

Relational Communication*

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Abstract

We study a communication game between an informed sender and an uninformed receiver with repeated interactions and voluntary transfers. Transfers motivate the receiver's decision-making and signal the sender's information. Although full separation can always be supported in equilibrium, partial or complete pooling is optimal if the receiver's decision-making is too responsive to information. In this case, the receiver's decision-making is disciplined by pooling extreme states, where she is most tempted to defect. In characterizing optimal equilibria, we establish new results on monotone persuasion.

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1 Introduction

Decision-makers and informed parties often develop relationships in which communication and decision-making are governed by informal agreements. We study how such interactions can be disciplined using relational contracts: discretionary compensation schemes that are self-enforcing in a repeated game. We characterize communication and decision-making patterns in optimal equilibria.

As an example of such relational communication, consider the interaction between lobbyists and politicians. Lobbyists seek to influence politicians' policy decisions.¹ They provide politicians with information about the electoral and economic consequences of various policy choices, such as focus group attitudes towards potential tobacco regulations, or the impact of cigarette smoking on health outcomes. Lobbyists also make transfers to politicians, in the form of political contributions. Such transfers serve as contingent contributions for favorable policy decisions (Grossman and Helpman 1994, 1996) and credible signals of lobbyists' information (Austen-Smith 1995 and Lohmann 1995).

While political contributions are legal in many countries, explicit payments for policy decisions usually constitute illegal bribery and political corruption. Consequently, agreements between politicians and lobbyists are largely implicit and supported by trust and reputation. Indeed, lobbyists often maintain long-standing relationships with politicians.

Another example of relational communication is decision-making in organizations, which are often governed by informal agreements — “firms are riddled with relational contracts” (Baker, Gibbons and Murphy 2002). Consider a subordinate who implements a project, and a superior who has relevant information. The superior may advise or even instruct the subordinate, but it is the subordinate who decides how to implement the project.² Besides giving advice, the superior often pays the subordinate to influence implementation. Payments may take the form of wages, bonuses, raises, and gifts. Payments may directly reward the subordinate for compliant implementation. Payments may also give credibility to the superior's advice — “the leader offers gifts to the followers ... because the leader's sacrifice convinces them that she must truly consider this to be a worthwhile activity” (Hermalin 1998).

Our analysis of relational communication is based on an infinitely-repeated communication game, played by a sender and a receiver who can make voluntary transfers to each other at any point in the game. In each period, the sender privately observes an

¹See Grossman and Helpman (2001) and Persson and Tabellini (2002) for reviews.

²Similar to us, Landier, Sraer and Thesmar (2009) and Van den Steen (2010) consider situations with a subordinate as a decision-maker implementing a project and a superior as an informed party giving advice. See Section 3.4 of Gibbons, Matouschek and Roberts (2013) for a review.

independent draw of the state and sends a message to the receiver, who then makes a decision. The players' preferred decisions are increasing in the state, but the magnitude and sign of the sender's bias may depend on the state.³

In relational communication, transfers allow the sender not only to reward the receiver for compliant decision-making, but also to credibly signal his private information. In particular, full separation can be supported in equilibrium, even when the players are impatient. Therefore, the essential incentive constraint is that the receiver is tempted to make decisions that benefit herself but hurt the sender.

We show that a message rule can be supported in equilibrium if and only if it is *monotone*: it induces a monotone partition of the set of states. In any (Pareto) optimal equilibrium, the decision rule simply maximizes, subject to the receiver's incentive constraint, the joint payoff for each message. Therefore, given this decision rule, the optimal message rule solves the monotone persuasion problem: it maximizes the expected joint payoff over all monotone message rules. We establish new results on optimal monotone persuasion and discuss how the monotonicity restriction affects the optimal message rule.

We completely characterize the optimal (second-best) equilibrium when the players' payoffs are quadratic. Our key insight is that the optimal equilibrium may involve (partial or complete) pooling of information. Pooling is optimal only if the receiver is too responsive to information (the slope of the receiver's preferred decision with respect to the state is sufficiently large relative to that of the sender). In this case, pooling occurs at states of extreme conflict (where the sender's bias is sufficiently large).

Intuitively, the receiver's incentive constraint requires that any feasible decision be close to the receiver's preferred decision. First-best decision-making cannot be supported for extreme states. Further, if the receiver is too responsive, then second-best decision-making is too responsive to information about extreme states relative to the first-best; so extreme states are optimally pooled.

The result that the sender reveals (hides) information when conflict of interest is moderate (extreme) seems to be a natural pattern of communication in relationships. Lobbyists often discuss in detail the costs and benefits of potential legislation with politicians, but may hide their private information in cases that are particularly controversial or consequential. For example, the tobacco lobby concealed and distorted evidence from internal studies that cigarettes caused lung cancer ([Hilts 1994](#) and [Harris 2008](#)), to soften regulation of tobacco products by Congress. In organizations, superiors provide honest

³We also argue that the main insights from our analysis apply in more general settings. Specifically, in Section 7, we discuss the following extensions: exogenous outside options for the players; imperfect monitoring of the receiver's decision; and correlation of states across time.

advice and subordinates comply when their preferences are largely aligned, but superiors may hide information when subordinates are most tempted to dissent or disobey.

An implication of our analysis is that in settings where voluntary transfers are available, incomplete information transmission does not imply a failure to motivate communication, but instead is a tool to discipline decision-making. In other words, the Pareto frontier cannot be expanded simply by introducing a technology for credible (monotone) communication.⁴ This point provides a rationale for the separation of information and control in organizations. Indeed, we show that increasing organizational transparency and delegating the decision right to an informed player generally decreases the efficiency of informal relationships.

1.1 Related Literature

Our analysis builds on an extensive literature on repeated interactions with transfers. The seminal papers by Bull (1987) and Macleod and Malcomson (1989) focus on settings with symmetric information. Levin (2003) characterizes the optimal relational contract in two important settings with asymmetric information: adverse selection and moral hazard. In these settings, only the decision-maker (agent) has private information, so there is no role for information transmission between the principal and agent. In contrast, our setting involves an informed sender and an uninformed decision-maker (receiver), in the vein of Crawford and Sobel (1982). In such relational communication, pooling serves to affect the receiver's beliefs and thus directly improves her decision-making. In contrast, the decision-maker (agent) in Levin (2003) is fully informed, so pooling has no such effect.

Alonso and Matouschek (2007) also consider repeated communication. In contrast to us, they disallow transfers and consider a sequence of short-lived senders rather than a single long-lived sender.⁵ In their setting, repeated interaction disciplines decision-making, in order to sustain more informative communication. In contrast, in our setting, credible communication is easy to achieve; so repeated interaction improves decision-making which in turn determines the informativeness of optimal communication.

In our model, transfers from the sender to the receiver are used to signal information.⁶ Austen-Smith and Banks (2000) and Kartik (2007) consider a related (albeit static)

⁴This is in contrast with the existing literature on cheap talk and delegation, where the receiver's expected payoff (which is the standard welfare criterion) unambiguously improves if credible communication can be costlessly achieved.

⁵Baker, Gibbons and Murphy (2011) consider a model of repeated decision-making with transfers between long-lived players, but assume symmetric information, so communication plays no role.

