linkage between studies on growth and exhaustible resources on the one hand, and the literature on growth and the environment on the other hand, by investigating a general problem of optimal growth with resource exhaustibility and pollution externality.[3] For this purpose, the present paper will provide a unified framework

[2]For a survey of theoretical models on the Environmental Kuznets Curve, see Kijima et al. (2010).

[3]While there are some studies that feature specifications of both exhaustible resources and accumulative pollutants as two stock variables for addressing resource and environmental economics problems (e.g., Withagen,

that represents augmentable man-made capital, exhaustible resource reserves, and accumulative environmental pollutants as three stock variables for optimal control analysis. By doing that, our present work is expected to play a positive role in moving forward most of the existing natural resource and environmental economics literature that offer theoretical insights based on a two-stock optimal control framework.

In general, an increase in the number of stock variables in an optimal control model will substantially complicate the analysis of their relations in a dynamic setting, and without simplifying assumptions it will be challenging to analytically characterize important relationships that underline the model with three stock variables. For this reason, most of the existing literature tend to address their research problems based on a two-stock dynamic framework, and there are only a few studies developing three-stock modelling frameworks to address resource and environmental economics issues, see Table 1, for example, Tahvonen and Kuuluvainen (1991), Bovenberg and Smulders (1995), Schou (2000), Tahvonen and Salo (2001), Tsur and Zemel (2005), Chakravorty et al. (2008) and Acemoglu et al. (2012).[4] To sum up, while the above-cited studies with different model specifications offer methodological insights into optimal control analysis with three stock variables to address natural resource and environmental economics, the problems specific to interactions between augmentable capital, exhaustible resources and accumulative pollutants are hardly explored in the literature. This research gap could be filled if a general problem of optimal growth with resource exhaustibility and pollution externality can be addressed by our study in this paper.

1994; Ulph and Ulph, 1994; Sinclair, 1994; Hoel and Kverndokk, 1996; Tahvonen, 1997; Chakravorty et al., 2006; van der Ploeg and Withagen, 2012a,b; Prieur et al., 2013), a somewhat weak aspect is that the scopes of these works are confined to natural environment, without casting the analysis in a growth context where resource extraction and environmental pollution are deem to be interactive with production, consumption and capital accumulation.

[4] Tahvonen and Kuuluvainen (1991) analyze the dynamic relations between economic growth, renewable resource harvesting and pollution emissions where man-made capital, renewable resources, and pollutants are considered as three stock variables. Bovenberg and Smulders (1995) investigate the conditions for a balanced growth path in an endogenous growth model where the three stock variables specified are natural capital, physical capital and pollution-augmenting technology. Schou (2000) also analyzes the condition of balanced growth paths based on an endogenous growth model with physical capital, human capital and non-renewable resources as three stock variables. Tahvonen and Salo (2001) investigate the transition between renewable to nonrenewable energy regime where man-made capital, exhaustible resources and technological knowledge are considered as three stock variables. Tsur and Zemel (2005) explore the dynamic relations between resource scarcity, economic growth and R&D, where physical capital, primary resources and technological knowledge are specified as three stock variables. Chakravorty et al. (2008) examine the order of extraction of non-renewable resources differentiated by pollution intensity, based on a three-stock optimal control model with two polluting non-renewable resource stocks and an aggregate stock of pollutants. Acemoglu et al. (2012) analyze directed technical change in a quality-improving Schumpeterian endogenous growth model with resource exhaustibility and environmental constraints, and the three stock variables specified are exhaustible natural resources, environmental quality and the time-varying quality of intermediate machines.

In this regard, our present work could be viewed as a helpful complement to the recent seminal works by Golosov et al. (2012) and van der Ploeg and Withagen (2014) which consider physical capital, exhaustible fossil energy and atmospherical concentration of carbon dioxide in their analysis of climate economics problems.[5] But the key difference is that our present work presents generalized treatments of man-made capital, exhaustible resources, and environmental pollutants as three separate stock variables throughout model specifications, characterizations, and analysis, while Golosov et al. (2012) and van der Ploeg and Withagen (2014) impose several simplifying assumptions to reduce the model into a system with two stock variables when it comes to analytical characterizations of the dynamic optimum and derivation of key results.[6] Furthermore, our study in this paper provides detailed expositions and methodological insights into the essential properties of steady state and transitional dynamic specific to optimal growth with resource exhaustibility and environmental externality, which is still missing in the current literature but would be helpful for future studies on the economics of growth, exhaustible resources, and environmental pollution.

The rest of this paper is structured as follows. Section 2 presents specifications of the model. Section 3 considers the social optimum problem with characterizations of the optimal growth path. Section 4 and 5 investigate steady state and transitional dynamics stability, respectively. Section 6 examines allocations in a decentralized market equilibrium and its consistency with the social optimum outcomes. Section 7 provides concluding remarks.

[5] Golosov et al. (2012) develop a discrete-time dynamic general equilibrium model with climate externality resulting from fossil energy use, and derive a simple formula for the marginal externality damage of emission. Fossil energy extracted from the exhaustible stock is productive to final goods production, carbon emissions from using fossil energy contribute to the accumulative stock of carbon concentration which is linked to climate damages externality on goods production. van der Ploeg and Withagen (2014) present a Green Ramsey growth model to investigate different regimes of switch from carbon-based exhaustible energy to carbon-free renewable backstops. Man-made capital and energy are inputs in the production process, energy from exhaustible oil reserve and renewable backstops are perfect substitutes, and convex global warming damages depend on the stock of atmospheric carbon.

[6] Golosov et al. (2012) assumes full depreciation of capital after use in each period and capital is simplified as an intermediates input without accumulation across periods, thus reducing the original three-stock system into a two-stock one (resource reserves and pollution stocks). van der Ploeg and Withagen (2014) simplify the specification of the accumulative stock of pollutants by omitting anthropogenic activities of pollution abatement, and the exhaustible stock of fossil energy and the accumulative stock of pollutants are fully correlated (i.e., the rates of change are of opposite sign but of equal absolute magnitude), and the model is thus reduced to a system with only two stock variables (man-made capital and exhaustible resource stock).

2 The Model

The model is formulated in continuous time over an infinite horizon $t \in [0,\infty)$. At each point in time anthropogenic activities of resource extraction and utilization deplete a finite stock of resource reserves, and the law of motion for the exhaustible resource stock is specified as

$$\dot{X}(t) = -E(t), \tag{1}$$

where $E(t)$ and $X(t)$ are resource extraction flows and the remaining stocks of resource reserves at time t, respectively. The "dot" above a variable corresponds to the time derivative of that variable. In addition to the physical exhaustibility and scarcity, extraction of natural resources also has underlying environmental impacts. Suppose that extraction and utilization of resources such as fossil energy is environmentally polluting, and extracting one unit of fossil-based resources generates one unit of pollutant emissions for normalization (the pollution intensity of resources is basically a constant physical property). The flows of pollutant emissions then contribute to accumulations of environmental pollution stocks via

$$\dot{Z}(t) = E(t) - B(t), \tag{2}$$

where Z is the stock of environmental pollutants which is increasing in pollution emissions resulting from resource extraction E, and decreasing by pollution abatement B.

The resources extracted from the exhaustible stock are used as intermediate inputs (that fully depreciate after use) to produce final goods. With an augmentable man-made capital as the other production input, the technology of final goods production is given by

$$Y(t) = \mathrm{F}(K(t), E(t)), \tag{3}$$

where Y and K are final goods outputs and man-made capital, respectively. Outputs of final goods are continuously increasing with man-made capital and resource inputs, and the marginal productivity of man-made capital and resources is diminishing. The production function $\mathrm{F}: \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ thus satisfies the assumptions as follows.[7]

[7]The sign of F_{KE} depends on whether K and E are substitutes $\mathrm{F}_{KE} < 0$ or complements $\mathrm{F}_{KE} > 0$. Time-series data classifies the two inputs as complements, while cross-section studies conclude that capital and energy are substitutes in production, (e.g., Apostolakis, 1990; Thompson and Taylor, 1995; Frondel and Schmidt, 2002; Koetse et al., 2008). Explanations are twofold. First, man-made capital and energy resources act more as substitutes

Assumption 1. $\mathrm{F}_K \equiv \frac{\partial Y}{\partial Y} > 0, \mathrm{F}_E \equiv \frac{\partial Y}{\partial E} > 0$, $\mathrm{F}_{KK} \equiv \frac{\partial^2 Y}{\partial K^2} < 0$, $\mathrm{F}_{EE} \equiv \frac{\partial^2 Y}{\partial E^2} < 0$, *and* $\mathrm{F}_{EE}\mathrm{F}_{KK} - \mathrm{F}_{KE}^2 > 0$.

Outputs of final goods are allocated to consumption, capital investment, resource extraction, and pollution abatement, and the stock of man-made capital is accumulated via

$$\dot{K}(t) = \mathrm{F}(K(t), E(t)) - \mathrm{M}(X(t), E(t)) - \mathrm{G}(B(t)) - C(t), \tag{4}$$

where outputs of final goods (the numéraire) Y are equal to the sum of consumption C, resource extraction expenditure $\mathrm{M}(X,E)$, pollution abatement expenditure $\mathrm{G}(B)$, and investment in man-made capital $\dot{K}$. Specifically, the costs of resource extraction are determined by both extraction flow and the remaining resource reserves, and the extraction cost function $\mathrm{M}: \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuously increasing with extraction flows E and decreasing with the remaining stock of resource reserves X. Suppose that the marginal cost of extraction M_E is non-decreasing with the extraction flows E and decreasing with the remaining stock of resources X. Meanwhile, the stock-dependent extraction costs are convex since a lower remaining stock of resource reserves makes the costs of further extraction rise at an increasing rate.[8] These assumptions are expressed as follows.

Assumption 2. $\mathrm{M}_E \equiv \frac{\partial \mathrm{M}}{\partial E} > 0$, $\mathrm{M}_X \equiv \frac{\partial \mathrm{M}}{\partial X} < 0$, $\mathrm{M}_{EE} \equiv \frac{\partial^2 \mathrm{M}}{\partial E^2} \geq 0$, $\mathrm{M}_{EX} \equiv \frac{\partial^2 \mathrm{M}}{\partial E \partial X} < 0$, *and* $\mathrm{M}_{XX} \equiv \frac{\partial^2 \mathrm{M}}{\partial X^2} > 0$.

Furthermore, the fraction of final goods outputs allocated to pollution abatement expenditure is measured by a convex function of pollution abatement costs $\mathrm{G}: \mathbb{R}_+ \mapsto \mathbb{R}_+$ which has a positive and increasing marginal cost, i.e.,

Assumption 3. $\mathrm{G}'(B) \equiv \frac{dG}{dB} > 0$, *and* $\mathrm{G}''(B) \equiv \frac{d^2 G}{dB^2} > 0$.

Finally, the economy admits a representative household with preferences valuing goods consumption and environmental stock, with her instantaneous utility function specified as

$$U(t) \equiv \mathrm{U}(C(t), Z(t)), \tag{5}$$

in the long run and more as complements in the short run. Second, the complementarity only occurs in cases where the cost share of energy resources in production is small (service sectors), and the substitutability is more likely to occur in cases where energy resources have large cost shares in production (electric utility sectors).

[8] Taking the cost function of resource extraction $\mathrm{M}(X, E) = \mathrm{mc}(X)E$ as an example, the total costs are determined by both extraction flows E and the marginal cost of resource extraction $\mathrm{mc}(X)$, where the latter depends on the remaining reserves of exhaustible resources X with $\mathrm{mc}'(X) > 0$ and $\mathrm{mc}''(X) > 0$.

where the utility function $\mathrm{U}:\mathbb{R}_+\times\mathbb{R}_+\mapsto\mathbb{R}_+$ is strictly increasingly in consumption but decreasing in pollution. The marginal utility of consumption is decreasing in both consumption and the stock of pollutants, and the marginal disutility of pollution is increasing in the stock of pollution (i.e., convex pollution damages). Thus, the following assumptions hold.

Assumption 4. $\mathrm{U}_C\equiv\frac{\partial U}{\partial C}>0$, $\mathrm{U}_Z\equiv\frac{\partial U}{\partial Z}<0$, $\mathrm{U}_{CC}\equiv\frac{\partial^2 U}{\partial C^2}<0$, $\mathrm{U}_{CZ}\equiv\frac{\partial^2 U}{\partial C\partial Z}<0$, $\mathrm{U}_{ZZ}\equiv\frac{\partial^2 U}{\partial Z^2}<0$, *and* $\mathrm{U}_{CC}\mathrm{U}_{ZZ}-\mathrm{U}_{CZ}^2>0$.

3 Social Optimum

Given the above-described model, we consider a social optimum problem to characterize the time path of consumption, resource extraction, pollution abatement, man-made capital, resource reserves, and pollution stocks $[C(t),E(t),B(t),K(t),X(t),Z(t)]_{t=0}^{\infty}$ that maximize discounted present values of social welfare given by

$$V(K_0,X_0,Z_0)\equiv\max_{[C(t),E(t),B(t)]_{t=0}^{\infty}}\int_0^{\infty}\exp(-\rho t)\mathrm{U}(C(t),Z(t))dt, \tag{6}$$

subject to the law of motion of man-made capital (4), resource reserves (1) and pollution stock (2), given their initial conditions $[K_0,X_0,Z_0]$. The present values of welfare are discounted by a rate of time preference $\rho>0$. Consumption, resource extraction, and pollution abatement $[C,E,B]$ are control variables, while man-made capital, exhaustible resources, and environmental pollutants $[K,X,Z]$ are state variables. The discount rate ρ and initial states $[K_0,X_0,Z_0]$ are exogenously given. $\mathrm{F}(K,E)$, $\mathrm{U}(C,Z)$, $\mathrm{M}(E,X)$ and $\mathrm{G}(B)$ are the function constraints on production, preference, extraction and abatement, respectively. Throughout the analysis, all these functions are assumed to be twice continuously differentiable.

A time path of goods consumption, resource extraction, pollution abatement, man-made capital, resource reserves, and environmental pollutant stocks $[C(t),E(t),B(t),K(t),X(t),Z(t)]_{t=0}^{\infty}$ is feasible if $[K(t),X(t),Z(t)]_{t=0}^{\infty}\geq 0$, $[K(0),X(0),Z(0)]=[K_0,X_0,Z_0]$, and $[C(t),E(t),B(t),K(t),X(t),Z(t)]_{t=0}^{\infty}$ satisfies (1), (2), and(4) for $\forall t\in[0,\infty)$. We simplify notations by surpassing the time argument t and denote $[C,E,B,K,X,Z]=[C(t),E(t),B(t),K(t),X(t),Z(t)]_{t=0}^{\infty}$. To characterize allocations in the social optimum, we define the ordinary current-value Hamiltonian function $\mathcal{H}^0:\mathbb{R}_+^9\mapsto\mathbb{R}_+$ that corresponds to the optimal control problem as

$$\mathcal{H}^0(C,E,B,K,X,Z,\lambda,\mu,\psi) \equiv \mathrm{U}(C,Z) + \lambda(\mathrm{F}(K,E) - \mathrm{M}(X,E) - \mathrm{G}(B) - C) - \mu E + \psi(E - B),$$

where λ, μ, ψ is the co-state variables corresponding to man-made capital K, exhaustible resources X, and environmental pollutants Z, respectively.

3.1 Necessary Conditions of Optimality

Following from the Pontryagin Maximum Principle, the necessary conditions of optimality are given as follows.

Lemma 1. *Suppose that $[C,E,B,K,X,Z]$ is an interior solution to the problem of maximizing (6) subject to (1), (2) and (4). Then there exist continuously differentiable co-state variables λ,μ,ψ, such that the optimal control variables C,E,B and the corresponding paths of state variables K,X,Z satisfy the necessary conditions of optimality given by*

$$0 = \mathrm{U}_C(C,Z) - \lambda, \tag{7a}$$

$$0 = \lambda(\mathrm{F}_E(K,E) - \mathrm{M}_E(X,E)) - \mu + \psi, \tag{7b}$$

$$0 = -\lambda \mathrm{G}'(B) - \psi, \tag{7c}$$

$$\rho\lambda - \dot{\lambda} = \lambda \mathrm{F}_K(K,E), \tag{7d}$$

$$\rho\mu - \dot{\mu} = -\lambda \mathrm{M}_X(X,E), \tag{7e}$$

$$\rho\psi - \dot{\psi} = \mathrm{U}_Z(C,Z), \tag{7f}$$

$$\dot{K} = \mathrm{F}(K,E) - \mathrm{M}(X,E) - \mathrm{G}(B) - C, \tag{7g}$$

$$\dot{X} = -E, \tag{7h}$$

$$\dot{Z} = E - B, \tag{7i}$$

and the transversality conditions

$$\lim_{t\to+\infty} \exp(-\rho t)\lambda K = 0, \quad \lim_{t\to+\infty} \exp(-\rho t)\mu X = 0, \quad \lim_{t\to+\infty} \exp(-\rho t)\psi Z = 0. \tag{8}$$

First, characterization given by (7a) states that consumption should reach socially optimal levels at which the marginal utility of consumption equals the shadow price of man-made capital.