⁶Ottaviani (2000), Krishna and Morgan (2008), and Ambrus and Egorov (2017) consider communication games with contractible transfers (in contrast to the voluntary transfers in our setting). In their set-

setting where the sender burns money to signal information.⁷ Unlike burning money, signaling information with transfers incurs no welfare cost. This leads to a clean characterization of the set of optimal equilibria; in particular, all optimal equilibria in our model produce identical communication outcomes.⁸ As a byproduct, we establish a general characterization of equilibria in games of cheap talk with burned money: a message rule is implementable if and only if it is monotone.

In our model, optimal equilibria are supported by carrot-and-stick strategies (Abreu 1986 and Goldlücke and Kranz 2012), in which a deviator is punished as harshly as possible but only for a single period. We show that the receiver is punished by complete pooling of information and the sender is punished by an extreme incentive compatible decision. These punishments also characterize the receiver’s and sender’s worst equilibria in games of cheap talk with burned money.

We establish equivalence between optimal relational communication and monotone persuasion. This is a Bayesian persuasion problem (Rayo and Segal 2010 and Kamenica and Gentzkow 2011) with a restriction to monotone message rules.⁹ Dworzak and Martini (2018), Kolotilin (2018), and Mensch (2018) derive conditions under which a monotone message rule is optimal among all (possibly nonmonotone) message rules. Kolotilin and Zapechelnyuk (2018) show that the monotone persuasion problem is equivalent to the delegation problem. In contrast to these papers, we explicitly characterize the optimal monotone message rule when it differs from the optimal (nonmonotone) message rule. Our approach relies on the representation of message rules as convex functions, as in Gentzkow and Kamenica (2016) and Kolotilin et al. (2017).

Our paper also contributes to the rapidly growing literature on Bayesian persuasion with transferable utility (Bergemann and Pesendorfer 2007, Eső and Szentes 2007, Li and Shi 2017, Bergemann, Bonatti and Smolin 2018, and Dworzak 2017). Similarly to these papers, we use tools from mechanism design and Bayesian persuasion. Unlike these papers, commitment power in our model is endogenous and thus imperfect.

tings, transfers are from the receiver to the sender, and thus cannot be used to signal information. Bester and Krämer (2017) consider a related setting with contractible transfer schemes and study the optimal allocation of authority, similar to Dessein (2002).

⁷Kartik, Ottaviani and Squintani (2007) and Kartik (2009) consider related models with lying costs instead of money burning.

⁸In the setting with burned money, equilibrium communication outcomes differ along the Pareto frontier because there is a tradeoff between the informativeness of communication and the costs of burning money. The receiver’s optimal equilibrium clearly involves full separation; Karamychev and Visser (2017) characterize the sender’s optimal equilibrium.

⁹A model of repeated (monotone) persuasion would reproduce many of the insights from our model of relational communication. The existing literature has studied dynamic Bayesian persuasion, albeit with persistent information (Kremer, Mansour and Perry 2014, Au 2015, Ely, Frankel and Kamenica 2015, Hörner and Skrzypacz 2016, Ely 2017, Che and Horner 2018, and Orlov, Skrzypacz and Zryumov 2018).

2 Model

2.1 Setup

A sender (S) and a receiver (R) play an infinitely repeated communication game with perfect monitoring and with voluntary transfers. Time is discrete and the players have a common discount factor $\delta \in [0, 1)$. In each period, the same stage game is played. The sender privately observes a state $\theta \in [0, 1]$ and sends a message $m \subset [0, 1]$ to the receiver, who then makes a decision $d \in \mathbb{R}$. The state θ is independently drawn each period from a prior distribution $F(\theta)$ with a strictly positive density $f(\theta)$ for all $\theta \in [0, 1]$. The sender's and receiver's payoffs are functions $u_S(d, \theta)$ and $u_R(d, \theta)$ that satisfy Crawford and Sobel (1982)'s assumptions:

Assumption 1. For each player $i \in \{S, R\}$,

1. $u_i(d, \theta)$ is twice differentiable in d and θ for all $d \in \mathbb{R}$ and $\theta \in [0, 1]$,
2. $\frac{\partial^2 u_i}{\partial d^2}(d, \theta) < 0$ for all $d \in \mathbb{R}$ and $\theta \in [0, 1]$,
3. $\frac{\partial u_i}{\partial d}(\rho_i(\theta), \theta) = 0$ for some function $\rho_i(\theta)$ and for all $\theta \in [0, 1]$,
4. $\frac{\partial^2 u_i}{\partial d \partial \theta}(d, \theta) > 0$ for all $d \in \mathbb{R}$ and $\theta \in [0, 1]$.

Parts 2 and 3 of Assumption 1 require that each player's payoff is strictly concave in the decision and each player $i \in \{S, R\}$ has a unique preferred decision $\rho_i(\theta)$ for each state $\theta \in [0, 1]$. Similarly, there is a unique first-best decision $\rho_{FB}(\theta)$ that maximizes the joint payoff $u(d, \theta) = u_S(d, \theta) + u_R(d, \theta)$. Part 4 is a sorting condition that ensures that $\rho_S(\theta)$, $\rho_R(\theta)$, and $\rho_{FB}(\theta)$ are strictly increasing in θ .

The players can make voluntary (non-contractible) transfers at any point in the game. Specifically, we enrich the stage game with three rounds of transfers: (i) an *ex-ante* round before the sender observes the state, (ii) an *interim* round after the sender observes the state and sends the message but before the receiver chooses a decision, and (iii) an *ex-post* round after the decision is chosen and the state is publicly observed. In each round, transfers are made sequentially, first by the sender and then by the receiver. Each player chooses a non-negative gross transfer to the other player and a non-negative amount of money to burn. The players' transfer choices in each round determine their net transfers in that round. Specifically, the sender's net transfer equals his gross transfer, minus the receiver's gross transfer, plus the sender's money burned (and similarly for the receiver). The net transfers by player $i \in \{S, R\}$ in the ex-ante, interim, and ex-post rounds are denoted by τ_i , t_i , and T_i ; so the stage game payoff of player i is $u_i(d, \theta) - \tau_i - t_i - T_i$. Note

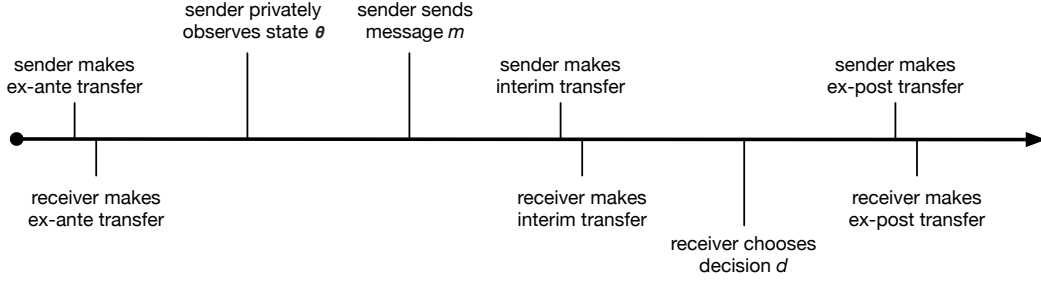


Figure 1: Timing of stage game

that net transfers in each round must satisfy $\tau_S + \tau_R \geq 0$, $t_S + t_R \geq 0$, and $T_S + T_R \geq 0$, with strict inequality in the case of burned money.¹⁰ Although we allow for both ex-ante and ex-post transfers, ex-ante transfers can substitute for ex-post transfers (and vice versa).¹¹

The game has perfect monitoring in that all actions (message, decision, and transfers) are immediately publicly observed, but the state is only observed by the sender. That is, the receiver never observes the state or her payoff.¹² Figure 1 summarizes the timing of each stage game.