(7a) implicitly determines the optimal control of consumption denoted by $\hat{C}$ as

$$\hat{C} = \mathrm{C}(Z,\lambda), \tag{9}$$

where the function of optimal consumption $\mathrm{C}:\mathbb{R}_+^2 \mapsto \mathbb{R}_+$ depends on pollution stock Z and the shadow price of man-made capital λ, with the first-order derivative properties given by

$$\mathrm{C}_Z \equiv \frac{\partial \hat{C}}{\partial Z} = -\frac{\mathrm{U}_{CZ}}{\mathrm{U}_{CC}} < 0, \qquad \mathrm{C}_\lambda \equiv \frac{\partial \hat{C}}{\partial \lambda} = \frac{1}{\mathrm{U}_{CC}} < 0. \tag{10}$$

Intuitively, the optimal growth path tends to have lower consumption if there is a larger stock of accumulated pollution because a heavily polluted environment lowers the marginal utility from consumption. Meanwhile, a larger shadow value of man-made capital, via offering an incentive to allocate final goods towards capital investment, will reduce consumption levels.

Second, extraction of the exhaustible resources along the social optimum path is characterized by (7b), i.e.,

$$\mathrm{F}_E(K,E) = \underbrace{\mathrm{M}_E(E,X)}_{\text{marginal extraction cost}} + \underbrace{\frac{\mu}{\lambda}}_{\text{resource exhaustibility cost}} + \underbrace{\left(\frac{-\psi}{\lambda}\right)}_{\text{pollution externality cost}}. \tag{11}$$

It is shown that resource extraction should reach an amount where the marginal product of resources equals the sum of marginal direct extraction cost and social cost (shadow values) related to resource exhaustibility and pollution externality (shadow costs are converted from utility units to final goods units by dividing the shadow value of man-made capital). The shadow value of the exhaustible resource reserves is determined as

$$\mu(t) = \int_{s=t}^{\infty} \exp[-\rho(s-t)][-\mathrm{U}_C(C(s),Z(s))\mathrm{M}_X(E(s),X(s))]ds > 0, \tag{12}$$

where $\mu > 0$ given that $\mathrm{U}_C > 0$ and $\mathrm{M}_X < 0$. The shadow value of the pollution stock is given as

$$\psi(t) = \int_{s=t}^{\infty} \exp[-\rho(s-t)]\mathrm{U}_Z(C(s),Z(s))ds < 0, \tag{13}$$

where $\psi < 0$ following from $\mathrm{U}_Z < 0$. Furthermore, equation (11) determines the optimal levels of resource extraction as

$$\hat{E} = \mathrm{E}(K,X,\lambda,\mu,\psi), \tag{14}$$

and the function of optimal resource extraction $\mathrm{E}:\mathbb{R}_+^5 \mapsto \mathbb{R}_+$ has the first-order derivative properties given by

$$\mathrm{E}_K \equiv \frac{\partial \hat{E}}{\partial K} = \frac{-\mathrm{F}_{EK}}{\mathrm{F}_{EE} - \mathrm{M}_{EE}}, \quad \mathrm{E}_X \equiv \frac{\partial \hat{E}}{\partial X} = \frac{\mathrm{M}_{EX}}{\mathrm{F}_{EE} - \mathrm{M}_{EE}}, \quad \mathrm{E}_\lambda \equiv \frac{\partial \hat{E}}{\partial \lambda} = \frac{-\mu + \psi}{\lambda^2(\mathrm{F}_{EE} - \mathrm{M}_{EE})},$$
$$\mathrm{E}_\mu \equiv \frac{\partial \hat{E}}{\partial \mu} = \frac{1}{\lambda(\mathrm{F}_{EE} - \mathrm{M}_{EE})}, \quad \mathrm{E}_\psi \equiv \frac{\partial \hat{E}}{\partial \psi} = \frac{-1}{\lambda(\mathrm{F}_{EE} - \mathrm{M}_{EE})}, \tag{15}$$

where $\mathrm{F}_{EE} - \mathrm{M}_{EE} < 0$ is the marginal effect of resource extraction flows E on marginal extraction revenues $\pi \equiv \mathrm{F}_E - \mathrm{M}_E$, and the negative sign follows from $\mathrm{F}_{EE} < 0$ and $\mathrm{M}_{EE} \geq 0$. Intuitively, the amounts of resource extraction will be raised by an increase in the remaining resource reserves ($\mathrm{E}_X > 0$), the shadow price of man-made capital ($\mathrm{E}_\lambda > 0$),[9] and the shadow value of pollution stock ($\mathrm{E}_\psi > 0$).[10] Furthermore, an increase in the shadow value of exhaustible resources would lower resource extraction such that larger resource reserves can be left in situ ($\mathrm{E}_\mu < 0$), but the marginal effect of man-made capital on resource extraction E_K depends on whether man-made capital and resources are substitutes or complements.

Third, characterizations given by (7c) suggest that pollution abatement should reach a level where the marginal cost of abatement is equal to the shadow value of pollution stock, i.e., $\mathrm{G}'(B) = -\frac{\psi}{\lambda}$, which determines optimal control of pollution abatement as

$$\hat{B} = \mathrm{B}(\lambda, \psi), \tag{16}$$

where the function of optimal abatement $\mathrm{B}: \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ has the first-order derivatives

$$\mathrm{B}_\lambda \equiv \frac{\partial \hat{B}}{\partial \lambda} = -\frac{\mathrm{G}'}{\lambda \mathrm{G}''} < 0, \qquad \mathrm{B}_\psi \equiv \frac{\partial \hat{B}}{\partial \psi} = -\frac{1}{\lambda \mathrm{G}''} < 0. \tag{17}$$

Intuitively, a larger shadow value of man-made capital induces the social planner to allocate more final goods to investment in physical capital and thus reduces allocations to pollution abatement. Meanwhile, as the shadow value of pollution stocks increases, there will be a lower level of abatement such that a greater stock of pollutants can be accumulated.

[9] A larger shadow price of man-made capital translates into lower shadow costs associated with resource exhaustibility and pollution externality, thus reducing marginal revenue of resource extraction and increasing the flows of resource extractions.

[10] An increase in the shadow value ψ is equivalent to a decrease in the shadow cost of pollution externality ψ, thus reducing marginal revenues of resource extraction and increasing the flows of resource extractions.

3.2 Sufficient Conditions of Optimality

Lemma 1 provides the necessary condition for the interior optimal solution. To consider the sufficient condition of optimality corresponding to our optimal control problem, we define the maximized Hamiltonian function as

$$\mathcal{H}(K,X,Z,\lambda,\mu,\psi)=\mathrm{U}(\hat{C},Z)+\lambda\Big[\mathrm{F}(K,\hat{E})-\mathrm{M}(\hat{E},X)-\mathrm{G}(\hat{B})-\hat{C}\Big]-\mu\hat{E}+\psi\Big(\hat{E}-\hat{B}\Big),$$

where the arguments of the maximized Hamiltonian function $\mathcal{H}:\mathbb{R}_+^6\mapsto\mathbb{R}_+$ are state and co-state variables K,X,Z,λ,μ,ψ, and $\hat{C},\hat{B},\hat{E}$ are the control variables optimally determined as a function of state and costate variables by (9), (14) and (16). Then, based on the Arrow's sufficiency theorem on the optimal control theory (Arrow, 1968; Kamien and Schwartz, 1971), we establish the sufficient condition of optimality corresponding to our optimal control problems as in the following lemma.

Lemma 2. *Suppose that there exists an interior continuous solution* $[C,E,B,K,X,Z]$ *that satisfies the necessary conditions of optimality (7). If the functions of production, preference, and resource extraction satisfy the conditions*

$$\frac{\mathrm{U}_{CC}\mathrm{U}_{ZZ}-\mathrm{U}_{CZ}^2}{\mathrm{U}_{CC}}<0,\quad \frac{\mathrm{M}_{XX}\mathrm{M}_{EE}-\mathrm{M}_{XE}^2}{\mathrm{M}_{XX}}<\frac{\mathrm{F}_{EE}\mathrm{F}_{KK}-\mathrm{F}_{KE}^2}{\mathrm{F}_{KK}}, \tag{18}$$

then the path $[C,E,B,K,X,Z]$ *that satisfies the necessary conditions of optimality (7) is the unique solution of the problem of maximizing (6) subject to (1), (2) and (4).*

Proof. See Appendix A. □

The sufficient conditions of optimality to the problem of optimal growth with resource exhaustibility and pollution externality have the following interesting features. First, the utility function representing preferences should be strictly concave with respect to its two arguments, i.e., $\mathrm{U}_{CC}<0$ and $\mathrm{U}_{CC}\mathrm{U}_{ZZ}-\mathrm{U}_{CZ}^2>0$. Second, given that final goods production function is strictly concave with respect to its two arguments, i.e., $\mathrm{F}_{KK}<0$, $\mathrm{F}_{EE}<0$ and $\mathrm{F}_{EE}\mathrm{F}_{KK}-\mathrm{F}_{KE}^2>0$, the second order leading principal minors of the resource extraction cost function must be negative, i.e., $\mathrm{M}_{XX}\mathrm{M}_{EE}-\mathrm{M}_{XE}^2<0$, such that the second sufficient condition can be established, i.e., the ratio of leading principal minors between the second and first order of $\mathrm{M}(E,X)$ is smaller than that of $\mathrm{F}(K,E)$. Hence, if the extraction cost function is convex with respect to its two arguments, i.e., $\mathrm{M}_{XX} > 0$, $\mathrm{M}_{EE} \geq 0$ and $\mathrm{M}_{XX}\mathrm{M}_{EE} - \mathrm{M}_{XE}^2 > 0$, then the sufficient condition for optimality would be violated.

3.3 Characterization of Dynamic Behaviors

Based on the necessary conditions of optimality (7a)-(7c), the dynamic behavior of consumption, resource extraction and pollution abatement can be described as

$$\frac{\dot{C}}{C} = \sigma\left[\mathrm{F}_K - \rho + \frac{\mathrm{U}_{CZ}}{\mathrm{U}_C}(E-B)\right], \tag{19a}$$

$$\dot{\pi} = \mathrm{F}_K\pi + \mathrm{M}_X + \frac{\mathrm{U}_Z}{\mathrm{U}_C}, \tag{19b}$$

$$\dot{B} = \frac{\mathrm{F}_K\mathrm{G}'}{\mathrm{G}''} + \frac{\mathrm{U}_Z}{\mathrm{U}_C\mathrm{G}''}, \tag{19c}$$

where $\sigma \equiv -\frac{\mathrm{U}_C}{C\mathrm{U}_{CC}}$ is the intertemporal elasticity of consumption substitution and $\pi \equiv \mathrm{F}_E - \mathrm{M}_E$ are marginal revenues of resource extraction.

(19a) expresses the Euler's consumption rule where both the marginal product of man-made capital F_K and the rate of time preference ρ have positive effects on the growth of consumption. Furthermore, if the connection between resource extraction and environmental pollution is factored into the model, then the negative effect of pollution on marginal utility of consumption $\mathrm{U}_{CZ} < 0$ and accumulation of the pollution stock $\dot{Z} = E - B$ will also affect the dynamic profiles of consumption.

(19b) extends the Hotelling rule on extraction of exhaustible resources: marginal revenues of resource extraction $\pi \equiv \mathrm{F}_E - \mathrm{M}_E$ increases at the rate of return on man-made capital F_K, adjusted by the marginal effect of in-situ resource reserves on extraction costs $\mathrm{M}_X < 0$ (i.e., the negative sign is due to that extractions reduce the remaining resource reserves and increase the stock-dependent extraction costs). Moreover, if the pollution externality resulting from resource extraction is taken into account, then accumulation of the pollution stock due to resource extraction would generate pollution damages $\frac{\mathrm{U}_Z}{\mathrm{U}_C} < 0$, which *ceteris paribus* would lower the growth rate of marginal revenues of resource extraction.

(19c), rewritten as $\dot{\mathrm{G}}' = \mathrm{F}_K\mathrm{G}' + \frac{\mathrm{U}_Z}{\mathrm{U}_C}$, gives the dynamic profile of pollution abatement. With the pollution externality taken into account, marginal costs of pollution abatement increases at the rate of return on man-made capital F_K. Furthermore, as the marginal cost of pollution abatement depends on the shadow cost of pollution externality in the social optimum, the rate of change in marginal costs of pollution abatement would be affected by marginal pollution damages $\frac{\mathrm{U}_Z}{\mathrm{U}_C}<0$.

As mentioned previously, investigations of growth and resource extraction would generate new results if the connection between natural resources and the environment is incorporated into a traditional growth framework with only resource exhaustibility. The following proposition summarizes this result.

Proposition 1. *For a given state of both man-made capital and resource reserves, the corresponding resource extraction flows generated in optimal growth with both resource exhaustibility and pollution externality are smaller than those in optimal growth with only resource exhaustibility. Furthermore, if man-made capital and natural resources are complements in final goods production, then taking account of pollution externality will reduce the growth rate of consumption.*

Proof. See Appendix B. □

Following from (19), the dynamic profile of optimal growth with both resource exhaustibility and pollution externality can feature the following three phases. First, the economy starts with a pristine environment and abundant resource reserves. Given this pristine environment with a sufficiently small stock of pollutants, the shadow cost related to pollution externality could be relatively smaller.[11] Meanwhile, an abundant supply of resource reserves leads to a lower shadow cost related to resource exhaustibility. With smaller shadow costs associated with resource exhaustibility and pollution externality, marginal revenues of resource extraction could thus be relatively lower and resource extraction flows could be relatively larger.[12] As the same time, marginal benefits of pollution abatement (this is equal to the shadow cost of pollution externality)

[11]Due to the convexity of pollution damage $\mathrm{U}_{ZZ}<0$, a small stock of pollutants translates into a lower level of marginal pollution damages, and shadow values of the pollution stock as given in (13) would be very small.

[12]Specifically, solving (19b) yields $\pi(t)=\int_{s=t}^{\infty}\exp\left[-\int_{s'=t}^{s}r(s')ds'\right]\left[-\mathrm{M}_X-\frac{\mathrm{U}_Z}{\mathrm{U}_C}\right]ds$, where a time-varying rate of return on capital is given by $r\equiv\mathrm{F}_K$, and marginal revenues of resource extraction are equal to discounted present value of shadow costs associated with resource exhaustibility (the first term within the integral) and pollution externality (the second term within the integral). Consider the marginal revenues of resource extraction at a point in time t_0 during initial growth phase which takes the form as

$$\pi(t_0)=\int_{t_0}^{t_1}\exp\left[-\int_{t_0}^{s}r(s')ds'\right]\left[-\mathrm{M}_X-\frac{\mathrm{U}_Z}{\mathrm{U}_C}\right]ds+\exp\left[-\int_{t_0}^{t_1}r(s')ds'\right]\pi(t_1) \tag{20}$$

Over the initial growth phase $[t_0,t_1]$, the pristine environment and abundant resource reserves leads to an outcome where the value of $-\mathrm{M}_X-\frac{\mathrm{U}_Z}{\mathrm{U}_C}$ is sufficiently small, thus $\pi(t_0)<\pi(t_1)$ due to $\exp\left[-\int_{t_0}^{t_1}r(s')ds'\right]<1$. Furthermore, given that $\mathrm{F}_{EE}-\mathrm{M}_{EE}<0$, marginal revenues of resource extraction $\pi\equiv\mathrm{F}_E-\mathrm{M}_E$ are negatively related to resource extraction flows E for a given state of K and X.

are relatively smaller and could be less than marginal costs of launching pollution abatement, i.e., $\frac{-\psi}{\lambda} < \mathrm{G}'(0)$, thus generating an outcome where there is no abatement $B=0$ and pollution accumulations are driven by emissions from resource extraction, i.e., $\dot{Z}=E$. Therefore, over the initial growth phase, the exhaustible stock of resources would be extracted at a higher flow rate $E\nearrow$, leading to rapid accumulation of pollutants in the environment.

Then the economy would enter the second growth stage at which the stock of pollutants has been accumulated to a larger level throughout the first growth phase and thus could generate a larger shadow cost associated with pollution externality. As a result, the marginal benefit of pollution abatement would rise and become equal to the marginal cost of abatement, i.e., $-\frac{\psi}{\lambda} = \mathrm{G}'(B)$ and thus launch positive abatement activities $B>0$ to slow pollution accumulation. Meanwhile, as rapid extractions in the initial phase has substantially reduced resource reserves, the shadow costs of resource exhaustibility could become larger. With the rising shadow costs associated with resource exhaustibility and pollution externality, marginal revenues of resource extraction could become larger and the amounts of resource extraction thus become relatively lower. Therefore, the second growth phase could be characterized by decreases in resource extraction $E\searrow$, increases in pollution abatement $B\nearrow$, and thus a lower rate of pollution accumulation in the environment $\dot{Z}=E-B\searrow$ (but the stock of pollution may still be augmented at a positive rate $\dot{Z}=E-B>0$). With an expanding stock of pollutants $Z\nearrow$, marginal benefits of pollution abatement (i.e., the shadow cost of pollution externality) could further increase, thus inducing higher levels of pollution abatement $B\nearrow$ on the one hand and reducing the amounts of resource extraction $E\searrow$ on the other hand. As a result, the stock of pollutants would peak and then begin to shrink $\dot{Z}=E-B<0$.

Finally, the economy will enter the final growth phase during which the stock of pollutants becomes decumulated $\dot{Z} = E - B < 0$, the shrinking stock of pollutants, through decreasing the shadow costs of pollution externality, would reduce the levels of pollution abatement $B\searrow$. Meanwhile, the decrease in shadow costs of pollution externality plays a role to increase resource extraction but that effect could be largely offset by substantial increases in social costs of resource exhaustibility (due to that the remaining resource reserves become much more scare after extractions over the previous two growth phases), thus making the amounts of resource extraction decrease. At the final stage, the high social costs incurred by resource scarcity could reduce resource extraction by an amount that is larger than the decrease in pollution abatement, thus further shrinking the stock of pollutants. Therefore, the final growth phase could be characterized by decreases in both pollution abatement $B\searrow$ and resource extraction $E\searrow$, and decumulation of both resource reserves $X\searrow$ and pollution stocks $Z\searrow$. As this process evolves, the final growth phase will reach the steady states, to which we now turn.

4 Steady State

In our model of optimal growth with resource exhaustibility and pollution externality, the steady state (SS) is defined as an optimal growth path along which final goods consumption, resource extraction, pollution abatement, man-made capital, resource reserves, and pollution stocks are stationary at the SS levels $(C^*,E^*,B^*,K^*,X^*,Z^*)$.

4.1 Existence of Steady State

Imposing stationary conditions on the necessary conditions of optimality (1), the SS corresponding to the social optimum is determined as follows. First, in the SS both resource extraction and pollution abatement would be equal to zero, i.e., $E^*=0$ and $B^*=0$, and outputs of final goods production are fully allocated to consumptions, i.e., $C^*=\mathrm{F}(K^*,0)$.[13] Second, the stock of man-made capital K^* would be accumulated to a level where the marginal product of man-made capital is equal to the rate of time preference, i.e.,

$$\mathrm{F}_K(K^*,0)=\rho, \tag{21}$$

which implies that the stock of man-made capital accumulated in the SS depends on the rate of time preference, i.e., $K^*=\mathrm{K}^*(\rho)$. Third, the remaining stock of exhaustible resource reserves X^* is determined by

$$\mathrm{F}_E(K^*,0)-\mathrm{M}_E(0,X^*)=-\frac{\mathrm{M}_X(0,X^*)}{\rho}+\mathrm{G}'(0). \tag{22}$$

Finally, the stock of pollutants accumulated in the environment Z^* is determined by

$$-\frac{\mathrm{U}_Z(C^*,Z^*)}{\rho\mathrm{U}_C(C^*,Z^*)}=\mathrm{G}'(0). \tag{23}$$

[13] This result implies that in the SS natural resources are not necessarily an essential input of production, and production of final goods could only reply on man-made capital without resource inputs $\mathrm{F}(K^*,0)>0$.

To establish the SS where both resource reserves and pollution stocks are stationary, there will be no resource extraction $E^*=0$ and pollution abatement $B^*=0$. With $E^*=0$ and $B^*=0$, there will be no spending on resource extraction and pollution abatement, i.e., $\mathrm{M}(0,X^*)=0$ and $\mathrm{G}(0)=0$, and outputs of final goods are fully allocated to consumption, i.e., $C^*=\mathrm{F}(K^*,0)$. With stationery consumption, the marginal product of man-made capital must be equal to the rate of time preference. Furthermore, revenues received from extracting a marginal unit of resources should cover the corresponding shadow cost associated with resource exhaustibility (the first term on the RHS of (22)) and pollution externality (the second term on the RHS of (22)), where the latter is equal to marginal abatement cost at zero abatement levels in the SS, i.e., $\mathrm{G}'(0)$ given in (23).

Equations 21)-(23) determine the SS of man-made capital, resource reserves and pollution stocks, and the sufficient conditions for the existence of SS for optimal growth with both exhaustibility and pollution externality are provided in the following result.

Proposition 2. *If marginal utility of consumption* U_C *is less elastic with respect to accumulations of the pollution stock* Z *than marginal disutility of environmental pollution* U_Z*, i.e.,* $\frac{\partial \ln \mathrm{U}_C}{\partial Z} < \frac{\partial \ln \mathrm{U}_Z}{\partial Z}$*, then there exists a steady state in which consumption, extraction, abatement, man-made capital, resource reserves, and pollution stocks satisfy (21)-(23).*

Proof. See Appendix C. □

4.2 Comparative Static Effects

As we aim to capture the growth implications of connecting natural resources with the environment, the following result gives insights into the effects of incorporating pollution externality into traditional growth models with only resource exhaustibility.

Proposition 3. *If pollution externality resulting from resource extraction is not taken into account, then in the SS resource reserves* X^* *and goods consumption* C^* *will decrease, but resource extraction* E^* *and man-made capital* K^* *would be unaffected.*

Proof. See Appendix D. □

Intuitively, if the connection between natural resource and environmental pollution is taken no account in optimal growth, then there would be no social cost associated with pollution externality in determining resource extraction. As a result, the sum of marginal direct extraction cost and the social cost of resource exhaustibility would increase. To establish that outcome, the SS would have a smaller stock of remaining resource reserves. Similarly, we can establish the following proposition that determines the effect of resource exhaustibility on the SS behavior of the social optimum.

Proposition 4. *If resource exhaustibility is not taken into account (resource reserves are infinitely large), then in the SS resource extraction E^* and pollution abatement B will increase, but changes in man-made capital K^*, consumption C^*, and pollution stock Z^* are ambiguous.*

Proof. See Appendix E. □

The analysis above captures the effect of resource exhaustibility and pollution externality on the SS of optimal growth. We now consider the comparative statics of the time preference,

Proposition 5. *The steady state levels of consumption, resource extraction, pollution abatement, man-made capital, resource reserves, and pollution stocks $(C^*, E^*, B^*, K^*, X^*, Z^*)$ are determined by (21)-(23). Comparative statics of the rate of time preference ρ are given by*

$$\frac{dC^*}{d\rho} < 0, \quad \frac{dE^*}{d\rho} = 0, \quad \frac{dB^*}{d\rho} = 0, \quad \frac{dK^*}{d\rho} < 0, \quad \frac{dZ^*}{d\rho} < 0, \tag{24}$$

and the comparative static effect of ρ on X^* is determined by,

$$\begin{cases} \frac{dX^*}{d\rho} \geq 0 \quad \textit{if} \quad \mathrm{M}_X \leq \rho^2 \frac{\mathrm{F}_{EK}}{\mathrm{F}_{KK}}, \\ \frac{dX^*}{d\rho} < 0 \quad \textit{if} \quad \mathrm{M}_X > \rho^2 \frac{\mathrm{F}_{EK}}{\mathrm{F}_{KK}}. \end{cases} \tag{25}$$

Proof. See Appendix F. □

Several interesting results arise from this proposition. First, change in the time preference has no effect on resource extraction and pollution abatement in the SS. Second, consumption, man-made capital and pollution stocks will decline in the SS if the rate of time preference increases. Intuitively, as the discounting rate rises, the marginal product of man-made capital F_K will increase. To yield a higher level of F_K, K^* will decrease because F_K is decreasing in K. When K^* decreases, outputs of final goods decline and consumption levels C^* drop. Meanwhile, if marginal utility of consumption is less elastic with respect to pollution accumulation (see Proposition 2), then C^* and Z^* will change in the same direction, and the SS would generate a lower pollution stock in the case of a higher time preference.

Furthermore, the effect of ρ on the SS levels of resource reserves X^* depends on the stock-dependent extraction cost, production technology of final goods, and time preference. On the one hand, increase in the rate of time preference will lower man-made capital in producing final goods. On the other hand, changes in ρ also have an impact on resource extraction accordingly to $\rho \mathrm{F}_{EK} dK^* + (\mathrm{M}_{XX} - \rho \mathrm{M}_{EX}) dX^* = \frac{\mathrm{M}_X}{\rho} d\rho$. If the marginal effect of resource scarcity on extraction costs is small (the first condition in (25)), then the resource stocks will increase in response to a larger level of ρ. Therefore, a larger rate of time preference helps preserve the remaining stock of resource reserves in the SS if the marginal effect of resource scarcity on extraction cost is small.

To establish additional interesting results, we further consider comparative statics of three parameters closely related to the connection between resource extraction and pollution emissions. 1) Resource use efficiency α: the effective use of resource inputs in final goods production $\mathrm{F}(K,\alpha E,Z)$; 2) resource extraction efficiency β: extraction costs $\mathrm{M}(\beta E,X)$ and the speed of resource depletion $\dot{X} = -\beta E$; and 3) pollution emission intensity γ: the amount of pollution emissions from extracting each unit of resources $\dot{Z} = \gamma E - B$. The comparative static effects are summarized in the following proposition,

Proposition 6. *Comparative static effects of α, β, and γ on the steady state of man-made capital K^*, resource reserves X^*, and pollution stocks Z^* are given by*

$$\frac{dK^*}{d\alpha} = 0, \quad \frac{dX^*}{d\alpha} < 0, \quad \frac{dZ^*}{d\alpha} = 0, \tag{26a}$$

$$\frac{dK^*}{d\beta} = 0, \quad \frac{dX^*}{d\beta} > 0, \quad \frac{dZ^*}{d\beta} = 0, \tag{26b}$$

$$\frac{dK^*}{d\gamma} = 0, \quad \frac{dX^*}{d\gamma} > 0, \quad \frac{dZ^*}{d\gamma} = 0. \tag{26c}$$

Proof. See Appendix G. □

It is shown that changes in resource use efficiency and emission intensity have no impact on man-made capital and pollutants tock in the SS, because these changes generate no effect on (21) and (23) that characterize the SS levels of man-made capital and pollutants. In comparison, the SS levels of resource reserves X^* are determined by (22) in which resource use efficiency, resource extraction efficiency and emission intensity can affect marginal products of resources, marginal costs of extraction, and social costs of pollution externality, respectively. Accordingly, changes in these three parameters can affect resource reserves in the SS X^*.

Specifically, when the resource use efficiency increases, marginal benefits of resource use will increase, and the higher return from resource extraction thus stimulate larger resource extraction, leaving a lower stock of remaining resource reserves, i.e., $\frac{dX^*}{d\alpha}<0$. Furthermore, when extraction efficiency improves, marginal costs of resource extraction will increase and thus slow resource extraction flows, leaving the remaining resource reserves larger, i.e., $\frac{dX^*}{d\beta}>0$. Finally, if emission intensity increases, then the social costs of pollution externality will rise, thus slowing the extraction process and leading to larger resource reserves, i.e., $\frac{dX^*}{d\gamma}>0$.

5 Transitional Dynamics

As our model extends the traditional DHHS framework by incorporating the connection between natural resources and the environment, it is interesting to determine the conditions under which a dynamical system with augmentable man-made capital, exhaustible resource reserves, and accumulative environmental pollutants has transitional dynamic stability.

To investigate the stability of transitional dynamics, we derive the modified Hamiltonian dynamic system (MHDS) corresponding to our growth model as

$$\mathcal{H}_\lambda = \mathrm{F}(K,\hat{E}) - \mathrm{M}(\hat{E},X) - \mathrm{G}(\hat{B}) - \hat{C} = \dot{K}, \tag{27a}$$

$$\mathcal{H}_\mu = -\hat{E} = \dot{X}, \tag{27b}$$

$$\mathcal{H}_\psi = \hat{E} - \hat{B} = \dot{Z}, \tag{27c}$$

$$\mathcal{H}_K = \lambda \mathrm{F}_K(K,\hat{E}) = \rho\lambda - \dot{\lambda}, \tag{27d}$$

$$\mathcal{H}_X = -\lambda \mathrm{M}_X(\hat{E},X) = \rho\mu - \dot{\mu}, \tag{27e}$$

$$\mathcal{H}_Z = \mathrm{U}_Z(\hat{C},Z) = \rho\psi - \dot{\psi}, \tag{27f}$$

where the MHDS includes three state K,X,Z and three costate variables λ,μ,ψ, and $\mathcal{H}_i$ is the derivative of the maximized Hamiltonian with respect to $i=K,X,Z,\lambda,\mu,\psi$, and $\hat{C},\hat{E},\hat{B}$ denote the control variables optimally determined by (9), (14) and (16).[14] To simplify notation, we denote by $\mathbf{S}\equiv[S_1,S_2,S_3]=[K,X,Z]$ a vector of the three state variables and by $\mathbf{P}\equiv[P_1,P_2,P_3]=[\lambda,\mu,\psi]$ a vector of the three costate variables. A 6×6 Jacobian matrix corresponding to the MHDS given

[14]The derivative of the maximized Hamiltonian with respect to a state or costate or variable is equal to the derivative of ordinary Hamiltonian with respect to that state or costate variable where the control variables within

in (27) can be written as

$$\mathcal{J}(\mathbf{S},\mathbf{P})\equiv\begin{bmatrix}\mathcal{H}_{\mathbf{PS}} & \mathcal{H}_{\mathbf{PP}}\\ -\mathcal{H}_{\mathbf{SS}} & -\mathcal{H}_{\mathbf{SP}}+\rho I_3\end{bmatrix}, \tag{28}$$

where ρ is the rate of time preference, I_3 is a 3×3 identity matrix, and $\mathcal{H}_{\mathbf{PS}},\mathcal{H}_{\mathbf{PP}},\mathcal{H}_{\mathbf{SS}}$ and $\mathcal{H}_{\mathbf{SP}}$ are 3×3 submatrix of $\mathcal{J}(\mathbf{S},\mathbf{P})$. The six row vectors of $\mathcal{J}(\mathbf{S},\mathbf{P})$ correspond to the derivatives with respect to K,X,Z,λ,μ,ψ of the six dynamic equations given by $\dot{K}=\mathcal{H}_\lambda$, $\dot{X}=\mathcal{H}_\mu$, $\dot{Z}=\mathcal{H}_\psi$, $\dot{\lambda}=-\mathcal{H}_K+\rho\lambda$, $\dot{\mu}=-\mathcal{H}_X+\rho\mu$, and $\dot{\psi}=-\mathcal{H}_Z+\rho\psi$, respectively.

5.1 Saddle-path Stability

Transitional dynamic stability of the MHDS is analyzed by finding eigenvalue of the Jacobian matrix $\mathcal{J}(\mathbf{S},\mathbf{P})$ corresponding to the MHDS, we first consider the following matrix

$$\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})\equiv\mathcal{J}(\mathbf{S},\mathbf{P})-\frac{\rho}{2}I_6=\begin{bmatrix}\mathcal{H}_{\mathbf{PS}}-\frac{\rho}{2}I_3 & \mathcal{H}_{\mathbf{PP}}\\ -\mathcal{H}_{\mathbf{SS}} & -\mathcal{H}_{\mathbf{SP}}+\frac{\rho}{2}I_3\end{bmatrix}, \tag{29}$$

where I_6 is a 6×6 identity matrix, and verify that $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$ takes a form of a Hamiltonian matrix, since the submatrix in first and third quadrants ($\mathcal{H}_{\mathbf{PP}}$ and $-\mathcal{H}_{\mathbf{SS}}$) are real-value symmetric matrix, and the submatrix in second and fourth quadrants satisfy $\mathcal{H}_{\mathbf{PS}}-\frac{\rho}{2}I_3+\left(-\mathcal{H}_{\mathbf{SP}}+\frac{\rho}{2}I_3\right)^T=0$ where the superscript T denotes the matrix transpose.[15] For the 6×6 Hamiltonian matrix $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$, the set of six eigenvalues of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$ are distributed symmetrically with respect to zero, i.e., $[\pm\hat{\xi}_1,\pm\hat{\xi}_2,\pm\hat{\xi}_3]$.[16]

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the ordinary Hamiltonian is optimally chosen (Envelope Theorem).

$$\mathcal{H}_i(K,X,Z,\lambda,\mu,\psi)\equiv\frac{\partial\left(\max\limits_{C,E,B}\mathcal{H}^0(C,E,B,K,X,Z,\lambda,\mu,\psi)\right)}{\partial i}=\frac{\partial\mathcal{H}^0(\hat{C},\hat{E},\hat{B},K,X,Z,\lambda,\mu,\psi)}{\partial i},$$

where $i=K,X,Z$ for state variables or $i=\lambda,\mu,\psi$ for co-state variables, and $\hat{C},\hat{E},\hat{B}$ are the control variables that are optimally determined by (9), (14) and (16).

[15]Formally, a real-value Hamiltonian matrix $\mathcal{M}$ takes a form of $\mathcal{M}=\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & -\mathcal{A}^T\end{bmatrix}$ with $\mathcal{A}\in\mathbb{R}^{n\times n}$, $\mathcal{B},\mathcal{C}\in\mathbb{S}^n$ where $\mathbb{S}^n$ denotes the set of all real-valued symmetric $n\times n$ matrices. The superscript T denotes matrix transpose. More generally, a matrix $\mathcal{M}\in\mathbb{R}^{2n\times 2n}$ is called Hamiltonian if $\mathcal{IM}$ is symmetric, i.e., $\mathcal{IM}=(\mathcal{IM})^T$, where $\mathcal{I}\in\mathbb{R}^{2n\times 2n}$ is a skew-symmetric matrix given by $\mathcal{I}=\begin{bmatrix}0_n & -I_n\\ I_n & 0_n\end{bmatrix}$ where $0_n,I_n$ are a $n\times n$ zero and identity matrix, respectively.