We study pure-strategy perfect Bayesian *equilibria*. For each period and each history, an equilibrium specifies a message rule $\mu(\theta)$ for the sender, a decision rule $\rho(m)$ for the receiver, and transfer rules $\tau_i, t_i(m), T_i(m)$ for each player $i \in \{S, R\}$.¹³

A (pure-strategy) message rule deterministically maps states to the messages they induce. Without loss of generality, we identify each message with the set of states that induce this message, $m = \{\theta : \mu(\theta) = m\}$. Thus, the range $\mu([0, 1])$ of a message rule μ is a partition of the set of states. A message rule μ is *monotone* if each $m \in \mu([0, 1])$ is a convex set (either a singleton or an interval).

We can now extend the definition of payoffs and preferred decisions from being state dependent to being message dependent. Specifically, $u_i(d, m) = \mathbb{E}_F[u_i(d, \theta) | m]$ and $\rho_i(m) = \arg \max_{d \in \mathbb{R}} u_i(d, m)$ for each player $i \in \{S, R\}$. Similarly, $u(d, m) = u_S(d, m) + u_R(d, m)$ and $\rho_{FB}(m) = \arg \max_{d \in \mathbb{R}} u(d, m)$. Assumption 1 ensures that $\rho_S(m)$, $\rho_R(m)$, and $\rho_{FB}(m)$ are well defined and are strictly increasing in m in the strong set order.

¹⁰Conversely, for any net transfers that satisfy these three constraints, we can construct gross transfers and burned money amounts that correspond to these net transfers.

¹¹Thus we may, for example, restrict attention to equilibria where the ex-ante transfers (τ_S, τ_R) are zero in every period except the first period. In this case, we may think of the first-period ex-ante transfers as ‘up-front’ payments that determine the division of surplus in the relationship.

¹²This assumption is common in the literature on repeated games with incomplete information (Aumann, Maschler and Stearns 1995), and is ubiquitous in models of repeated communication (Renault, Solan and Vieille 2013, Frankel 2016, Margaria and Smolin 2018, and Lipnowski and Ramos 2018).

¹³The functions μ, ρ, t_i , and T_i are required to be measurable.

2.2 Stationarity

An equilibrium is *stationary* if on the equilibrium path, the message rule μ , the decision rule ρ , and the transfer rules τ_i , t_i , and T_i for $i \in \{S, R\}$ are identical in every period. An equilibrium is *optimal* if it is not Pareto dominated by any other equilibrium. An equilibrium is *sequentially optimal* if the continuation equilibrium following any history on the equilibrium path is optimal.

Lemma 1. *There exist \underline{v}_S , \underline{v}_R , and \bar{v} such that the set of equilibrium payoffs $V \subset \mathbb{R}^2$ is a simplex of the form*

$$V = \{(v_S, v_R) : v_S \geq \underline{v}_S, v_R \geq \underline{v}_R, v_S + v_R \leq \bar{v}\}.$$

Any optimal equilibrium is sequentially optimal and involves no burned money. Further, there exists a stationary optimal equilibrium σ_ such that any $(v_S, v_R) \in V$ can be supported by an equilibrium that differs from σ_* only in the first-period ex-ante transfers.*

Lemma 1 extends some of [Levin \(2003\)](#)'s and [Goldlücke and Kranz \(2012\)](#)'s results to our setting. Because players' payoffs are quasi-linear in money, payoffs are fully transferable, and contingent transfers can substitute for contingent continuation payoffs. Consequently, we can restrict attention to stationary equilibria, and all optimal equilibria induce the message and decision rules that maximize joint payoff $v = v_S + v_R$. Further, due to free disposal (both players can burn money), the set of equilibrium payoffs is a simplex.

Optimal equilibria do not involve burned money, because burning money would only tighten incentive constraints and reduce the joint payoff. Therefore, the Pareto frontier would not change if we modified the model by disallowing money burning.

3 Equilibrium

3.1 Implementability

Define the receiver's temptation to deviate from decision d given message m as

$$w(d, m) = u_R(\rho_R(m), m) - u_R(d, m),$$

and the (net) *discounted surplus* given joint payoff v as

$$L(v) = \frac{\delta}{1 - \delta}(v - \underline{v}_S - \underline{v}_R). \quad (1)$$

Proposition 1. *A message rule μ and a decision rule ρ that produce a joint payoff v can be supported in a stationary equilibrium if and only if*

$$\rho(\mu(\theta)) \text{ is nondecreasing in } \theta, \quad (2)$$

$$w(\rho(m), m) \leq L(v) \text{ for all } m \in \mu([0, 1]). \quad (3)$$

We first argue that (2) and (3) are necessary. In any equilibrium, the message rule $\mu(\theta)$ must be incentive compatible for the sender. Since the sender's payoff is quasi-linear in money and satisfies a sorting condition, a standard characterization of incentive compatibility in mechanism design (see, for example, [Rochet 1987](#)) implies that $\rho(\mu(\theta))$ must be nondecreasing in θ .

Also, in any equilibrium, the decision rule ρ must be incentive compatible for the receiver. Therefore, given a message m , the receiver's one-period payoff gain from choosing her preferred decision $\rho_R(m)$ instead of equilibrium decision $\rho(m)$ must be less than the maximum available punishment equal to the discounted surplus.

We now argue that (2) and (3) are sufficient. Ignoring the sender's incentive compatibility constraint, any decision rule ρ that satisfies (3) can be made incentive compatible for the receiver by giving all surplus to the receiver ($v_R = v - \underline{v}_S$) and threatening her with her worst equilibrium payoff ($v_R = \underline{v}_R$) following any deviation from $\rho(m)$.

In such a construction, the sender receives his worst equilibrium payoff \underline{v}_S and thus cannot be punished for deviating. But for any message rule μ that satisfies (2), we can separately construct a (voluntary) interim transfer rule that makes μ incentive compatible for the sender.

The revenue equivalence theorem (see, for example, [Milgrom and Segal 2002](#)) implies that there exists a unique (up to a constant C) interim transfer rule t_S such that the sender prefers to induce $\rho(\mu(\theta))$ and pay $t_S(\mu(\theta))$ rather than to induce $\rho(\mu(\hat{\theta}))$ and pay $t_S(\mu(\hat{\theta}))$ for all $\hat{\theta} \neq \theta$,

$$t_S(m) = u_S(\rho(m), \theta(m)) - \int_0^{\theta(m)} \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} + C, \quad (4)$$

where $\theta(m)$ is an arbitrary state $\theta \in m$.¹⁴ The constant C can be chosen in such a way that the sender does not want to deviate to any out-of-equilibrium message-transfer pair (\hat{m}, \hat{t}_S) . Specifically, choose C such that the minimum transfer is equal to zero and is

¹⁴Since $\rho(\mu(\theta))$ is nondecreasing in θ , $t_S(m)$ is independent of the choice of a representative state $\theta \in m$.

achieved for some punishment message m^p ,¹⁵

$$t_S(m) \geq 0 \text{ for all } m \in \mu([0, 1]), \text{ with equality for some } m^p \in \mu([0, 1]). \quad (5)$$

If following any out-of-equilibrium pair (\hat{m}, \hat{t}_S) , the receiver believes that the state is in m^p and chooses the punishment decision $d^p = \rho(m^p)$, then the sender prefers to report m^p and pay $t_S(m^p) = 0$ rather than to report \hat{m} and pay \hat{t}_S . Thus, the sender's incentive compatibility constraint is satisfied.