[16]The proof is as follows. Suppose that $\hat{\xi}$ and $\hat{v}$ are the eigenvalue and eigenvector corresponding to $\hat{\mathcal{J}}$, i.e., $\hat{\mathcal{J}}\hat{v}=\hat{\xi}\hat{v}$, we obtain $\hat{\mathcal{I}}(\hat{\xi}\hat{v})=\hat{\mathcal{I}}(\hat{\mathcal{J}}\hat{v})=\hat{\mathcal{J}}^T\hat{\mathcal{I}}^T\hat{v}$, following from that the product of a skew-symmetric matrix $\hat{\mathcal{I}}$ and a Hamiltonian matrix $\hat{\mathcal{J}}$ is symmetric, i.e., $\hat{\mathcal{I}}\hat{\mathcal{J}}=(\hat{\mathcal{I}}\hat{\mathcal{J}})^T=\hat{\mathcal{J}}^T\hat{\mathcal{I}}^T$. Since the skew symmetric matrix satisfied $\hat{\mathcal{I}}^T=-\hat{\mathcal{I}}$, we have $\hat{\mathcal{J}}^T(-\hat{\mathcal{I}})\hat{v}=\hat{\mathcal{I}}(\hat{\xi}\hat{v})$, and rearranging yields $\hat{\mathcal{J}}^T(\hat{\mathcal{I}}\hat{v})=-\hat{\xi}(\hat{\mathcal{I}}\hat{v})$, implying that $-\hat{\xi}$ is also an eigenvalue corresponding to the eigenvector $\hat{\mathcal{I}}\hat{v}$.

Furthermore, let $\hat{\xi}$ be an eigenvalue of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$ with the corresponding eigenvector $\hat{v}$, we have

$$\hat{\xi}\hat{v}=\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})\hat{v}=\left[\mathcal{J}(\mathbf{S},\mathbf{P})-\frac{\rho}{2}I_6\right]\hat{v}=\mathcal{J}(\mathbf{S},\mathbf{P})\hat{v}-\frac{\rho}{2}\hat{v}\quad\Rightarrow\quad\mathcal{J}(\mathbf{S},\mathbf{P})\hat{v}=\left(\hat{\xi}+\frac{\rho}{2}\right)\hat{v}. \tag{30}$$

We derive that if $\hat{\xi}$ and $\hat{v}$ are the eigenvalue and eigenvector of the Hamiltonian matrix $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$, then $\hat{\xi}+\frac{\rho}{2}$ and $\hat{v}$ are the eigenvalue and eigenvector of the Jacobian matrix $\mathcal{J}(\mathbf{S},\mathbf{P})$. That is, if $-\hat{\xi}$ is an eigenvalue of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$, then $-\hat{\xi}+\frac{\rho}{2}$ is the eigenvalue of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$. Therefore, for each symmetrical pair of the eigenvalues $\pm\hat{\xi}_i(i=1,2,3)$ of the Hamiltonian matrix $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$, $\xi_i^{1,2}=[-\hat{\xi}_i+\frac{\rho}{2},\hat{\xi}_i+\frac{\rho}{2}]$ is a pair of eigenvalues for the Jacobian matrix $\mathcal{J}(\mathbf{S},\mathbf{P})$.

Based on the relation (30), we establish the condition under which optimal growth with resource exhaustibility and pollution externality has saddle-path stability as follows.

Proposition 7. *Denote by ρ the rate of time preference and by $\pm\hat{\xi}_i$ each symmetric pair of eigenvalues of the Hamiltonian matrix given by (29). If the condition*

$$\frac{\rho}{2}<|\pm\hat{\xi}_i|\quad\forall i=1,2,3 \tag{31}$$

holds, then there is a three-dimensional saddle-path stable manifold. Given the initial levels of state variables $[K(0),X(0),Z(0)]$, the three-dimensional stable manifold determines the initial levels of costate variables $[\lambda(0),\mu(0),\psi(0)]$. Starting from this initial condition, transitional dynamics evolve along the growth path given by the MHDS (27) and converge to the steady state with $[K(t),X(t),Z(t),\lambda(t),\mu(t),\psi(t)]\rightarrow[K^*,X^*,Z^*,\lambda^*,\mu^*,\psi^*]$ as $t\rightarrow\infty$.

Proof. The saddle-path stability condition (31) implies that $\hat{\xi}_i>\frac{\rho}{2}$ and $-\hat{\xi}_i<-\frac{\rho}{2}$ for $\forall i=1,2,3$. Following from $\xi_i^{1,2}=[-\hat{\xi}_i+\frac{\rho}{2},\hat{\xi}_i+\frac{\rho}{2}]$, the corresponding pairs of eigenvalues ξ_i of the Jacobian matrix (28) thus satisfy $\xi_i^1=\hat{\xi}+\frac{\rho}{2}>0$ and $\xi_i^2=-\hat{\xi}+\frac{\rho}{2}<0$ for $\forall i=1,2,3$. Accordingly, the six eigenvalues $\xi_i^{1,2}=[-\hat{\xi}_i+\frac{\rho}{2},\hat{\xi}_i+\frac{\rho}{2}]$ $(i=1,2,3)$ of the 6×6 matrix $\mathcal{J}(\mathbf{S},\mathbf{P})$ are three positive and three negative, establishing that the MHDS is saddle-path stable. □

5.2 Global Stability

Proposition (7) has established the conditions for saddle-path dynamic stability. As a further step, we will examine the conditions under which transitional dynamics of our model has global stability. Specifically, global stability is defined as follows.

Definition 1. *The transitional dynamics are globally asymptotically stable (GAS) for every bounded solution to the problem of maximizing (6) subject to (1), (2) and (4), if starting from any initial condition* $[K(0),X(0),Z(0),\lambda(0),\mu(0),\psi(0)]$, *transitional dynamics can evolve along the optimal path characterized by the MHDS (27) and converge to the steady state with* $[K(t),X(t),Z(t),\lambda(t),\mu(t),\psi(t)]\to[K^*,X^*,Z^*,\lambda^*,\mu^*,\psi^*]$ *as* $t\to\infty$.

For transitional dynamics of the growth path characterized by (27), the sufficient condition of GAS can be established if there is a 3×3 real-value symmetric matrix $\mathcal{Q}\in\mathbb{R}^{3\times3}$ such that the following generalized curvature matrix

$$\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})\equiv\begin{bmatrix}\mathcal{H}_{\mathbf{SS}}+\mathcal{H}_{\mathbf{SP}}\mathcal{Q}+\mathcal{Q}\mathcal{H}_{\mathbf{PS}}+\mathcal{Q}\mathcal{H}_{\mathbf{PP}}\mathcal{Q}-\rho\mathcal{Q} & -\frac{\rho}{2}I_3\\ -\frac{\rho}{2}I_3 & -\mathcal{H}_{\mathbf{PP}}\end{bmatrix}\prec 0 \tag{32}$$

is negative definite. This sufficient condition of GAS draws on the seminal work of Sorger (1989), which generalizes previous studies on the sufficient conditions for the convergence of optimal growth paths to a steady state (e.g., Brock and Scheinkman, 1976; Cass and Shell, 1976; Magill, 1977; Rockafellar, 1976). Using the Schur decomposition, we derive that the curvature matrix $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})$ is negative definite if the fourth quadrant submatrix $-\mathcal{H}_{\mathbf{PP}}$ and its corresponding Schur complement $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})$ are both negative definite, i.e.,

$$-\mathcal{H}_{\mathbf{PP}}\prec 0 \quad \text{and} \quad \mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})\equiv\mathcal{H}_{\mathbf{SS}}+\mathcal{H}_{\mathbf{SP}}\mathcal{Q}+\mathcal{Q}\mathcal{H}_{\mathbf{PS}}+\mathcal{Q}\mathcal{H}_{\mathbf{PP}}\mathcal{Q}-\rho\mathcal{Q}+\frac{\rho^2}{4}\mathcal{H}_{\mathbf{PP}}^{-1}\prec 0.$$

The following result firstly establishes the negative definiteness of the submatrix $-\mathcal{H}_{\mathbf{PP}}$.

Lemma 3. *For optimal growth with resource exhaustibility and pollution externality, if the model assumptions given in 1-4 hold, then the submatrix* $-\mathcal{H}_{\mathbf{PP}}$ *is negative definite, i.e.,*

$$-\mathcal{H}_{\mathbf{PP}}\equiv\begin{bmatrix}-\mathcal{H}_{\lambda\lambda} & -\mathcal{H}_{\lambda\mu} & -\mathcal{H}_{\lambda\psi}\\ -\mathcal{H}_{\mu\lambda} & -\mathcal{H}_{\mu\mu} & -\mathcal{H}_{\mu\psi}\\ -\mathcal{H}_{\psi\lambda} & -\mathcal{H}_{\psi\mu} & -\mathcal{H}_{\psi\psi}\end{bmatrix}\prec 0,$$

Proof. See Appendix H. □

Intuitively, the second-order derivatives of the maximized Hamiltonian with respect to three costate variables $\mathcal{H}_{\mathbf{PP}}$ characterizes marginal effects of the shadow values λ, μ and ψ on the dynamics of man-made capital, resource reserves and pollution stocks, i.e., $\dot{K}=\mathcal{H}_\lambda$, $\dot{X}=\mathcal{H}_\mu$ and

$\dot{Z} = \mathcal{H}_{\psi}$. The negative definiteness of $-\mathcal{H}_{\mathbf{PP}}$ is equivalent to the positive definiteness of $\mathcal{H}_{\mathbf{PP}}$, thus implying that the growth rate of the three stock variables are positively affected by their corresponding shadow values.

Furthermore, the following proposition establishes negative definiteness of the curvature matrix $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})$, and hence provides the sufficient condition of GAS for optimal growth with resource exhaustibility and pollution externality.

Proposition 8. *If the model of optimal growth with resource exhaustibility and pollution externality satisfy the conditions*

$$\mathrm{F}_{EK} = 0, \quad \mathrm{F}_K(\mathrm{F}_E - \mathrm{M}_E) + \mathrm{M}_X + \frac{\mathrm{U}_{CZ}}{\mathrm{U}_{CC}} = 0, \tag{33}$$

then the curvature matrix given in (32) is negative definite and the sufficient conditions for GAS are satisfied. Accordingly, starting from any initial condition $[K(0),X(0),Z(0),\lambda(0),\mu(0),\psi(0)]$, *transitional dynamics can evolve along the path given by the MHDS (27) and converge to the steady state with* $[K(t),X(t),Z(t),\lambda(t),\mu(t),\psi(t)] \to [K^*,X^*,Z^*,\lambda^*,\mu^*,\psi^*]$ *as* $t \to \infty$.

Proof. See Appendix I. □

Proposition 8 provides two conditions for establishing the GAS. First, production function of final goods should be separable in its two arguments K and E ($\mathrm{F}_{EK} = 0$), i.e., one of the production inputs - man-made capital or natural resources - has no effect on marginal products of the other. Second, returns from marginal revenues of resource extraction $\pi \equiv \mathrm{F}_E - \mathrm{M}_E$ (marginal product of man-made capital F_K as the rate of return) should be equal to the sum of marginal effects of resource depletion on extraction costs $\mathrm{M}_X < 0$ and marginal effect of pollution accumulation on optimal consumption $\frac{\partial \hat{C}}{\partial Z} = \frac{\mathrm{U}_{CZ}}{\mathrm{U}_{CC}} < 0$.

Following from (33), an interesting result is that transitional dynamics with the GAS will feature a growth path along which changes in marginal revenues of resource extraction depend on consumption and its marginal effect on the marginal rate of substitution between consumption and environmental pollution as given by

$$\dot{\pi} = \sigma C \frac{\partial \left(\frac{\mathrm{U}_Z}{\mathrm{U}_C} \right)}{\partial C},$$

where $\sigma \equiv -\frac{\mathrm{U}_C}{C\mathrm{U}_{CC}}$ is the intertemporal elasticity of consumption substitution, and the marginal effect of C on the marginal rate of substitution between C and Z is given by $\frac{\partial(\mathrm{U}_Z/\mathrm{U}_C)}{\partial C} = \frac{\mathrm{U}_{ZC}\mathrm{U}_C - \mathrm{U}_Z\mathrm{U}_{CC}}{\mathrm{U}_C^2} < 0$ with a negative sign.[17]

6 The Market Equilibrium

6.1 Equilibrium without Pollution Pricing

This section will investigate whether allocations in a market equilibrium (ME) are consistent with the social optimum (SO), and if not which policy interventions can be implemented to achieve the consistency. In the ME we consider a representative household who solves a problem of maximizing discounted present values of utility streams subject to her budget constraint as

$$\begin{aligned} \max_{[C(t),K(t)]_{t=0}^{\infty}} & \int_0^{\infty} \exp(-\rho t)\mathrm{U}(C(t),Z(t))dt \\ s.t., \quad & \dot{K}(t) = \Pi(t) + r(t)K(t) - C(t), \end{aligned} \tag{34}$$

where the instantaneous utility $\mathrm{U}(C,Z)$ is discounted by the constant rate of time preference ρ, C and K is consumption and man-made capital assets owned by the household, and the household receives remunerations by renting capital K at the market interest rate r. The household has an ownership of the firms operating in the economy and receives corporate profits Π at each instantaneous time point as another source of income in her budget constraint.[18] Corporate profits are determined by solving the firm problems and not controllable by the household, and the utility-maximizing household only optimally chooses consumption C and capital asset K.[19]

Furthermore, using both durable man-made capital K and intermediate energy inputs E as production factors, a representative firm producing final goods solves the profit maximization problem

$$\max_{[E(t),K(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right] [\mathrm{F}(K(t),E(t)) - r(t)K(t) - \kappa(t)E(t)]dt, \tag{35}$$

where rK is the cost of renting capital K at the market interest rate r, and κE is the costs of using

[17] For derivations, substituting the modified Hotelling rule for resource extraction (19b) into the global stability condition yields $\dot{\pi} = \frac{\mathrm{U}_Z}{\mathrm{U}_C} - \frac{\mathrm{U}_{ZC}}{\mathrm{U}_{CC}} = \frac{\partial\left(\frac{\mathrm{U}_Z}{\mathrm{U}_C}\right)}{\partial C} C\left(-\frac{\mathrm{U}_C}{C\mathrm{U}_{CC}}\right)$.

[18] The aggregate profits are given by $\Pi = \Pi_1 + \Pi_2$, where Π_1 and Π_2 denotes the flow profit of the firms producing final goods and extracting exhaustible resources respectively, which is given by (35) and (36).

[19] In our specification of the representative household problem, the size of the household is normalized to unity and inelastically supplies, thus there is no endogenous treatment of labor input and wage remuneration.

resource inputs E at the unit cost κ paid to resource extraction firms. Given the market interest rate r and the price of resource inputs κ, the final goods producer optimally chooses the demand for physical capital K and resource inputs E for maximizing discounted values of profits streams, which is equivalent to maximization of static flow profits at each instantaneous point in time.

Finally, the ME considers that a representative firm that extracts resources and supplies them to the final goods producer solves the problem

$$\begin{aligned} &\max_{[E(t),X(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right][\kappa(t)E(t)-\mathrm{M}(E(t),X(t))]dt, \\ &s.t., \quad \dot{X}(t)=-E(t), \end{aligned} \tag{36}$$

where the resource extraction firm optimally chooses extraction flows E and the remaining stock of resource reserves X for maximizing discounted present values of profit streams, subject to depletions of the exhaustible resource stock. Instantaneous flow profits are obtained by receiving revenues from selling extracted resources to the final goods producer at a price level κ, minus the resource extraction cost denoted by $\mathrm{M}(E,X)$.

The market equilibrium is defined as an allocation in which 1) the household optimally chooses consumption C and man-made capital K to maximize discounted present values of utility streams subject to her budget constraint (34); 2) the final goods producer optimally chooses demands for man-made capital K and resource inputs E for maximizing profits (35); and 3) the resource extraction firm optimally chooses resource extraction flows E and the remaining stock of resource reserves X for maximizing discounted present values of profit streams (36).

To characterize allocations in the ME, we first solve the household problem (34) and yield the necessary condition of optimality: $\mathrm{U}_C(C,Z)-\lambda=0$, and $\rho\lambda-\dot{\lambda}=r\lambda$, where λ corresponds to the shadow value of man-made capital K. Then solving the problem of the final goods firm yields: $\mathrm{F}_K(K,E)=r$, and $\mathrm{F}_E(K,E)=\kappa$. The optimal behavior of consumption and capital accumulation in the ME is thus characterized as

$$\lambda^M=\mathrm{U}_C(C^M,Z^M), \tag{37a}$$

$$\frac{\dot{\lambda}^M}{\lambda^M}=\rho-\mathrm{F}_K(K^M,E^M), \tag{37b}$$

where the superscript "M" corresponds to allocations in the ME. Furthermore, solving the problem of the resource extraction firm yields: $\kappa - \mathrm{M}_E - \mu = 0$, and $r\mu - \dot{\mu} = -\mathrm{M}_X$, where μ corresponds to the shadow value of exhaustible resources. Substituting $\mathrm{F}_E = \kappa$ and rearranging yields the following equations characterizing extraction flows and the remaining stock of natural resources,

$$\mu^M = \mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M), \tag{38a}$$

$$\frac{\dot{\mu}^M}{\mu^M} = r + \frac{\mathrm{M}_X(E^M, X^M)}{\mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M)}, \tag{38b}$$

where the shadow value of the exhaustible resource stock is equal to marginal revenues of resource extraction, and the growth rate of the shadow values depends on the market interest rate and the stock-dependent extraction costs. Without pollution pricing to internalize pollution externality resulting from resource extraction, allocations in the ME are characterized by (37) and (38), and its consistency with the SO outcome is given by the following result.