This argument implies that voluntary interim transfers are powerful in signaling information, even if the players are myopic.¹⁶

Corollary 1. *Suppose $\delta = 0$. A message rule μ and a decision rule ρ can be supported in an equilibrium if and only if μ is monotone and $\rho(m) = \rho_R(m)$ for all $m \in \mu([0, 1])$.*

Corollary 1 is closely connected to existing results from the literature on cheap talk and burned money (Austen-Smith and Banks 2000, Kartik 2007, and Karamychev and Visser 2017). In the myopic setting, interim transfers serve the same signaling role as burned money. In fact, the set of implementable message and decision rules does not depend on whether the sender transfers money to the receiver ($t_R = -t_S$) or whether the sender burns money ($t_R = 0$).¹⁷

In contrast to burned money, interim transfers are not wasteful: the sender's loss is the receiver's gain. Further, since ex-ante transfers are available, the use of interim transfers does not create a distributional imbalance. Any surplus obtained by the receiver from interim transfers can be redistributed to the sender using ex-ante transfers. Such ex-ante transfers can be supported by the threat of playing a babbling equilibrium. Consequently, as we show next, the sender can effectively commit at no welfare cost to any monotone message rule.

¹⁵In the proof, we allow for the possibility that $\inf t_S(m)$ is not attained by any m^p .

¹⁶Although interim transfers are powerful, messages are still used to convey information. For example, suppose the players' preferred decision rules intersect at some state. Then in any fully separating equilibrium, the interim transfer function is non-monotone and takes the same value for multiple state realizations. Messages are thus used to distinguish between these realizations.

¹⁷Karamychev and Visser (2017)'s Proposition 1 characterizes implementable outcomes with money burning. Our mechanism design approach to characterization provides a much simpler proof of the result and removes the assumptions that the sender's bias has constant sign and that the receiver's payoff satisfies a sorting condition. Indeed, if the receiver's payoff did not satisfy part 4 of Assumption 1, our Proposition 1 and its proof would still hold, but Corollary 1 would require that $\rho_R(\mu(\theta))$ is nondecreasing in θ , rather than that μ is monotone.

3.2 Optimality

Define the second-best decision given message m as

$$\begin{aligned} \rho_*(m) &= \arg \max_d u(d, m) \\ &\text{subject to } w(d, m) \leq L(\bar{v}), \end{aligned} \tag{6}$$

and the joint payoff under the second-best decision as

$$u_*(m) = u(\rho_*(m), m) \text{ for all } m \in [0, 1]. \tag{7}$$

Proposition 2. *In an optimal equilibrium, the message rule is*

$$\begin{aligned} \mu_* &\in \arg \max_{\mu} \mathbb{E}[u_*(\mu(\theta))] \\ &\text{subject to } \mu \text{ is monotone,} \end{aligned} \tag{8}$$

and the decision rule is $\rho_*(m)$ for all $m \in \mu_*([0, 1])$.

The intuition for Proposition 2 is as follows. By Proposition 1, an optimal equilibrium maximizes v jointly over message and decision rules that satisfy (2) and (3). The constraint (2) implies that we can restrict attention to monotone message rules. Consider a relaxed problem in which the constraint (2) is replaced with the restriction that the message rule is monotone. It is easy to see that ρ_* given by (6) and μ_* given by (8) solve this relaxed problem. Further, we show that $\rho_*(m)$ is nondecreasing in m because the sender's and receiver's payoffs satisfy the sorting condition (part 4 of Assumption 1). Therefore, $\rho_*(\mu_*(\theta))$ is nondecreasing in θ , the constraint (2) is automatically satisfied, and ρ_* and μ_* constitute an optimal equilibrium.

Proposition 2 shows that the decision rule and message rule in any optimal equilibrium can be calculated in two steps. First, the decision rule is characterized without reference to the message rule. The decision rule is point-wise equal to the second-best decision $\rho_*(m)$ given by (6). For each message m , the second-best decision $\rho_*(m)$ can be found as follows. If $d = \rho_{FB}(m)$ satisfies the constraint of (6), then $\rho_*(m) = \rho_{FB}(m)$. Otherwise $\rho_*(m)$ is such that $d = \rho_*(m)$ satisfies the constraint of (6) with equality. Second, given ρ_* and thus u_* , the message rule μ_* solves the *monotone persuasion* problem (8): it maximizes the expected joint payoff $\mathbb{E}[u_*(\mu(\theta))]$ over all monotone message rules μ .

3.3 Punishment

Consider strategy profiles where the ex-post transfers are zero ($T_S = T_R = 0$) and money is never burned ($\tau_S + \tau_R = 0$ and $t_S + t_R = 0$). Denote the sender's ex-ante transfer τ_S by τ and the sender's interim transfer rule t_S by t (correspondingly, $\tau_R = -\tau$ and $t_R = -t$).

A *single-period punishment* strategy profile is characterized by: normal as well as penal ex-ante transfers, $\underline{\tau}_0$ as well as $\underline{\tau}_S$ and $\underline{\tau}_R$; normal as well as penal message rules, $\underline{\mu}_0$ as well as $\underline{\mu}_S$ and $\underline{\mu}_R$; normal as well as penal decision rules, $\underline{\rho}_0$ as well as $\underline{\rho}_S$ and $\underline{\rho}_R$.

Play proceeds as follows. The ex-ante transfer is $\underline{\tau}_i$ if player $i \in \{0, S, R\}$ deviated last in the previous period, where $i = 0$ denotes that no player deviated. The message rule, interim transfer rule, decision rule, and punishment message are $\underline{\mu}_j$, \underline{t}_j , $\underline{\rho}_j$, and \underline{m}_j^p if player $j \in \{0, S, R\}$ deviated from the ex-ante transfer in this period, where \underline{t}_j and \underline{m}_j^p are defined by (4) and (5) given $\underline{\mu}_j$ and $\underline{\rho}_j$. The punishment decision is $\underline{d}_j^p = \underline{\rho}_j(\underline{m}_j^p)$ if the sender deviated to some $(\hat{m}, \hat{t}) \notin (\underline{\mu}_j, \underline{t}_j)([0, 1])$ in this period.

Proposition 3. *There exists an optimal equilibrium in single-period punishment strategies where*

1. $\underline{\mu}_0 = \mu_*$, $\underline{\rho}_0 = \rho_*$, and $\bar{v} = \mathbb{E}[u_*(\mu_*(\theta))]$;
2. $\underline{\mu}_R = [0, 1]$, $\underline{\rho}_R = \rho_R$, and $\underline{v}_R = u_R(\rho_R([0, 1]), [0, 1])$;
3. $\underline{\mu}_S$ and $\underline{\rho}_S$ solve

$$\begin{aligned} \underline{v}_S = \min_{\mu, \rho, \theta^p} & \left\{ u_S(\rho(m^p), \theta^p) + \mathbb{E} \left[\int_{\theta^p}^{\theta} \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \right] \right\} \\ & \text{subject to } \rho(\mu(\theta)) \text{ is nondecreasing in } \theta, \\ \rho(m) & \begin{cases} = \rho_-(m), & \text{if } m > m^p, \\ \in \{\rho_-(m), \rho_+(m)\}, & \text{if } m = m^p, \\ = \rho_+(m), & \text{if } m < m^p, \end{cases} \end{aligned} \quad (9)$$

where $m^p = \mu(\theta^p)$ and $[\rho_-(m), \rho_+(m)] = \{d : w(d, m) \leq L(\bar{v})\}$ for all $m \in \mu([0, 1])$.

Proposition 3 specifies optimal punishments for the receiver and the sender: a deviator is punished as harshly as possible for a single period, and then optimal play resumes. The deviator's worst equilibrium payoff equals his or her payoff in the punishment period. Following a deviation from τ by the receiver, the message rule is completely uninformative and no transfers are made. Following a deviation from τ by the sender, the receiver makes either the highest or lowest incentive compatible decision, and the message and interim transfer rules are chosen to minimize the sender's expected payoff.

4 Monotone Persuasion

The monotone persuasion problem (8) is of independent interest to the persuasion literature set in motion by [Kamenica and Gentzkow \(2011\)](#). For generality, we treat u_* as a primitive rather than being given by (7). Assume that $u_*(m)$ depends on a message m only through the induced posterior mean state $\mathbb{E}[\theta|m]$:

Assumption 2. $u_*(m) = u_*(\mathbb{E}[\theta|m])$ for all $m \subset [0, 1]$.

Therefore, without loss of generality, we identify each message m with the induced posterior mean state, $m = \mathbb{E}[\theta|m]$. This simplifies the previous convention that identified each message with the set of states that induce it.

To solve the monotone persuasion problem (8), it is convenient to define the pooling set $P \subset [0, 1]$ of a monotone message rule μ as the set of states that are not separated by μ . Since the prior distribution $F(\theta)$ has a density, without loss of generality, each message $m \in \mu([0, 1])$ of a monotone message rule μ is either a singleton or an open interval. Thus, each open pooling set uniquely determines a corresponding monotone message rule. This pooling set is a union of some set of disjoint open intervals, $P = \bigcup_i (\xi_i, \zeta_i)$.¹⁸ The distribution F of states induces a distribution G_P of posterior mean states given by

$$G_P(\theta) = \begin{cases} F(\theta), & \text{if } \theta \notin (\xi_i, \zeta_i) \text{ for all } i, \\ F(\xi_i), & \text{if } \theta \in (\xi_i, \mathbb{E}[\theta|(\xi_i, \zeta_i)]) \text{ for some } i, \\ F(\zeta_i), & \text{if } \theta \in [\mathbb{E}[\theta|(\xi_i, \zeta_i)], \zeta_i) \text{ for some } i, \end{cases} \quad (10)$$

and the expected payoff may be written as $\mathbb{E}[u_*(\mu(\theta))] = \int_0^1 u_*(\theta) dG_P(\theta)$. Solving the monotone persuasion problem (8) is thus equivalent to finding the optimal pooling set P_* that maximizes $\int_0^1 u_*(\theta) dG_P(\theta)$. Define the integral of G_P as

$$\Gamma_P(\theta) = \int_0^\theta G_P(\tilde{\theta}) d\tilde{\theta} \text{ for all } \theta \in [0, 1]. \quad (11)$$

Notice that each such function Γ_P uniquely determines a corresponding open set P .

Assume that $u_*(\theta)$ is continuously differentiable in θ for all $\theta \in [0, 1]$ and is twice continuously differentiable in θ for almost all $\theta \in [0, 1]$.

¹⁸We define open sets in $[0, 1]$ rather than in \mathbb{R} ; so $[0, 1/2) \cup (1/2, 1]$ is also an open subset of $[0, 1]$.

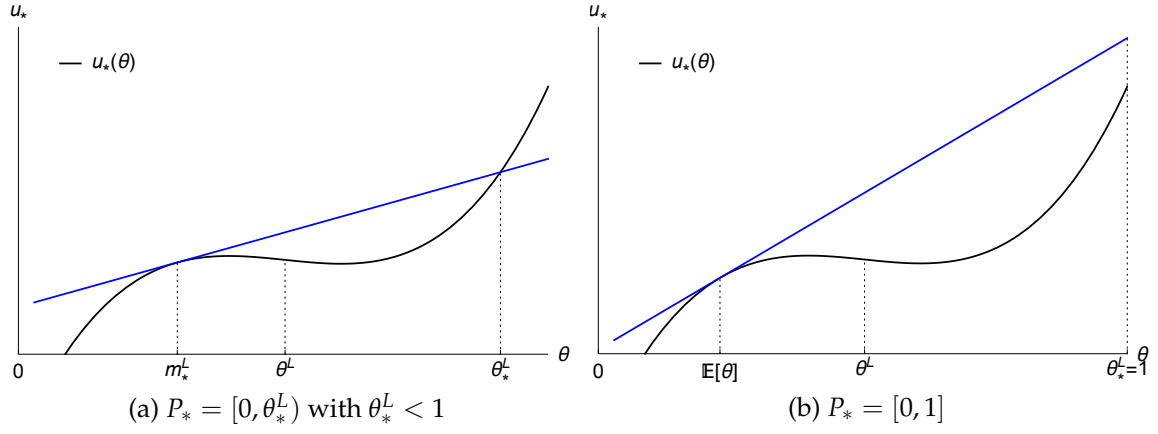


Figure 2: P_* when $u_*''(\theta)$ changes sign once

Lemma 2. *The optimal pooling set is*

$$P_* \in \arg \max_P \int_0^1 u_*''(\theta) \Gamma_P(\theta) d\theta \quad (12)$$

subject to P is an open subset of $[0, 1]$.

As (12) suggests, the optimal pooling set P_* should be chosen to make $\Gamma_P(\theta)$ large (small) at states θ where $u_*''(\theta)$ is positive (negative). Separating (pooling) state θ increases (decreases) $\Gamma_P(\theta)$, so full separation is optimal ($P_* = \emptyset$) if and only if $u_*(\theta)$ is convex in θ (see Lemmas 4 and 5 in Appendix C).

We next characterize P_* when $u_*''(\theta)$ changes sign once (see Figure 2).

Proposition 4. *Suppose there exists $\theta^L \in (0, 1)$ such that*

$$u_*''(\theta) \begin{cases} < 0, & \text{if } \theta \in [0, \theta^L), \\ > 0, & \text{if } \theta \in (\theta^L, 1]. \end{cases} \quad (13)$$

If there exists $\theta_^L \in (\theta^L, 1)$ such that*

$$u_*(m_*^L) + u_*'(m_*^L)(\theta_*^L - m_*^L) = u_*(\theta_*^L), \quad (14)$$

where $m_^L = \mathbb{E}[\theta | [0, \theta_*^L]]$,*

then $P_ = [0, \theta_*^L]$ and $m_*^L \in (0, \theta_L)$. Else, $P_* = [0, 1]$ and $m_*^L = \mathbb{E}[\theta] \in (0, \theta_L)$.*

If (13) holds, the optimal pooling set is an interval $[0, \theta_*^L]$. Figures 2a and 2b respectively illustrate when incomplete pooling ($\theta_*^L < 1$) and complete pooling ($\theta_*^L = 1$) are

optimal. Complete pooling is optimal whenever the prior distribution F puts sufficient weight on low states.