Proposition 9. *Without taking into account the externality of pollution resulting from resource extraction, allocations in the ME are not consistent with the SO outcomes.*

Proof. See Appendix J. □

6.2 Flow-based Pollution Pricing

To achieve consistency with the SO allocations, the ME needs to introduce a way of internalizing the externality of pollution resulting from resource extraction. We consider implementation of a polluters pay system that makes resource extraction firms generating emissions pay for the costs of abating pollution. With such a system established, the resource extraction firm solves the problem

$$\begin{aligned} \max_{[E(t),X(t)]_{t=0}^{\infty}} & \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right] [\kappa(t)E(t) - \mathrm{M}(E(t),X(t)) - \omega(t)E(t)]dt, \\ s.t., \quad & \dot{X}(t) = -E(t), \end{aligned} \tag{39}$$

where the cost burden facing this resource extraction firm involves both direct extraction costs denoted by $\mathrm{M}(E,X)$ and the costs of abating pollution emissions resulting from resource extraction ωE, with ω denoting the unit cost of abatement. The total costs of abatement are charged based on resource extraction flows E (one unit of resource extraction leads to one unit of pollution emissions).

Meanwhile, pricing environmental pollutants is implemented by creating a market for pollution where pollution abatement firms, upon receiving the price signal of pollutants, has an incentive to provide abatement services to resource extraction firms. Specifically, the pollution abatement firm solves the problem

$$\max_{[B(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right][\omega(t)B(t) - \mathrm{G}(B(t))]dt, \tag{40}$$

where pollution abatement B are optimally chosen for maximizing intertemporal profit streams, which is equivalent to maximization of static profits at instantaneous time points.

To characterize allocations with flow-based pollution pricing, we solve the resource extraction firm problem (39) and obtain the necessary condition of optimality: $\kappa - \mathrm{M}_E - \omega - \mu = 0$, and $r\mu - \dot{\mu} = -\mathrm{M}_X$, where ω is the unit cost of pollution in the flow-based pricing system, and μ is the shadow value of exhaustible resources. Substituting $\mathrm{F}_E = \kappa$ given in (35) and rearranging yields the static and dynamic profiles of the shadow price of resource reserves

$$\mu^M = \mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M) - \omega, \tag{41a}$$

$$\frac{\dot{\mu}^M}{\mu^M} = r + \frac{\mathrm{M}_X(E^M, X^M)}{\mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M) - \omega}. \tag{41b}$$

Then solving the problem of pollution abatement firms (40) yields $\omega = \mathrm{G}'(B^M)$. We obtain the following result concerning allocation in a ME with flow-based pollution pricing.

Proposition 10. *With a flow-based pollution pricing system, allocations in the ME can achieve consistency with SO outcomes. Furthermore, with pollution costs charged based on resource extraction flows, the unit pricing of pollution emissions is given as the ratio of shadow values between pollution stocks and man-made capital in the SO, i.e.,* $\omega = \frac{-\psi^S}{\lambda^S}$.

Proof. See Appendix K. □

6.3 Stock-based Pollution Pricing

To compare with the flow-based pricing system, we consider an alternative pollution pricing mechanism based on the stock of polluting resources rather than extraction flows. Within a stock-based pollution pricing system, the resource extraction firm solves the problem

$$\begin{aligned} &\max_{[E(t),X(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right][\kappa(t)E(t) - \mathrm{M}(E(t),X(t)) - \bar{\omega}(t)X(t)]dt, \\ &s.t., \quad \dot{X}(t) = -E(t), \end{aligned} \tag{42}$$

where the total cost of abating pollution from resource extraction is charged based on the remaining stock of resources X (that will generate an equivalent stock of environmental pollutants), with the unit price of pollutants denoted by $\bar{\omega}$. Meanwhile, the pollution abatement firm solves the problem

$$\begin{aligned} &\max_{[B(t),Z(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left[-\int_0^t r(t')dt'\right][\bar{\omega}(t)Z(t)-\mathrm{G}(B(t))]dt, \\ &s.t., \quad \dot{Z}(t)=E(t)-B(t), \end{aligned} \tag{43}$$

where the pollution abatement firm optimally chooses abatement levels B and the pollution stock Z for maximizing intertemporal profit streams, subject to the law of motion for pollution accumulation. Instantaneous flow profits are obtained by receiving payments from the resource extraction firm that is responsible for the accumulated pollution stock Z, minus the abatement costs given by $\mathrm{G}(B)$.

To characterize allocations with stock-based pollution pricing, we solve the resource extraction firm problem (42) and yield the necessary condition of optimality: $\kappa - \mathrm{M}_E - \mu = 0$, and $r\mu - \dot{\mu} = -\mathrm{M}_X - \bar{\omega}$. Substituting $\mathrm{F}_E = \kappa$ and rearranging yields the shadow price of resource reserves in a stock-based pollution pricing system

$$\mu^M = \mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M), \tag{44a}$$

$$\frac{\dot{\mu}^M}{\mu^M} = r + \frac{\mathrm{M}_X(E^M, X^M) - \bar{\omega}}{\mathrm{F}_E(K^M, E^M) - \mathrm{M}_E(E^M, X^M)}. \tag{44b}$$

Meanwhile, solving the problem of pollution abatement firms yields

$$\psi^M = -\mathrm{G}'(B^M), \tag{45a}$$

$$r\psi^M - \dot{\psi}^M = \bar{\omega}, \tag{45b}$$

where ψ^M is the shadow value of pollution stocks. We hence obtain the following result for allocations in the ME with stock-based pollution pricing.

Proposition 11. *Allocations in the ME with stock-based pollution pricing can achieve consistency with SO outcomes. With pollution costs charged based on the remaining stock of resource reserves, the unit pricing of pollution emissions is determined as a ratio between the marginal disutility of pollution and the marginal utility of consumption, i.e.,* $\bar{\omega} = -\frac{\mathrm{U}_Z}{\mathrm{U}_C}$.

Proof. See Appendix L. □

7 Conclusions

One of the grand challenges facing the world is the endeavor to sustain economic growth without depleting exhaustible natural resources and without jeopardizing the environment. This paper provides a general theory of optimal growth with resource exhaustibility and pollution externality, based on an optimal control framework that explicitly considers augmentable man-made capital, exhaustible resource reserves and accumulative environmental pollutants as three stock variables.

First, characterization of social optimum shows that resource extraction should reach a level where marginal products of natural resources equal marginal direct costs of extraction plus social costs associated with resource exhaustibility and pollution externality. With pollution externality taken into account, for any given states of man-made capital and resource reserves the resource extraction flows generated in optimal growth with both resource exhaustibility and pollution externality are smaller than those with only resource exhaustibility. Moreover, if man-made capital and resource inputs are complements in final goods production, then taking account of pollution externality will reduce the growth rate of consumption.

Second, we shows that if marginal utility of consumption is less elastic with respect to pollution stock accumulation than marginal disutility of pollution, then there is a unique steady state where the marginal product of man-made capital is equal to the rate of time preference, and marginal revenues of resource extraction are equal to marginal extraction costs of resource depletion plus marginal disutility of pollution discounted by the rate of time preference. Furthermore, the steady state levels of resource reserves and consumption will decrease if pollution externality resulting from resource extraction is not taken into account. Comparative statics show that in the steady state an increase in the rate of time preference will reduce man-made capital and pollution stocks, but that effect on remaining resource reserves is ambiguous depending on the stock-dependent resource extraction costs, production technology of final goods, and the rate of time preference. An increase in resource use efficiency reduces the remaining resource reserves, but an increase in marginal extraction cost and pollution emission intensity leads to a larger stock of remaining resource reserves.

Third, the analysis of transitional dynamics shows that if the rate of time preference is sufficiently small, then there is always a three-dimensional stable saddle path along which transitional dynamics evolve and converge to the steady state. Furthermore, if production technology of final goods is separable in man-made capital and resource inputs, and return from marginal revenues of resource extraction is equal to the sum of marginal effects of resource depletion on extraction costs and marginal effects of pollution accumulation on optimal consumption, the transitional dynamics are globally asymptotically stable.

Finally, we find that the market equilibrium without pricing pollution resulting from resource extraction can not achieve consistency with allocations in the social optimum. To achieve consistency, the ME needs to consider pricing pollution externality by establishing a polluters pay system that makes resource extraction firms generating emissions pay for the costs of pollution abatement. Specifically, in a flow-based pollution pricing system where pollution costs are charged based on resource extraction flows, the unit pricing of pollution emissions is determined as the ratio of shadow values between pollution stocks and man-made capital. In a stock-based pollution pricing system where pollution costs are charged based on the stock of resource reserves, the unit pricing of pollution stocks is determined as a ratio between the marginal disutility of pollution and the marginal utility of consumption.

Appendix A Proof of Lemma 2

The proof involves two steps. First, we will show if the maximized Hamiltonian $\mathcal{H}$ is strictly concave with respect to state variables $[K,X,Z]$, then the solution $[\hat{C},\hat{E},\hat{B},\hat{K},\hat{X},\hat{Z}]$ satisfying the necessary conditions of optimality corresponds to a unique global maximum. Specifically, we have $\mathcal{H}(K,X,Z,\lambda,\mu,\psi)\geq\mathcal{H}^0(C,E,B,K,X,Z,\lambda,\mu,\psi)\equiv\mathrm{U}(C,Z)+\lambda\dot{K}+\mu\dot{X}+\psi\dot{Z}$ and multiplying the exponential discounting factor $e^{-\rho t}$ and integration yields

$$\int_0^\infty e^{-\rho t}\mathrm{U}(C,Z)dt+\int_0^\infty e^{-\rho t}\left[\lambda\dot{K}+\mu\dot{X}+\psi\dot{Z}\right]dt\leq\int_0^\infty e^{-\rho t}\mathcal{H}(K,X,Z,\lambda,\mu,\psi)dt.$$

If the maximized Hamiltonian $\mathcal{H}$ is strictly concave with respect to state variables $[K,X,Z]$ given the resulting costate variables $[\lambda,\mu,\psi]$, then we have

$$\mathcal{H}(K,X,Z,\lambda,\mu,\psi)<\mathcal{H}(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)+(\rho\lambda-\dot{\lambda})(K-\hat{K})+(\rho\mu-\dot{\mu})(X-\hat{X})+(\rho\psi-\dot{\psi})(Z-\hat{Z}),$$

where $[K,X,Z]$ corresponds to a non-optimal feasible path, and $[\hat{K},\hat{X},\hat{Z}]$ to the optimal path for a given costate variables $[\lambda,\mu,\psi]$, and the inequality follows from the necessary conditions of optimality: $\mathcal{H}_K(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)=\rho\lambda-\dot{\lambda}$, $\mathcal{H}_X(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)=\rho\mu-\dot{\mu}$, and $\mathcal{H}_Z(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)=\rho\psi-\dot{\psi}$. Combining the two inequality given above yields

$$\begin{aligned}&\int_0^\infty e^{-\rho t}\mathrm{U}(C,Z)dt+\int_0^\infty e^{-\rho t}\Big[\lambda\dot{K}+\mu\dot{X}+\psi\dot{Z}\Big]dt\leq\int_0^\infty e^{-\rho t}\mathcal{H}(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)dt\\&+\int_0^\infty e^{-\rho t}\Big[(\rho\lambda-\dot{\lambda})(K-\hat{K})+(\rho\mu-\dot{\mu})(X-\hat{X})+(\rho\psi-\dot{\psi})(Z-\hat{Z})\Big]dt.\end{aligned}\tag{A.1}$$

The maximized Hamiltonian evaluated at the optimal path $[\hat{K},\hat{X},\hat{Z}]$ is given by $\mathcal{H}(\hat{K},\hat{X},\hat{Z},\lambda,\mu,\psi)\equiv\mathrm{U}(\hat{C},\hat{Z})+\lambda\dot{\hat{K}}+\mu\dot{\hat{X}}+\psi\dot{\hat{Z}}$, and substituting it into (A.1) and rearranging yields

$$\begin{aligned}&\int_0^\infty e^{-\rho t}\Big[\mathrm{U}(C,Z)-\mathrm{U}(\hat{C},\hat{Z})\Big]dt<\int_0^\infty e^{-\rho t}\Big[\lambda(\dot{\hat{K}}-\dot{K})+\mu(\dot{\hat{X}}-\dot{X})+\psi(\dot{\hat{Z}}-\dot{Z})\Big]dt\\&+\int_0^\infty e^{-\rho t}\Big[(\rho\lambda-\dot{\lambda})(K-\hat{K})+(\rho\mu-\dot{\mu})(X-\hat{X})+(\rho\psi-\dot{\psi})(Z-\hat{Z})\Big]dt\\&=e^{-\rho t}\lambda(K-\hat{K})|_{t=0}^{\infty}+e^{-\rho t}\lambda(X-\hat{X})|_{t=0}^{\infty}+e^{-\rho t}\lambda(Z-\hat{Z})|_{t=0}^{\infty}=0,\end{aligned}$$

where the last line boils down to zero following from both transversality and initial conditions, and we hence have

$$\int_0^\infty e^{-\rho t}\mathrm{U}(C,Z)dt\leq\int_0^\infty e^{-\rho t}\mathrm{U}(\hat{C},\hat{Z})dt.$$

This establishes that the solution $[\hat{C},\hat{E},\hat{B},\hat{K},\hat{X},\hat{Z}]$ satisfying the necessary conditions of optimality corresponds to a unique global maximum.

Second, to establish the concavity of the maximized Hamiltonian with respect to three state variables, the corresponding 3×3 Hessian matrix should be negative definite, which requires the leading principal minors of the Hessian matrix alternate their signs, i.e.,

$$\mathcal{H}_{KK}<0,\quad\begin{vmatrix}\mathcal{H}_{KK}&\mathcal{H}_{KX}\\\mathcal{H}_{XK}&\mathcal{H}_{XX}\end{vmatrix}>0,\quad\begin{vmatrix}\mathcal{H}_{KK}&\mathcal{H}_{KX}&\mathcal{H}_{KZ}\\\mathcal{H}_{XK}&\mathcal{H}_{XX}&\mathcal{H}_{XZ}\\\mathcal{H}_{ZK}&\mathcal{H}_{ZX}&\mathcal{H}_{ZZ}\end{vmatrix}<0.\tag{A.2}$$

Using the Envelop Theorem yields the first-order derivative of $\mathcal{H}$ with respect to three state variables: $\mathcal{H}_K=\lambda\mathrm{F}_K(K,\hat{E})$, $\mathcal{H}_X=-\lambda\mathrm{M}_X(\hat{E},X)$, and $\mathcal{H}_Z=\mathrm{U}_Z(\hat{C},Z)$. Using (10), (15) and (17) to simplify the first inequality of the conditions (A.2). as $\frac{\lambda}{\mathrm{M}_{EE}-\mathrm{F}_{EE}}[\mathrm{F}_{KK}(\mathrm{M}_{EE}-\mathrm{F}_{EE})+\mathrm{F}_{KE}^2]<0$, and the second inequality as $\frac{\lambda^2}{\mathrm{M}_{EE}-\mathrm{F}_{EE}}\{\mathrm{F}_{KK}\mathrm{M}_{XE}^2-\mathrm{M}_{XX}[\mathrm{F}_{KE}^2+\mathrm{F}_{KK}(\mathrm{M}_{EE}-\mathrm{F}_{EE})]\}>0$. Given that Z only affect $\mathcal{H}_Z$, the last inequality boils down to

$$\mathcal{H}_{ZZ}\begin{vmatrix}\mathcal{H}_{KK} & \mathcal{H}_{KX}\\ \mathcal{H}_{XK} & \mathcal{H}_{XX}\end{vmatrix}=\left[\mathrm{U}_{ZZ}-\frac{\mathrm{U}_{CZ}^2}{\mathrm{U}_{CC}}\right]\frac{\lambda^2}{\mathrm{M}_{EE}-\mathrm{F}_{EE}}\left\{\mathrm{F}_{KK}\mathrm{M}_{XE}^2-\mathrm{M}_{XX}\left[\mathrm{F}_{KE}^2+\mathrm{F}_{KK}(\mathrm{M}_{EE}-\mathrm{F}_{EE})\right]\right\}<0.$$

Given that $\mathrm{M}_{EE}-\mathrm{F}_{EE}>0$, $\mathrm{F}_{KK}<0$ and $\mathrm{M}_{XX}>0$, the sufficient conditions can be simplified as $\mathrm{F}_{KE}^2+\mathrm{F}_{KK}(\mathrm{M}_{EE}-\mathrm{F}_{EE})<\mathrm{F}_{KK}\frac{\mathrm{M}_{XE}^2}{\mathrm{M}_{XX}}$ and $\mathrm{U}_{ZZ}\mathrm{U}_{CC}-\mathrm{U}_{CZ}^2<0$.

Appendix B Proof of Proposition 1

Rewriting (19c) as $\dot{\mathrm{G}}'=\mathrm{F}_K\mathrm{G}'+\frac{\mathrm{U}_Z}{\mathrm{U}_C}$, and substituting it into (19b), the dynamic profiles of resource extraction in an optimal growth model with both resource exhaustibility and pollution externality (named Model 1) are given by

$$\frac{d}{dt}[\pi(K,X,E)-\mathrm{G}'(B)]=\mathrm{F}_K(K,E)[\pi(K,X,E)-\mathrm{G}'(B)]+\mathrm{M}_X(X,E), \tag{B.1}$$

where $\pi(K,X,E)\equiv\mathrm{F}(K,E)-\mathrm{M}(X,E)$ is marginal revenue of resource extraction, $\pi(K,X,E)-\mathrm{G}'(B)$ is marginal extraction revenue minus marginal pollution abatement costs. In contrast, for an optimal growth model with only resource exhaustibility (named Model 2), the terms representing pollution disutility and abatements vanish, and marginal revenues of resource extraction would evolve according to

$$\frac{d}{dt}\left[\pi(\hat{K},\hat{X},\hat{E})\right]=\mathrm{F}_K(\hat{K},\hat{E})\pi(\hat{K},\hat{X},\hat{E})+\mathrm{M}_X(\hat{X},\hat{E}), \tag{B.2}$$

where the "hat" over the underlying arguments corresponds to the case of optimal growth with only resource exhaustibility.

We prove by contradiction that resource extraction flows derived from Model 1 are smaller than that from Model 2, i.e., $E<\hat{E}$, for any given man-made capital and resource reserves, i.e., $K=\hat{K}$ and $X=\hat{X}$. First, suppose that $E=\hat{E}$, with $K=\hat{K}$ and $X=\hat{X}$, we have $\mathrm{F}_K(K,E)=\mathrm{F}_K(\hat{K},\hat{E})$ and $\mathrm{M}_X(X,E)=\mathrm{M}_X(\hat{X},\hat{E})$. Then from (B.1)-(B.2), we derive $\pi(K,X,E)-\mathrm{G}'(B)=\pi(\hat{K},\hat{X},\hat{E})$. However, it is straightforward to verify that this equality cannot hold with $E=\hat{E}$, $K=\hat{K}$, $X=\hat{X}$ and $\mathrm{G}'(B)>0$, which thus leads to contradiction.

Moreover, suppose that $E>\hat{E}$, with $K=\hat{K}$ and $X=\hat{X}$, we have $\mathrm{F}_K(K,E)<\mathrm{F}_K(\hat{K},\hat{E})$ and

$\mathrm{M}_X(X,E) < \mathrm{M}_X(\hat{X},\hat{E})$ given that $\mathrm{F}_{EK} < 0$ and $\mathrm{M}_{EX} < 0$. Then from (B.1)-(B.2), we have

$$\pi(K,X,E) - \mathrm{G}'(B) > \pi(\hat{K},\hat{X},\hat{E}). \tag{B.3}$$

However, with $E > \hat{E}$, $K = \hat{K}$, $X = \hat{X}$ and π is negatively related to E, we have $\pi(K,X,E) < \pi(\hat{K},\hat{X},\hat{E})$ which contradicts with (B.3).

Finally, suppose that $E < \hat{E}$, with $K = \hat{K}$ and $X = \hat{X}$, we have $\mathrm{F}_K(K,E) > \mathrm{F}_K(\hat{K},\hat{E})$ and $\mathrm{M}_X(X,E) > \mathrm{M}_X(\hat{X},\hat{E})$ given that $\mathrm{F}_{EK} < 0$ and $\mathrm{M}_{EX} < 0$. Then from (B.1)-(B.2), we derive

$$\pi(K,X,E) - \mathrm{G}'(B) < \pi(\hat{K},\hat{X},\hat{E}). \tag{B.4}$$

Given that $E < \hat{E}$, $K = \hat{K}$, $X = \hat{X}$ and ψ is negatively related to E, we can have $\pi(K,X,E) > \pi(\hat{K},\hat{X},\hat{E})$. Therefore, with $\mathrm{G}'(B) > 0$, $\pi(K,X,E) - \mathrm{G}'(B) < \pi(\hat{K},\hat{X},\hat{E})$ can possibly hold and thus not contradict with (B.4). Further check when $\mathrm{F}_{EK} > 0$, we have $\mathrm{F}_K(K,E) < \mathrm{F}_K(\hat{K},\hat{E})$ for $E < \hat{E}$ with $K = \hat{K}$ and $X = \hat{X}$. Then from (B.1)-(B.2), the following inequality can hold

$$\pi(K,X,E) - \mathrm{G}'(B) > \pi(\hat{K},\hat{X},\hat{E}). \tag{B.5}$$

Given that $E < \hat{E}$, $K = \hat{K}$, $X = \hat{X}$ and ψ is negatively related to E, we can have $\pi(K,X,E) > \pi(\hat{K},\hat{X},\hat{E})$. With $\mathrm{G}'(B) > 0$, $\pi(K,X,E) - \mathrm{G}'(B) > \pi(\hat{K},\hat{X},\hat{E})$ can hold and thus not contradict with (B.4). Therefore, it can be concluded that only $E < \hat{E}$ does not lead to contradiction. With $E < \hat{E}$, F_K will decrease if $\mathrm{F}_{EK} > 0$, and together with the negative pollution effect on utility, the growth rate of consumption would be lower.

Appendix C Proof of Proposition 2

Substituting $C = \mathrm{F}(K^*,0)$ into (23) yields the following equation that characterizes the steady state of pollution stock

$$-\frac{\mathrm{U}_Z(\mathrm{F}(K^*,0),Z^*)}{\rho \mathrm{U}_C(\mathrm{F}(K^*,0),Z^*)} = \mathrm{G}'(0), \tag{C.1}$$

where the marginal cost of abatement at zero abatement levels is exogenously given $\mathrm{G}'(0)$. Combined with (21), the two equations can pin down two endogenous variables K^* and Z^*. To establish the existence of K^* and Z^*, we use (21) to pin down K^*. Then total differentiating (C.1) yields

$$\left[\frac{\mathrm{U}_C\mathrm{U}_{ZC}-\mathrm{U}_Z\mathrm{U}_{CC}}{\mathrm{U}_C^2}\right]dC^*+\left[\frac{\mathrm{U}_C\mathrm{U}_{ZZ}-\mathrm{U}_Z\mathrm{U}_{CZ}}{\mathrm{U}_C^2}\right]dZ^*=0 \quad\Rightarrow\quad \Omega dC^*+\Psi dZ^*=0,$$

where we define two terms: $\Omega\equiv\frac{\mathrm{U}_C\mathrm{U}_{ZC}-\mathrm{U}_Z\mathrm{U}_{CC}}{\mathrm{U}_C^2}$ and $\Psi\equiv\frac{\mathrm{U}_C\mathrm{U}_{ZZ}-\mathrm{U}_Z\mathrm{U}_{CZ}}{\mathrm{U}_C^2}$. Following from $\mathrm{U}_C>0,\mathrm{U}_{ZC}<0,\mathrm{U}_Z<0,\mathrm{U}_{CC}<0$, we have $\Omega<0$. Furthermore, suppose that

$$\frac{\partial \ln \mathrm{U}_C}{\partial Z}<\frac{\partial \ln \mathrm{U}_Z}{\partial Z},$$

we would obtain $\mathrm{U}_C\mathrm{U}_{ZZ}-\mathrm{U}_Z\mathrm{U}_{CZ}>0$ and $\Psi>0$. Substituting $dC^*=\mathrm{F}_KdK^*$ (derived from $C^*=\mathrm{F}(K^*,0)$) yields $\Omega\mathrm{F}_KdK^*+\Psi dZ^*=0$ and rearranging yields

$$\frac{dZ^*}{dK^*}=-\frac{\Omega\mathrm{F}_K}{\Psi}>0.$$

where the positive sign is derived following from $\Omega<0$, $\Psi>0$, $\mathrm{F}_K>0$, implying that Z^* is strictly increasing in K^*. Given that K^* is uniquely determined by (21), there exists a unique value of K^* and Z^* that satisfies (21) and (23).

Given the existence of K^* and Z^*, the steady state levels of consumption are given by $C^*=\mathrm{F}(K^*,0)$, and the steady state levels of resource reserves X^* are determined by

$$\mathrm{F}_E(K^*,0)=\mathrm{M}_E(0,X^*)-\frac{\mathrm{M}_X(0,X^*)}{\rho}+\mathrm{G}'(0). \tag{C.2}$$

where the uniqueness of K^* and Z^* implies that $\mathrm{F}_E(K^*,0)$ is uniquely determined, and the right-hand side of (C.2) is also uniquely determined. Given that $\mathrm{M}_{EX}<0$ and $\mathrm{M}_{XX}>0$, the right-hand side of (C.2) is monotonically decreasing with the only argument X^*, so the right-hand side of (C.2) uniquely determines the steady state of resource reserves X^*.

Appendix D Proof of Proposition 3

Since (21) and (23) has no argument of Z, pollution externality has no effect on E^* and K^*. From (22), the term $\mathrm{G}'(B^*)$ will vanish if the pollution externality is not taken into account. With $\mathrm{G}'(B^*)$ decrease, the term $\mathrm{M}_E(E^*,X^*)-\frac{\mathrm{M}_X(E^*,X^*)}{\rho}$ will increase such that the right-hand side of (22) will be equal to the left-hand side F_E which is unaffected. Given that E^* is unchanged, X^* will decrease when $\mathrm{M}_E(E^*,X^*)-\frac{\mathrm{M}_X(E^*,X^*)}{\rho}$ increases because it is monotonically decreasing in X^*.

Finally, since $\mathrm{F}(K^*,E^*)$ is unchanged and $\mathrm{G}(B^*)=\mathrm{G}(0)=0$, an increase in $\mathrm{M}(E^*,X^*)$ leads to a decrease in $C^*=\mathrm{F}(K^*,E^*)-\mathrm{M}(E^*,X^*)-\mathrm{G}(B^*)$.

Appendix E Proof of Proposition 4

Assuming that the supply of resources is infinitely large, i.e., $X^*\to\infty$, the law of motion for resource depletion vanishes and (22) characterizing the steady state of resource extraction becomes

$$\mathrm{F}_E(\mathrm{K}(E^*),E^*)=\mathrm{M}_E(E^*,X^*)+\frac{\mu^*}{\lambda^*}+\mathrm{G}'(E^*), \tag{E.1}$$

where $\mathrm{F}_K(K^*,E^*)=\rho$ implicitly determines $K^*=\mathrm{K}(E^*)$ that relates K^* as a function of E^* with $\frac{dK^*}{dE^*}=-\frac{\mathrm{F}_{EK}}{\mathrm{F}_{KK}}$, and $B^*=E^*$ holds such that the pollution stock can be stabilized in the steady state. Total differentiating (E.1) yields

$$\left[\frac{\mathrm{F}_{EE}\mathrm{F}_{KK}-\mathrm{F}_{EK}^2}{\mathrm{F}_{KK}}-\mathrm{M}_{EE}-\mathrm{G}''\right]dE^*=\mathrm{M}_{EX}dX^*+d\left(\frac{\mu^*}{\lambda^*}\right). \tag{E.2}$$

With resource reserves increase to an infinitely large level $X^*\to\infty$, we have $dX^*>0$ and the shadow value of resources due to exhaustibility will decrease and vanish $d\left(\frac{\mu^*}{\lambda^*}\right)<0$. Following from $\mathrm{F}_{EE}\mathrm{F}_{KK}-\mathrm{F}_{EK}^2>0$, $\mathrm{F}_{KK}<0$, $\mathrm{M}_{EE}>0$ and $\mathrm{G}''>0$, we have $dE^*>0$, $dB^*>0$, and $d\mathrm{G}(B^*)>0$. Furthermore, while $X^*\to\infty$ lowers the marginal cost of extraction, the increase in extraction flows $dE^*>0$ makes the total cost of extraction $\mathrm{M}(E^*,X^*)$ ambiguous. Depending on whether K and E are complementary in final goods production, we have $dK^*>0$ if $\mathrm{F}_{KE}>0$ and $dK^*<0$ if $\mathrm{F}_{KE}<0$. Given that $\mathrm{F}(K^*,E^*)$ and $\mathrm{M}(E^*,X^*)$ are ambiguous, changes in C^* are ambiguous, implying that Z^* is undetermined following from (23).

Appendix F Proof of Proposition 5

With respect to the exogenous parameter ρ, totally differentiating $\mathrm{F_K}(\mathrm{K}^*,0)=\rho$, (C.1) and (C.2) that characterize K^*,X^*,Z^* yields the following linear equation system

$$\begin{bmatrix} \mathrm{F}_{KK} & 0 & 0 \\ \rho\mathrm{F}_{EK} & 0 & \mathrm{M}_{XX}-\rho\mathrm{M}_{EX} \\ \Omega\mathrm{F}_K & \Psi & 0 \end{bmatrix}\begin{bmatrix} dK^* \\ dZ^* \\ dX^* \end{bmatrix}=\begin{bmatrix} 1 \\ \mathrm{M}_E-\mathrm{F}_E+\mathrm{G}'(0) \\ 0 \end{bmatrix}d\rho,$$

and using the Cramer's Rule to solve and yield $\frac{dK^*}{d\rho}=\frac{\mathcal{D}_{K^*}}{\mathcal{D}}$, $\frac{dZ^*}{d\rho}=\frac{\mathcal{D}_{Z^*}}{\mathcal{D}}$, $\frac{dX^*}{d\rho}=\frac{\mathcal{D}_{X^*}}{\mathcal{D}}$, where the sign of $\mathcal{D}$ is established by $\mathcal{D}=-(\mathrm{M}_{XX}-\rho\mathrm{M}_{EX})\mathrm{F}_{KK}\Psi>0$, following from $\mathrm{M}_{XX}-\rho\mathrm{M}_{EX}>0$, $\mathrm{F}_{KK}<0$ and $\Psi>0$. The sign of $\mathcal{D}_{K^*}$ is given by $\mathcal{D}_{K^*}=-(\mathrm{M}_{XX}-\mathrm{F}_K\mathrm{M}_{EX})\Psi<0$ following from $\mathrm{M}_{XX}-\rho\mathrm{M}_{EX}>0$ and $\Psi>0$. The sign of $\mathcal{D}_{Z^*}$ is given by $\mathcal{D}_{Z^*}=-(\mathrm{M}_{XX}-\mathrm{F}_K\mathrm{M}_{EX})(0-\Omega\mathrm{F}_K)<0$ due to $\mathrm{M}_{XX}-\rho\mathrm{M}_{EX}>0$ and $\Omega\mathrm{F}_K<0$. The sign of $\mathcal{D}_{X^*}$ is given by $\mathcal{D}_{X^*}=\rho\mathrm{F}_{EK}\Psi-\mathrm{F}_{KK}\Psi(\mathrm{M}_E-\mathrm{F}_E+\mathrm{G}'(0))$. Using (C.2) $\mathrm{M}_E-\mathrm{F}_E+\mathrm{G}'(0)=\frac{\mathrm{M}_X}{\rho}$ to simplify notations and rearranging yields $\mathcal{D}_{X^*}=\left[\rho\mathrm{F}_{EK}-\frac{\mathrm{M}_X}{\rho}\mathrm{F}_{KK}\right]\Psi$. Given that $\Psi>0$, we yield (25). Finally, E^* and B^* are not affected by changes in ρ, so $\frac{dE^*}{d\rho}=0$ and $\frac{dB^*}{d\rho}=0$. Given that $\frac{dK^*}{d\rho}<0$ and $C^*=\mathrm{F}(K^*,0)$, we have $\frac{dC^*}{d\rho}<0$.