Further, if (13) holds, then the optimal unrestricted message rule that solves the unrestricted persuasion problem is monotone, as shown by Kolotilin (2018). Thus, Proposition 4 also characterizes the optimal unrestricted message rule.

We next characterize P_* when $u''_*(\theta)$ changes sign twice (see Figure 3).

Proposition 5. *Suppose that for some $\theta^L, \theta^H \in (0, 1)$ such that $\theta^L < \theta^H$,*

$$u''(\theta) \begin{cases} < 0, & \text{if } \theta \in [0, \theta^L), \\ > 0, & \text{if } \theta \in (\theta^L, \theta^H), \\ < 0, & \text{if } \theta \in (\theta^H, 1]. \end{cases} \quad (15)$$

1. *If there exist $\theta_*^L, \theta_*^H \in (\theta^L, \theta^H)$ such that $\theta_*^L < \theta_*^H$ and*

$$u_*(m_*^L) + u'_*(m_*^L)(\theta_*^L - m_*^L) = u_*(\theta_*^L), \quad (16)$$

$$u_*(m_*^H) + u'_*(m_*^H)(\theta_*^H - m_*^H) = u_*(\theta_*^H), \quad (17)$$

where $m_^L = \mathbb{E}[\theta | [0, \theta_*^L]]$ and $m_*^H = \mathbb{E}[\theta | (\theta_*^H, 1]]$,*

then $P_ = [0, \theta_*^L) \cup (\theta_*^H, 1]$. Also, $m_*^L \in (0, \theta_L)$ and $m_*^H \in (\theta_H, 1)$.*

2. *Else if there exists $\theta_*^M \in (0, 1)$ such that*

$$u_*(m_*^L) + u'_*(m_*^L)(\theta_*^M - m_*^L) = u_*(m_*^H) + u'_*(m_*^H)(\theta_*^M - m_*^H), \quad (18)$$

$$u_*(m_*^L)F(\theta_*^M) + u_*(m_*^H)(1 - F(\theta_*^M)) \geq u_*(\mathbb{E}[\theta]), \quad (19)$$

where $m_^L = \mathbb{E}[\theta | [0, \theta_*^M]]$ and $m_*^H = \mathbb{E}[\theta | (\theta_*^M, 1]]$,*

then $P_ = [0, \theta_*^M) \cup (\theta_*^M, 1]$ for some θ_*^M that satisfies (18) and (19). Also, $m_*^L \in (0, \theta_L)$ and $m_*^H \in (\theta_H, 1)$.*

3. *Else, $P_* = [0, 1]$.*

If (15) holds, the optimal pooling set takes one of three forms: (i) pooling of low states, separation of intermediate states, and pooling of high states (Figure 3a); (ii) pooling of low states and pooling of high states (Figures 3b and 3c); and (iii) pooling of all states (Figure 3d). Moving along Figures 3a \rightarrow 3b \rightarrow 3c \rightarrow 3d, the prior distribution F puts increasingly more weight on low and high states (and less weight on intermediate states).

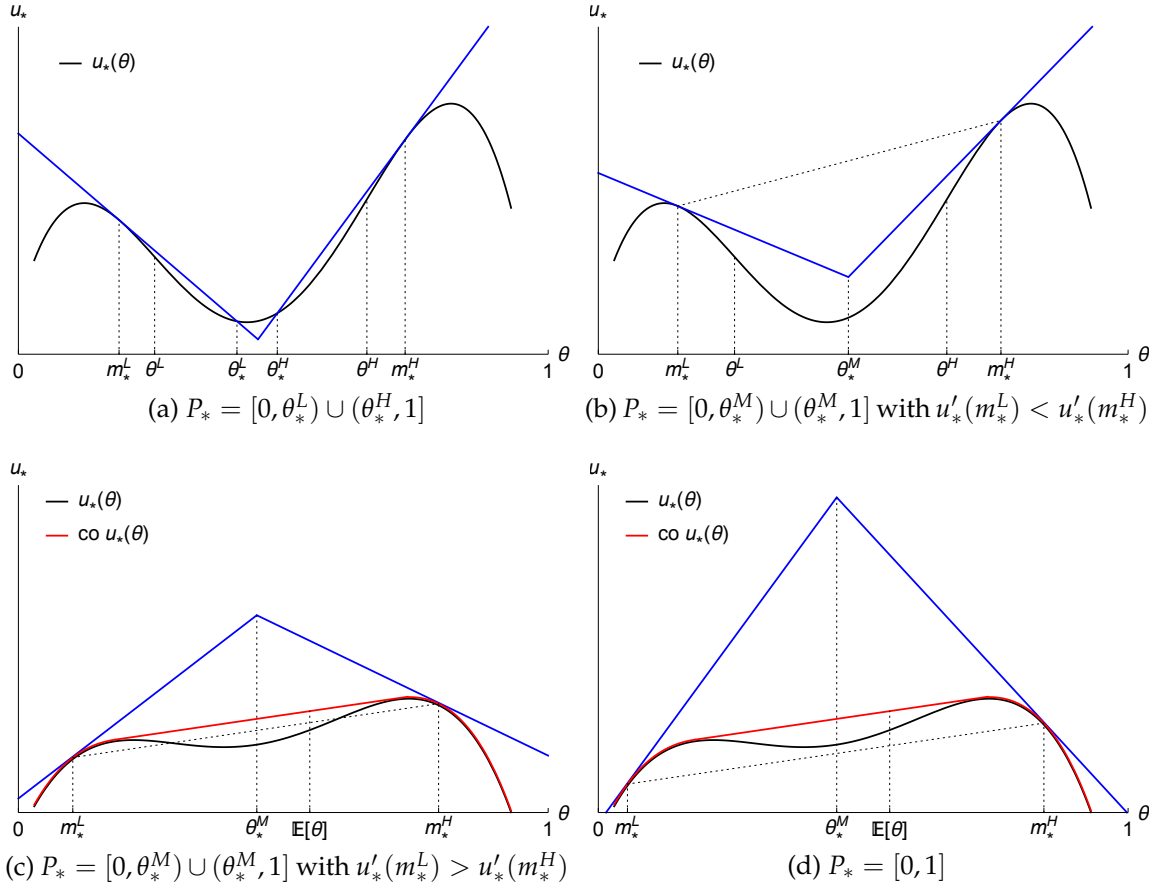


Figure 3: P_* when $u''_*(\theta)$ changes sign twice

Further, if (15) holds, then there exists a prior distribution F such that the optimal unrestricted message rule is nonmonotone, as shown by Dworczak and Martini (2018). Thus, in this case, the existing approaches to the unrestricted persuasion problem do not address the monotone persuasion problem, but Proposition 5 does.

We now compare the optimal monotone and unrestricted message rules, characterized respectively by Proposition 5 above and Proposition 3 of Kolotilin (2018). If (16) and (17) hold (Figure 3a), then the optimal unrestricted message rule is monotone and is represented by $P_* = [0, \theta_*^L] \cup (\theta_*^H, 1]$. If $u'_*(m_*^L) \leq u'_*(m_*^H)$ and (18) and (19) hold (Figure 3b), then the optimal unrestricted message rule is likewise monotone and is represented by $P_* = [0, \theta_*^M] \cup (\theta_*^M, 1]$. Otherwise (Figures 3c and 3d), the optimal unrestricted message rule is nonmonotone and yields the expected payoff $\text{co } u_*(\mathbb{E}[\theta])$ where $\text{co } u_*$ is the concavification of u_* ;¹⁹ but the optimal monotone message rule yields a strictly lower expected payoff than $\text{co } u_*(\mathbb{E}[\theta])$ and is represented by either $P_* = [0, \theta_*^M] \cup (\theta_*^M, 1]$ or $P_* = [0, 1]$.