Appendix G Proof of Proposition 6

With respect to the exogenous parameter α, totally differentiating $\mathrm{F_K(K^*,0)}=\rho$, (C.1) and (C.2) that characterize K^*,X^*,Z^* yields

$$\begin{bmatrix} \mathrm{F}_{KK} & 0 & 0 \\ \alpha\mathrm{F}_{EK} & 0 & \frac{\mathrm{M}_{XX}}{\rho}-\mathrm{M}_{EX} \\ \Omega\mathrm{F}_K & \Psi & 0 \end{bmatrix} \begin{bmatrix} dK^* \\ dZ^* \\ dX^* \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathrm{F}_E \\ 0 \end{bmatrix} d\alpha. \tag{G.1}$$

Using the Cramer's Rule to solve the linear equation system and yields $\frac{dK^*}{d\alpha}=\frac{\mathcal{D}_{K^*}}{\mathcal{D}}$, $\frac{dZ^*}{d\alpha}=\frac{\mathcal{D}_{Z^*}}{\mathcal{D}}$, $\frac{dX^*}{d\alpha}=\frac{\mathcal{D}_{X^*}}{\mathcal{D}}$, where the signs of $\mathcal{D},\mathcal{D}_{K^*},\mathcal{D}_{Z^*},\mathcal{D}_{X^*}$ are given by $\mathcal{D}=(-1)\left(\frac{\mathrm{M}_{XX}}{\rho}-\mathrm{M}_{EX}\right)\mathrm{F}_{KK}\Psi>0$, $\mathcal{D}_{K^*}=0$, $\mathcal{D}_{Z^*}=0$, and $\mathcal{D}_{X^*}=\mathrm{F}_E\mathrm{F}_{KK}\Psi<0$. For the exogenous parameter β and γ, we follow a similar procedure of total differentiation $\mathrm{F_K(K^*,0)}=\rho$, (C.1) and (C.2), and the matrix of linear equations is given by

$$\begin{bmatrix} \mathrm{F}_{KK} & 0 & 0 \\ \mathrm{F}_{EK} & 0 & \frac{\beta\mathrm{M}_{XX}}{\rho}-\mathrm{M}_{EX} \\ \Omega\mathrm{F}_K & \Psi & 0 \end{bmatrix} \begin{bmatrix} dK^* \\ dZ^* \\ dX^* \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{M}_E-\frac{\mathrm{M}_X}{\rho} \\ 0 \end{bmatrix} d\beta,$$

$$\begin{bmatrix} \mathrm{F}_{KK} & 0 & 0 \\ \mathrm{F}_{EK} & 0 & \frac{\mathrm{M}_{XX}}{\rho}-\mathrm{M}_{EX} \\ \Omega\mathrm{F}_K & \Psi & 0 \end{bmatrix} \begin{bmatrix} dK^* \\ dZ^* \\ dX^* \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{G}'(0) \\ 0 \end{bmatrix} d\gamma,$$

from which the signs of the comparative static effects of β and γ can be established.

Appendix H Proof of Lemma 3

We prove this lemma by verifying the negative definiteness of the matrix $-\mathcal{H}_{\mathbf{PP}}$. The first leading principal minor is given by

$$-\mathcal{H}_{\lambda\lambda}=-\left[\frac{(\mathrm{F}_E-\mathrm{M}_E)^2}{\lambda\Delta}+\frac{(\mathrm{G}')^2}{\lambda\mathrm{G}''}-\frac{1}{\mathrm{U}_{CC}}\right]<0,$$

where the negative sign is determined following from $\lambda>0$, $\Delta\equiv\mathrm{M}_{EE}-\mathrm{F}_{EE}>0$ and $\mathrm{U}_{CC}<0$. The second leading principal minor of the matrix (3) is determined by

$$\begin{vmatrix} -\mathcal{H}_{\lambda\lambda} & -\mathcal{H}_{\lambda\mu} \\ -\mathcal{H}_{\mu\lambda} & -\mathcal{H}_{\mu\mu} \end{vmatrix}=\mathcal{H}_{\lambda\lambda}\mathcal{H}_{\mu\mu}-\mathcal{H}_{\lambda\mu}^2=\left[\frac{(\mathrm{G}')^2}{\lambda\mathrm{G}''}-\frac{1}{\mathrm{U}_{CC}}\right]\frac{1}{\lambda\Delta}>0.$$

The third leading principal minor can be rewritten as

$$\begin{aligned}
&(-\mathcal{H}_{\lambda\psi})\begin{vmatrix} -\mathcal{H}_{\mu\lambda} & -\mathcal{H}_{\mu\mu} \\ -\mathcal{H}_{\psi\lambda} & -\mathcal{H}_{\psi\mu} \end{vmatrix}-(-\mathcal{H}_{\mu\psi})\begin{vmatrix} -\mathcal{H}_{\lambda\lambda} & -\mathcal{H}_{\lambda\mu} \\ -\mathcal{H}_{\psi\lambda} & -\mathcal{H}_{\psi\mu} \end{vmatrix}+(-\mathcal{H}_{\psi\psi})\begin{vmatrix} -\mathcal{H}_{\lambda\lambda} & -\mathcal{H}_{\lambda\mu} \\ -\mathcal{H}_{\mu\lambda} & -\mathcal{H}_{\mu\mu} \end{vmatrix}\\
&=-\left[\frac{\mathrm{F}_E-\mathrm{M}_E}{\lambda\Delta}+\frac{\mathrm{G}'}{\lambda\mathrm{G}''}\right]\left[\frac{\mathrm{F}_E-\mathrm{M}_E}{\lambda\Delta}\frac{1}{\lambda\Delta}-\left(\frac{\mathrm{F}_E-\mathrm{M}_E}{\lambda\Delta}+\frac{\mathrm{G}'}{\lambda\mathrm{G}''}\right)\frac{1}{\lambda\Delta}\right]\\
&\quad-\left(\frac{1}{\lambda\Delta}\right)\left[\frac{1}{\lambda\Delta}\left[\frac{1}{\mathrm{U}_{CC}}-\frac{(\mathrm{G}')^2}{\lambda\mathrm{G}''}\right]+\frac{(\mathrm{F}_E-\mathrm{M}_E)\mathrm{G}'}{\lambda^2\Delta\mathrm{G}''}\right]-\left[\frac{1}{\lambda\Delta}+\frac{1}{\lambda\mathrm{G}''}\right]\left[\frac{(\mathrm{G}')^2}{\lambda\mathrm{G}''}-\frac{1}{\mathrm{U}_{CC}}\right]\frac{1}{\lambda\Delta}\\
&=\frac{1}{\lambda^2\Delta\mathrm{G}''\mathrm{U}_{CC}}<0,
\end{aligned}$$

where cancelling common terms and simplifying yields the last line, and the negative sign is determined given that $\Delta\equiv\mathrm{M}_{EE}-\mathrm{F}_{EE}>0$, $\mathrm{G}''>0$ and $\mathrm{U}_{CC}<0$.

Appendix I Proof of Proposition 8

The proof begins by defining two Hamiltonian matrix

$$\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})\equiv\begin{bmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{W} & -\mathcal{R}^T \end{bmatrix}=\begin{bmatrix} \mathcal{H}_{\mathbf{PS}}-\frac{\varrho}{2}I_3 & \mathcal{H}_{\mathbf{PP}} \\ -\mathcal{H}_{\mathbf{SS}} & -\mathcal{H}_{\mathbf{SP}}+\frac{\varrho}{2}I_3 \end{bmatrix}, \tag{I.1a}$$

$$\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})\equiv\begin{bmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{W}-\left(\frac{\varrho^2}{4}+\tau\right)\mathcal{T}^{-1} & -\mathcal{R}^T \end{bmatrix}=\begin{bmatrix} \mathcal{H}_{\mathbf{PS}}-\frac{\varrho}{2}I_3 & \mathcal{H}_{\mathbf{PP}} \\ -\mathcal{H}_{\mathbf{SS}}-\left(\frac{\varrho^2}{4}+\tau\right)\mathcal{H}_{\mathbf{PP}}^{-1} & -\mathcal{H}_{\mathbf{SP}}+\frac{\varrho}{2}I_3 \end{bmatrix}, \tag{I.1b}$$

where τ is an exogenous parameter, and we verify that if $\mathcal{H}_{\mathbf{PS}}\mathcal{H}_{\mathbf{PP}}=\mathcal{H}_{\mathbf{PP}}\mathcal{H}_{\mathbf{SP}}$ holds, then we have $\mathcal{R}^T\mathcal{T}^{-1}=\mathcal{T}^{-1}\mathcal{R}$ and $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})^2=\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})^2-\left(\frac{\rho^2}{4}+\tau\right)I_6$. Hence, the eigenvalues of $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})$ and $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$ satisfy the condition $(\pm\bar{\xi}_i)^2=(\pm\hat{\xi}_i)^2-\left(\frac{\rho^2}{4}+\tau\right)\quad\forall i=1,2,3$, where $\pm\bar{\xi}_i$ and $\pm\hat{\xi}_i$ denote the symmetric pair of eigenvalue of $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})$ and $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$, respectively.

Now consider the case of the saddle path stability, the eigenvalues of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$ satisfy $|\pm\hat{\xi}_i|>\frac{\rho}{2}\ \forall i=1,2,3$, there thus exists $\bar{\tau}=\min_{i=1,2,3}\left(\hat{\xi}_i^2-\frac{\rho^2}{4}\right)>0$ such that for $\forall\tau\in(0,\bar{\tau})$, $(\pm\bar{\xi}_i)^2=(\pm\hat{\xi}_i)^2-\left(\frac{\rho^2}{4}+\tau\right)>0\Longrightarrow|\bar{\xi}_i|>0$, i.e., $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})$ has nonzero eigenvalues (three positive and three negative, symmetrically distributed with respect to zero). Given the symmetric distribution of nonzero eigenvalues, there exists a 3×3 matrix $\mathcal{Z}\in\mathbb{R}^{3\times3}$ and a 6×3 matrix $[\mathcal{X},\mathcal{Y}]^T\in\mathbb{R}^{6\times3}$, such that the 6×6 Hamiltonian matrix $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})$ given by (I.1b) can be diagonalized as

$$\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}\equiv\begin{bmatrix}\mathcal{R} & \mathcal{T}\\ \mathcal{V} & -\mathcal{R}^T\end{bmatrix}\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}=\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}\mathcal{Z},\tag{I.2}$$

where the three columns of the 6×3 matrix $[\mathcal{X},\mathcal{Y}]^T\in\mathbb{R}^{6\times3}$ correspond to the three eigenvectors associated with three negative eigenvalues of $\bar{\mathcal{J}}(\mathbf{S},\mathbf{P})$, and $\mathcal{Z}\equiv\operatorname{diag}(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^{3\times3}$ is the eigenvalue diagonalized matrix where ξ_1,ξ_2,ξ_3 denote the three negative eigenvalues of $\hat{\mathcal{J}}(\mathbf{S},\mathbf{P})$.

Matrix manipulation on the second row of (I.2) yields, $\mathcal{V}-\mathcal{R}^T\mathcal{Y}\mathcal{X}^{-1}=\mathcal{Y}\mathcal{Z}\mathcal{X}^{-1}=\mathcal{Y}\mathcal{X}^{-1}\mathcal{X}\mathcal{Z}\mathcal{X}^{-1}$. Define $\mathcal{Q}=\mathcal{Y}\mathcal{X}^{-1}$ and substituting the first row of (I.2) yields, $\mathcal{V}-\mathcal{R}^T\mathcal{Q}=\mathcal{Q}(\mathcal{R}\mathcal{X}+\mathcal{T}\mathcal{Y})\mathcal{X}^{-1}=\mathcal{Q}\mathcal{R}+\mathcal{T}\mathcal{Q}$. Rearranging yields a matrix Riccati equation

$$-\mathcal{V}+\mathcal{R}^T\mathcal{Q}+\mathcal{Q}\mathcal{R}+\mathcal{Q}\mathcal{T}\mathcal{Q}=0,\tag{I.3}$$

where there exists a symmetric matrix $\mathcal{Q}=\mathcal{Y}\mathcal{X}^{-1}$ ($\mathcal{X}$ is nonsingular) that solves (I.3).[20] Substituting the expressions of $\mathcal{R},\mathcal{T},\mathcal{V}$ given by (I.1b) into the matrix Riccati equation (I.3) yields

$$\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})\equiv\mathcal{H}_{\mathbf{SS}}+\frac{\rho^2}{4}\mathcal{H}_{\mathbf{PP}}^{-1}+\mathcal{H}_{\mathbf{SP}}\mathcal{Q}+\mathcal{Q}\mathcal{H}_{\mathbf{PS}}+\mathcal{Q}\mathcal{H}_{\mathbf{PP}}\mathcal{Q}-\rho\mathcal{Q}=-\tau\mathcal{H}_{\mathbf{PP}}^{-1}.\tag{I.4}$$

[20] To establish the matrix $\mathcal{Q}=\mathcal{Y}\mathcal{X}^{-1}$ is symmetric, from (I.2) we have

$$\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}^T\mathcal{I}\bar{\mathcal{J}}\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}=\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}^T\mathcal{I}\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}\mathcal{Z}\quad\Rightarrow\quad\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}^T\begin{bmatrix}-\mathcal{V} & \mathcal{R}^T\\ \mathcal{R} & \mathcal{T}\end{bmatrix}\begin{bmatrix}\mathcal{X}\\ \mathcal{Y}\end{bmatrix}=(\mathcal{Y}^T\mathcal{X}-\mathcal{X}^T\mathcal{Y})\mathcal{Z},$$

where $\mathcal{I}\in\mathbb{R}^{6\times6}$ is a skew-symmetric matrix. As the left-hand side is symmetric, the right-hand side is also symmetric, i.e., $(\mathcal{Y}^T\mathcal{X}-\mathcal{X}^T\mathcal{Y})\mathcal{Z}=\mathcal{Z}^T(\mathcal{X}^T\mathcal{Y}-\mathcal{Y}^T\mathcal{X})$. Rearranging yields $(\mathcal{Y}^T\mathcal{X}-\mathcal{X}^T\mathcal{Y})(\mathcal{Z}+\mathcal{Z}^T)=0\Rightarrow\mathcal{Y}^T\mathcal{X}=\mathcal{X}^T\mathcal{Y}$. We can accordingly establish that the matrix $\mathcal{Q}=\mathcal{Y}\mathcal{X}^{-1}$ is symmetry, i.e., $\mathcal{Q}=\mathcal{Y}\mathcal{X}^{-1}=(\mathcal{X}^{-1})^T\mathcal{X}^T\mathcal{Y}\mathcal{X}^{-1}=(\mathcal{X}^{-1})^T\mathcal{Y}^T\mathcal{X}\mathcal{X}^{-1}=(\mathcal{Y}\mathcal{X}^{-1})^T=\mathcal{Q}^T$.

Therefore, if $\mathcal{H}_{\mathbf{PS}}\mathcal{H}_{\mathbf{PP}}=\mathcal{H}_{\mathbf{PP}}\mathcal{H}_{\mathbf{SP}}$ holds, then $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})$ has the same negative definiteness as $-\mathcal{H}_{\mathbf{PP}}^{-1}$. Given that $-\mathcal{H}_{\mathbf{PP}}\prec 0$, we have $-\mathcal{H}_{\mathbf{PP}}^{-1}\prec 0$,[21] and $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})\prec 0$. With both $-\mathcal{H}_{\mathbf{PP}}$ and its Schur complement $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})/(-\mathcal{H}_{\mathbf{PP}})$ are negative definite, the curvature matrix is negative definite, i.e., $\mathcal{C}^{\mathcal{Q}}(\mathbf{S},\mathbf{P})\prec 0$. The negative definiteness of the curvature matrix given in (32) thus established the condition for GAS.

The second part of proof is to establish the condition $\mathcal{H}_{\mathbf{PS}}\mathcal{H}_{\mathbf{PP}}=\mathcal{H}_{\mathbf{PP}}\mathcal{H}_{\mathbf{SP}}$, i.e.,

$$\begin{bmatrix}\mathcal{H}_{\lambda K} & \mathcal{H}_{\lambda X} & \mathcal{H}_{\lambda Z}\\ \mathcal{H}_{\mu K} & \mathcal{H}_{\mu X} & \mathcal{H}_{\mu Z}\\ \mathcal{H}_{\psi K} & \mathcal{H}_{\psi X} & \mathcal{H}_{\psi Z}\end{bmatrix}\begin{bmatrix}\mathcal{H}_{\lambda\lambda} & \mathcal{H}_{\lambda\mu} & \mathcal{H}_{\lambda\psi}\\ \mathcal{H}_{\mu\lambda} & \mathcal{H}_{\mu\mu} & \mathcal{H}_{\mu\psi}\\ \mathcal{H}_{\psi\lambda} & \mathcal{H}_{\psi\mu} & \mathcal{H}_{\psi\psi}\end{bmatrix}=\begin{bmatrix}\mathcal{H}_{\lambda\lambda} & \mathcal{H}_{\lambda\mu} & \mathcal{H}_{\lambda\psi}\\ \mathcal{H}_{\mu\lambda} & \mathcal{H}_{\mu\mu} & \mathcal{H}_{\mu\psi}\\ \mathcal{H}_{\psi\lambda} & \mathcal{H}_{\psi\mu} & \mathcal{H}_{\psi\psi}\end{bmatrix}\begin{bmatrix}\mathcal{H}_{K\lambda} & \mathcal{H}_{K\mu} & \mathcal{H}_{K\psi}\\ \mathcal{H}_{X\lambda} & \mathcal{H}_{X\mu} & \mathcal{H}_{X\psi}\\ \mathcal{H}_{Z\lambda} & \mathcal{H}_{Z\mu} & \mathcal{H}_{Z\psi}\end{bmatrix}, \tag{I.5}$$

where the equality corresponding to the main diagonal (1,1), (2,2) and (3,3) elements always holds since $\mathcal{H}(K,X,Z,\lambda,\mu,\psi)$ satisfies the equality of mixed second-order partial derivatives, i.e., $\frac{\partial^2\mathcal{H}}{\partial i\partial j}=\frac{\partial^2\mathcal{H}}{\partial j\partial i}\quad\forall i,j\in\{K,X,Z,\lambda,\mu,\psi\}$. Meanwhile, the off-diagonal elements in (I.5) are symmetric, and we thus only need to verify the equalities corresponding to the (1,2), (1,3) and (2,3) element as they are the same with that corresponding to the (2,1), (3,1) and (3,2) element respectively.