¹⁹The concavification of u_* is defined as $\text{co } u_*(\theta) = \min_{u \in \mathcal{U}} u(\theta)$, for $\theta \in [0, 1]$, where \mathcal{U} is the set of all concave functions u on $[0, 1]$ such that $u(\theta) \geq u_*(\theta)$ for all $\theta \in [0, 1]$.

5 Quadratic Payoffs

Assume that the players' payoffs are quadratic:

Assumption 3. $u_R(d, \theta) = \lambda_R(\theta d - d^2/2)$ and $u_S(d, \theta) = \lambda_S((a\theta + b)d - d^2/2)$ for all $d \in \mathbb{R}$ and $\theta \in [0, 1]$, where $a > 0$, $b \in \mathbb{R}$, $\lambda_R > 0$, $\lambda_S > 0$, and $\lambda_R + \lambda_S = 1$.

Assumption 3 satisfies Assumption 1, so Propositions 1, 2, and 3 hold. It also satisfies Assumption 2, so Propositions 4 and 5 hold with $u_*(m) = u(\rho_*(m), m)$ for all $m \in [0, 1]$. Under Assumption 3, the receiver's preferred decision is $\rho_R(\theta) = \theta$, and the sender's preferred decision is $\rho_S(\theta) = a\theta + b$. Moreover, given the normalization $\lambda_S + \lambda_R = 1$, the first-best decision is $\rho_{FB}(\theta) = \lambda_R\rho_R(\theta) + \lambda_S\rho_S(\theta) = \alpha\theta + \beta$ and the joint payoff is $u(d, \theta) = \rho_{FB}(\theta)d - d^2/2$, where $\alpha = \lambda_R + \lambda_S a$ and $\beta = \lambda_S b$.²⁰

5.1 Preliminaries

Under Assumption 3, the second-best decision (6) given message m pushes d as close to $\rho_{FB}(m)$ as possible, while still keeping d within distance ℓ from the receiver's preferred decision $\rho_R(m)$:

$$\begin{aligned} \rho_*(m) &= \arg \max_d u(d, m) \\ \text{subject to } |d - \rho_R(m)| &\leq \ell = \sqrt{\frac{\delta}{1 - \delta} \frac{\bar{v} - \underline{v}_S - \underline{v}_R}{\lambda_R}}, \end{aligned} \quad (20)$$

where we call ℓ the relational *leeway*. The second-best decision rule ρ_* is parallel to ρ_R at extreme states and coincides with ρ_{FB} at non-extreme states. Formally, a state is *extreme* if the first-best decision is not enforceable at this state. The set of extreme states is thus

$$X = \{\theta : |\rho_{FB}(\theta) - \rho_R(\theta)| > \ell\}. \quad (21)$$

If nonempty, the set X consists of one or two intervals (see Figures 4 and 5).

An immediate implication of Proposition 2 is that pooling can be optimal only if there exist extreme states ($X \neq \emptyset$). Moreover, if pooling is optimal, then each non-singleton message m must contain such extreme states; otherwise it would be optimal to separate all states $\theta \in m$ and implement the first-best decision $\rho_{FB}(\theta)$ for each $\theta \in m$.

²⁰These payoff functions nest two special cases. First, in Crawford and Sobel (1982)'s example, the sender has constant upward bias, so that $\rho_S(\theta) = \theta + b$. Second, in Kamenica and Gentzkow (2011)'s lobbying example, the sender is biased towards a specific decision $d_c > 1$, so that $\rho_S(\theta) = a\theta + (1 - a)d_c$ with $a \in (0, 1)$.

Further, Assumption 3 imposes a tight link between the set of extreme states and the curvature of the joint payoff. It is easy to show that u_* is convex on the set of non-extreme states. Further, define the receiver to be *over-responsive* if $\rho'_R(\theta) > 2\rho'_{FB}(\theta)$; equivalently, $\alpha < 1/2$. Then u_* is concave on the set of extreme states if and only if the receiver is over-responsive.

The rest of Section 5 will exploit this link between X and $u''_*(\theta)$ by using results from Section 4 to characterize the optimal pooling set. We set the stage by restating the fact that full separation is optimal if and only if the joint payoff u_* is convex on $[0, 1]$.

Corollary 2. *The optimal pooling set P_* is nonempty if and only if the set of extreme states X is nonempty and the receiver is over-responsive.*

To understand the benefits of pooling, consider the choice between the following two message rules which differ only on an interval $(\bar{\zeta}, \zeta)$ of extreme states. One message rule μ_p completely pools this interval into one message $\mathbb{E}[(\bar{\zeta}, \zeta)]$, and the other μ_s fully separates this interval. Both message rules induce the same expected decision on $(\bar{\zeta}, \zeta)$,

$$\mathbb{E}[\rho_*(\mu_p(\theta)) | (\bar{\zeta}, \zeta)] = \mathbb{E}[\rho_*(\mu_s(\theta)) | (\bar{\zeta}, \zeta)] = \rho_*(\mathbb{E}[\theta | (\bar{\zeta}, \zeta)]).$$

But, decisions are less responsive to the state on $(\bar{\zeta}, \zeta)$ under μ_p than under μ_s :

$$0 = \rho'_*(\mu_p(\theta)) < \rho'_*(\mu_s(\theta)) = \rho'_R(\theta) = 1.$$

So, ρ_{FB} is closer to $\rho_*(\mu_p)$ than to $\rho_*(\mu_s)$ if α is closer to 0 than to 1. That is, pooling is optimal if and only if, at extreme states, the receiver is over-responsive ($\alpha < 1/2$), in which case all extreme states are optimally pooled: the closure of P_* contains X .

5.2 Constant-Sign Bias

Suppose the receiver is over-responsive and the sender is upwardly biased, $\rho_S(\theta) > \rho_R(\theta)$ for all $\theta \in [0, 1]$.²¹ In this case, X consists of up to one interval that shrinks and eventually vanishes as the players become more patient.²² Specifically, there exist $\delta^A, \delta^B \in (0, 1)$ such that $\delta^A > \delta^B$ and

$$X = \begin{cases} \emptyset, & \text{if } \delta \in (\delta^A, 1), \\ [0, \theta^L) \text{ for some } \theta^L \in (0, 1), & \text{if } \delta \in (\delta^B, \delta^A), \\ [0, 1], & \text{if } \delta \in [0, \delta^B). \end{cases}$$

²¹The case of a downwardly biased sender is symmetric and omitted.

²²This is because the relational leeway ℓ increases with δ , as shown in Lemma 7 in Appendix D.