Making LHS to equalize RHS in (I.5) and cancelling the common terms, the equality corresponding to (1,2) element boils down to the condition

$$-\mathrm{F}_K\frac{\partial\hat{E}}{\partial\lambda}+\mathrm{M}_X\frac{\partial\hat{E}}{\partial\mu}+\frac{\partial\hat{C}}{\partial Z}\frac{\partial\hat{E}}{\partial\psi}=\frac{\partial\hat{C}}{\partial\lambda}\frac{\partial\hat{E}}{\partial K}+\mathrm{G}'\frac{\partial\hat{B}}{\partial\lambda}\frac{\partial\hat{E}}{\partial K}, \tag{I.6}$$

where the LHS and RHS are given by $\nabla_{K,X,Z}\mathcal{H}_\lambda\cdot[\nabla_{\lambda,\mu,\psi}\mathcal{H}_\mu]^T=\left\langle \mathrm{F}_K+\pi\frac{\partial\hat{E}}{\partial K},-\mathrm{M}_X+\pi\frac{\partial\hat{E}}{\partial X},-\frac{\partial\hat{C}}{\partial Z}\right\rangle\cdot\left\langle -\frac{\partial\hat{E}}{\partial\lambda},-\frac{\partial\hat{E}}{\partial\mu},-\frac{\partial\hat{E}}{\partial\psi}\right\rangle^T$, and $\nabla_{\lambda,\mu,\psi}\mathcal{H}_\lambda\cdot[\nabla_{K,X,Z}\mathcal{H}_\mu]^T=\left\langle \pi\frac{\partial\hat{E}}{\partial\lambda}-\mathrm{G}'\frac{\partial\hat{B}}{\partial\lambda}-\frac{\partial\hat{C}}{\partial\lambda},\pi\frac{\partial\hat{E}}{\partial\mu},\pi\frac{\partial\hat{E}}{\partial\mu}-\mathrm{G}'\frac{\partial\hat{B}}{\partial\psi}\right\rangle\cdot\left\langle -\frac{\partial\hat{E}}{\partial X},-\frac{\partial\hat{E}}{\partial X},-\frac{\partial\hat{E}}{\partial Z}\right\rangle^T$, respectively. Furthermore, the equality corresponding to (1,3) element boils down to the condition

$$-\mathrm{F}_K\left(\frac{\partial\hat{E}}{\partial\lambda}-\frac{\partial\hat{B}}{\partial\lambda}\right)+\pi\frac{\partial\hat{E}}{\partial K}\frac{\partial\hat{B}}{\partial\lambda}+\mathrm{M}_X\frac{\partial\hat{E}}{\partial\mu}+\frac{\partial\hat{C}}{\partial Z}\left(\frac{\partial\hat{E}}{\partial\psi}-\frac{\partial\hat{B}}{\partial\psi}\right)=\frac{\partial\hat{C}}{\partial\lambda}\frac{\partial\hat{E}}{\partial K}+\mathrm{G}'\frac{\partial\hat{B}}{\partial\lambda}\frac{\partial\hat{E}}{\partial K}, \tag{I.7}$$

where the LHS and RHS are given by $\nabla_{K,X,Z}\mathcal{H}_\lambda\cdot[\nabla_{\lambda,\mu,\psi}\mathcal{H}_\psi]^T=\left\langle \mathrm{F}_K+\pi\frac{\partial\hat{E}}{\partial K},-\mathrm{M}_X+\pi\frac{\partial\hat{E}}{\partial X},-\frac{\partial\hat{C}}{\partial Z}\right\rangle\cdot\left\langle \frac{\partial\hat{E}}{\partial\lambda}-\frac{\partial\hat{B}}{\partial\lambda},\frac{\partial\hat{E}}{\partial\mu},\frac{\partial\hat{E}}{\partial\psi}-\frac{\partial\hat{B}}{\partial\psi}\right\rangle^T$, and $\nabla_{\lambda,\mu,\psi}\mathcal{H}_\lambda\cdot[\nabla_{K,X,Z}\mathcal{H}_\psi]^T=\left\langle \pi\frac{\partial\hat{E}}{\partial\lambda}-\mathrm{G}'\frac{\partial\hat{B}}{\partial\lambda}-\frac{\partial\hat{C}}{\partial\lambda},\pi\frac{\partial\hat{E}}{\partial\mu},\pi\frac{\partial\hat{E}}{\partial\psi}-\mathrm{G}'\frac{\partial\hat{B}}{\partial\psi}\right\rangle\cdot\left\langle \frac{\partial\hat{E}}{\partial K},\frac{\partial\hat{E}}{\partial X},\frac{\partial\hat{E}}{\partial Z}\right\rangle^T$, respectively. Finally, the equality corresponding to (2,3) element can be simplified

[21] If a nonsingular symmetric matrix is negative definite $-\mathcal{H}_{\mathbf{PP}}\prec 0$, so is an inverse matrix of this symmetric matrix $-\mathcal{H}_{\mathbf{PP}}^{-1}\prec 0$.

as the condition

$$\frac{\partial \hat{E}}{\partial K}\frac{\partial \hat{B}}{\partial \lambda}=0. \tag{I.8}$$

where the LHS and RHS are given by $\nabla_{K,X,Z}\mathcal{H}_\mu\cdot[\nabla_{\lambda,\mu,\psi}\mathcal{H}_\psi]^T=\left\langle -\frac{\partial \hat{E}}{\partial K},-\frac{\partial \hat{E}}{\partial X},-\frac{\partial \hat{E}}{\partial Z}\right\rangle\cdot\left\langle \frac{\partial \hat{E}}{\partial \lambda}-\frac{\partial \hat{B}}{\partial \lambda},\frac{\partial \hat{E}}{\partial \mu},\frac{\partial \hat{E}}{\partial \psi}-\frac{\partial \hat{B}}{\partial \psi}\right\rangle^T$, and $\nabla_{\lambda,\mu,\psi}\mathcal{H}_\mu\cdot[\nabla_{K,X,Z}\mathcal{H}_\psi]^T=\left\langle -\frac{\partial \hat{E}}{\partial \lambda},-\frac{\partial \hat{E}}{\partial \mu},-\frac{\partial \hat{E}}{\partial \psi}\right\rangle\cdot\left\langle \frac{\partial \hat{E}}{\partial K},\frac{\partial \hat{E}}{\partial X},\frac{\partial \hat{E}}{\partial Z}\right\rangle^T$, respectively.

The three conditions (I.6), (I.7) and (I.8) can be further simplified as $\mathrm{F}_K\frac{\partial \hat{E}}{\partial \lambda}-\mathrm{M}_X\frac{\partial \hat{E}}{\partial \mu}-\frac{\partial \hat{C}}{\partial Z}\frac{\partial \hat{E}}{\partial \psi}+\frac{\partial \hat{C}}{\partial \lambda}\frac{\partial \hat{E}}{\partial K}=0$, $\mathrm{F}_K\frac{\partial \hat{B}}{\partial \lambda}-\frac{\partial \hat{C}}{\partial Z}\frac{\partial \hat{B}}{\partial \psi}=0$, and $\frac{\partial \hat{E}}{\partial K}\frac{\partial \hat{B}}{\partial \lambda}=0$. Substituting the specific function forms given in (10), (15) and (17) into these conditions, we obtain $\mathrm{F}_K(\mathrm{F}_E-\mathrm{M}_E)+\mathrm{M}_X-\mathrm{F}_K\mathrm{G}'+\frac{\mathrm{U}_C}{\mathrm{U}_{CC}}\mathrm{F}_{EK}=0$, $\mathrm{F}_K\mathrm{G}'+\frac{\mathrm{U}_{CZ}}{\mathrm{U}_{CC}}=0$, and $\mathrm{F}_{EK}\mathrm{G}'=0$. Given that $\mathrm{G}'\neq 0$, we obtain (33).

Appendix J Proof of Proposition 9

Following from (7a) and (7d), the static and dynamic profiles of the shadow price of man-made capital in the SO are characterized by $\lambda^S=\mathrm{U}_C(C^S,Z^S)$ and $\frac{\dot{\lambda}^S}{\lambda^S}=\rho-\mathrm{F}_K(K^S,E^S)$, where the superscript "S" correspond to allocations in the SO. To establish static consistency with (7b) characterizing resource extraction in the SO, i.e., $\frac{\mu^S-\psi^S}{\lambda^S}=\mathrm{F}_E-\mathrm{M}_E$, the shadow value of resource stocks μ^M should satisfy $\mu^M=\frac{\mu^S-\psi^S}{\lambda^S}$. Taking time derivatives yields,

$$\frac{\dot{\mu^M}}{\lambda^M}=\frac{\dot{\mu}^S-\dot{\psi}^S}{\mu^S-\psi^S}-\frac{\dot{\lambda}^S}{\lambda^S}=r+\frac{\mathrm{M}_X(E^S,X^S)+\frac{\mathrm{U}_Z(C^S,Z^S)}{\lambda^S}}{\mathrm{F}_E(K^S,E^S)-\mathrm{M}_E(E^S,X^S)}, \tag{J.1}$$

where $\dot{\lambda}^S=\rho\lambda^S-\lambda^S\mathrm{F}_K(K^S,E^S)$, $\dot{\mu}^S=\rho\mu^S+\lambda^S\mathrm{M}_X(E^S,X^S)$, and $\dot{\psi}^S=\rho\psi^S-\mathrm{U}_Z(C^S,Z^S)$ are used to derive (J.1). However, with $\frac{\mathrm{U}_Z(C^S,Z^S)}{\lambda^S}\neq 0$, (38b) and (J.1) imply that the ME can not achieve dynamic consistency with SO if the pollution externality is not considered.

Appendix K Proof of Proposition 10

To establish a consistency with (7b) characterizing the shadow price of resources in the SO, i.e., $\frac{\mu^S}{\lambda^S}=\mathrm{F}_E-\mathrm{M}_E+\frac{\psi^S}{\lambda^S}$, we need to have $\mu^M=\frac{\mu^S}{\lambda^S},\omega=\frac{-\psi^S}{\lambda^S}$. Taking time derivatives yields

$$\frac{\dot{\mu^M}}{\lambda^M}=\frac{\dot{\mu}^S}{\mu^S}-\frac{\dot{\lambda}^S}{\lambda^S}=r+\frac{\mathrm{M}_X(E^S,X^S)}{\mathrm{F}_E(K^S,E^S)-\mathrm{M}_E(E^S,X^S)+\frac{\psi^S}{\lambda^S}}, \tag{K.1}$$

where $\dot{\lambda}^S=\rho\lambda^S-\lambda^S \mathrm{F}_K(K^S,E^S)$, $\dot{\mu}^S=\rho\mu^S+\lambda^S \mathrm{M}_X(E^S,X^S)$ are used to derive (K.1). With $\omega=\frac{-\psi^S}{\lambda^S}$, (41b) and (K.1) imply that allocations in the ME can satisfy the dynamic consistency with SO in characterizing resource extraction. Furthermore, with the unit pricing of pollutants $\omega=\frac{-\psi^S}{\lambda^S}$, characterization of pollution abatement in the ME is given by $\mathrm{G}'(B^M)=\omega=\frac{-\psi^S}{\lambda^S}$, which is consistent with (7c) characterizing abatement in the SO.

Appendix L Proof of Proposition 11

From (7b) and (7e) resource extraction and the remaining stocks in the SO are given by

$$\frac{\mu^S-\psi^S}{\lambda^S}=\mathrm{F}_E(K^S,E^S)-\mathrm{M}_E(E^S,X^S), \tag{L.1a}$$

$$\frac{\dot{\mu}^S-\dot{\psi}^S}{\mu^S-\psi^S}-\frac{\dot{\lambda}^S}{\lambda^S}=r+\frac{\mathrm{M}_X(E^S,X^S)+\frac{\mathrm{U}_Z(C^S,Z^S)}{\lambda^S}}{\mathrm{F}_E(K^S,E^S)-\mathrm{M}_E(E^S,X^S)}. \tag{L.1b}$$

Comparing (44a) with (L.1a), a static consistency between ME and SO can be established if $\mu^M=\frac{\mu^S-\psi^S}{\lambda^S}$ holds. Furthermore, from (44b) and (L.1b), a dynamic version of $\mu^M=\frac{\mu^S-\psi^S}{\lambda^S}$, i.e., $\frac{\dot{\mu}^M}{\mu^M}=\frac{\dot{\mu}^S-\dot{\psi}^S}{\mu^S-\psi^S}-\frac{\dot{\lambda}^S}{\lambda^S}$ satisfies dynamic consistency between ME and SO if the unit price of pollutants ω satisfies $\bar{\omega}=-\frac{\mathrm{U}_Z}{\lambda}$. The shadow price of pollutants in the ME given in (45) is thus characterized by

$$\psi^M=-\mathrm{G}'(B^M),\quad \mathrm{F}_K(K^M,E^M)\psi^M-\dot{\psi}^M=\frac{\mathrm{U}_Z(C^M,Z^M)}{\lambda^M}. \tag{L.2}$$

In comparison, from (7f) the static and dynamic profiles of the shadow price of pollution stocks in the SO is characterized by[22]

$$\frac{\psi^S}{\lambda^S}=-\mathrm{G}'(B^S),\quad \mathrm{F}_K(K^S,E^S)\left(\frac{\psi^S}{\lambda^S}\right)-\left(\dot{\overline{\frac{\psi^S}{\lambda^S}}}\right)=\frac{\mathrm{U}_Z(C^S,Z^S)}{\lambda^S}. \tag{L.3}$$

Comparing (L.2) with (L.3), we find that by setting the ME shadow price of pollutants as the shadow price ratio between pollution stocks and man-made capital in the SO, i.e., $\psi^M=\frac{\psi^S}{\lambda^S}$, the ME can establish consistency with SO in characterizing pollution abatement and stocks.

[22]Rearranging $\mathrm{U}_Z=\rho\psi-\dot{\psi}$ given in (7f) yields $\frac{\mathrm{U}_Z}{\lambda}=\rho\left(\frac{\psi}{\lambda}\right)-\left(\dot{\overline{\frac{\psi}{\lambda}}}\right)-\left(\frac{\psi}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$, then substituting $\frac{\dot{\lambda}}{\lambda}=\rho-\mathrm{F}_K$ given in (7d) and simplifying yields (L.3).

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Table 1: Model specifications in selected natural resource and environmental economics literature

Reference	Production[1]	Utility[2]	Aggregate constraints[3]	Pollution emissions[4]	Pollution damages[5]	Man-made capital[6]	Resource reserves[7]	Pollutant stocks[8]
Bovenberg and Smulders (1995)	K,Q	C,Q	Y=C+I	P	U,Y	✓	✓	✗
Mohtadi (1996)	K,Q	C,Q	Y=C+I	K	U,Y	✓	✗	✓
Byrne (1997)	K,L,A	C,Q	Y=C	K,B	U	✗	✗	✓
Stokey (1998)	K	C,Q	Y=C+I	Y,A	U	✓	✗	✓
Jones and Manuelli (2001)	K,L	C,Q	Y=C+I	K,A	U	✓	✗	✗
van der Ploeg and Withagen (2001)	K	C,Q	Y=C+I+B	Y,B	U	✓	✗	✓
Bartz and Kelly (2008)	K,L	C,Q	Y=C+I+B	Y,B	U	✓	✗	✓
Smulders et al. (2012)	K	C,Q	Y=C+I+B	K,B	U,Y	✓	✗	✗
Tahvonen and Kuuluvainen (1991)	K,E,Q	C,Q	Y=C+I	P	U,X	✓	✓	✓
Schou (2000)	K,L,R,Q	C	Y=C+I	R	Y	✓	✓	✗
Chakravorty et al. (2008)	R	C	Y=C	R	Z	✗	✓	✓
Acemoglu et al. (2012)	L,R,A	C,Q	Y=C+I+M	Y	U	✗	✓	✓
Golosov et al. (2012)	K,L,R,Q	C	Y=C+I	R	Y	✓	✓	✓
van der Ploeg and Withagen (2014)	K,R	C,Q	Y=C+I+M	R	U	✓	✓	✓

Note:
1: Production inputs: physical capital “K”, labour “L”, resources “R”, environmental quality “Q”, and technology “A”
2: Preference: final goods consumption “C” and environmental quality “Q”
3: Final goods outputs “Y” allocated to consumption “C”, capital investment “I”, pollution abatement “B” and resource extraction “M”
4: Emissions as a function of production “Y”, capital “K”, technology “A”, abatement “B”, resource use ”R”, and pollution “P”
5: Pollution affects utility “U”, final goods production “Y”, resource regeneration “X”, and environmental regulation “Z”
6: “✓”: man-made capital as a stock variable; “✗”: man-made capital not specified or as a flow variable
7: “✓”: resource reserves as a stock variable; “✗”: resource reserve not specified or as a flow variable
8: “✓”: environmental pollutants as a stock variable; “✗”: environmental pollutants not specified or as a flow variable