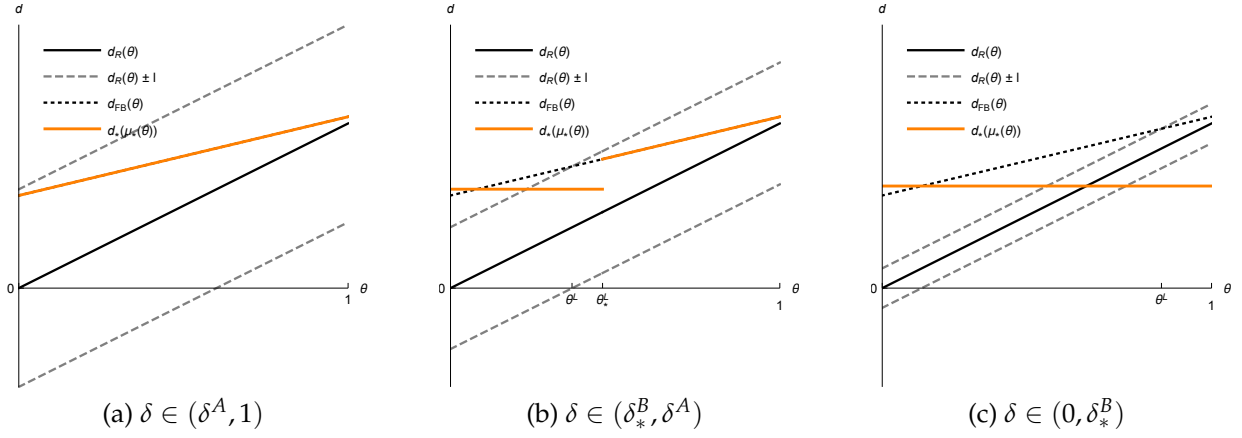


Figure 4: Over-responsive receiver, constant-sign bias

Proposition 6. *Suppose the receiver is over-responsive, and the sender is upwardly biased. There exists $\delta_*^B \in (\delta^B, \delta^A)$ such that the optimal pooling set is*

$$P_* = \begin{cases} \emptyset, & \text{if } \delta \in (\delta^A, 1), \\ [0, \theta_*^L) \text{ for some } \theta_*^L \in (\theta^L, 1), & \text{if } \delta \in (\delta_*^B, \delta^A), \\ [0, 1], & \text{if } \delta \in [0, \delta_*^B]. \end{cases}$$

Further, $d\theta_*^L/d\delta < 0$ if $\delta \in (\delta_*^B, \delta^A)$.

Proposition 6 highlights what we call *over-pooling*: all extreme states are optimally pooled with some adjacent non-extreme states (see Figure 4). The proof relies on Proposition 4. To build intuition, consider the effects of marginally expanding the pooling interval from $[0, \theta^L)$ to $[0, \theta^L + d\theta)$, and thus increasing the pooling message from $m^L = \mathbb{E}[\theta | [0, \theta^L)]$ to $m^L + dm = \mathbb{E}[\theta | [0, \theta^L + d\theta)]$. The cost of this expansion is that mass $f(\theta^L)d\theta$ of newly-added states $[\theta^L, \theta^L + d\theta)$ switch from the first-best decision $\rho_{FB}(\theta^L)$ to a lower decision $\rho_*(m^L)$, resulting in a loss of $(u_*(\theta^L) - u_*(m^L)) f(\theta^L)d\theta$. The benefit of this expansion is that mass $F(\theta^L)$ of existing states $[0, \theta^L)$ switch from decision $\rho_*(m^L)$ to a higher decision $\rho_*(m^L + dm)$, resulting in a gain of $u'_*(m^L)dm F(\theta^L)$. The net benefit of this expansion is thus²³

$$\left(u'_*(m^L)(\theta^L - m^L) - (u_*(\theta^L) - u_*(m^L)) \right) f(\theta^L)d\theta. \quad (22)$$

If the receiver is over-responsive, then u_* is concave on $[0, \theta^L)$; see Figure 2. In this case,

²³Here, we use the fact that $dm/d\theta = (\theta^L - m^L)f(\theta^L)/F(\theta^L)$ to rewrite the benefit term of (22).

(22) indicates that the benefit of the marginal expansion outweighs the cost, leading to over-pooling: the optimal threshold θ_*^L is greater than θ^L . Further, as the players become more patient, the set of extreme states shrinks, and the optimal pooling interval shrinks with it.²⁴

5.3 Changing-Sign Bias

Now, suppose the receiver is over-responsive, and the players' preferred decision rules intersect at some state $\theta_0 \in [1/2, 1)$, so that the sender's bias changes sign at θ_0 . In this case, X consists of up to two intervals that shrink and eventually vanish as the players become more patient. Then there exist $\delta^A, \delta^B \in (0, 1)$ such that $\delta^A \geq \delta^B$ (with equality if and only if $\theta_0 = 1/2$) and

$$X = \begin{cases} \emptyset, & \text{if } \delta \in (\delta^A, 1), \\ [0, \theta^L) \text{ for some } \theta^L \in (0, 1), & \text{if } \delta \in (\delta^B, \delta^A), \\ [0, \theta^L) \cup (\theta^H, 1] \text{ for some } \theta^L, \theta^H \in (0, 1) \text{ such that } \theta^L < \theta^H, & \text{if } \delta \in (0, \delta^B). \end{cases}$$

Proposition 7. *Suppose the receiver is over-responsive and $\rho_R(\theta_0) = \rho_S(\theta_0)$ for some $\theta_0 \in [1/2, 1)$. There exist $\delta_*^B, \delta_*^C, \delta_*^D \in (0, 1)$ with either $\delta^B < \delta_*^D = \delta_*^C = \delta_*^B < \delta_A$ or $\delta_*^D < \delta_*^C < \delta_*^B = \delta^B$ such that the optimal pooling set is*

$$P_* = \begin{cases} \emptyset, & \text{if } \delta \in (\delta^A, 1), \\ [0, \theta_*^L) \text{ for some } \theta_*^L \in (\theta^L, 1), & \text{if } \delta \in (\delta_*^B, \delta^A), \\ [0, \theta_*^L) \cup (\theta_*^H, 1] \text{ for some } \theta_*^L, \theta_*^H \in (\theta^L, \theta^H) \text{ such that } \theta_*^L < \theta_*^H, & \text{if } \delta \in (\delta_*^C, \delta_*^B), \\ [0, \theta_*^M) \cup (\theta_*^M, 1] \text{ for some } \theta_*^M \in (0, 1), & \text{if } \delta \in (\delta_*^D, \delta_*^C), \\ [0, 1], & \text{if } \delta \in [0, \delta_*^D). \end{cases}$$

Further, $d\theta_*^L/d\delta < 0$ if $\delta \in (\delta_*^C, \delta^A)$, and $d\theta_*^H/d\delta > 0$ if $\delta \in (\delta_*^C, \delta_*^B)$.

The proof of Proposition 7 relies on Proposition 5. Consider how the optimal pooling set P_* changes as we decrease δ from δ^A to 0. For $\delta \in (\delta^B, \delta^A)$, the set X consists of one interval $[0, \theta^L)$ and, as in Proposition 6, over-pooling occurs: $P_* = [0, \theta_*^L)$ with $\theta_*^L > \theta^L$. As δ decreases towards δ^B , the optimal pooling threshold θ_*^L increases. If complete pooling becomes optimal (θ_*^L reaches 1) at $\delta_*^B > \delta^B$, then complete pooling remains optimal for all $\delta \in [0, \delta_*^B)$.

²⁴Relatedly, as Corollary 3 in Appendix D shows, the optimal pooling interval shrinks as the receiver becomes less responsive to the state (that is, as $\rho'_R(\theta) - \rho'_{FB}(\theta) = 1 - \alpha$ decreases).

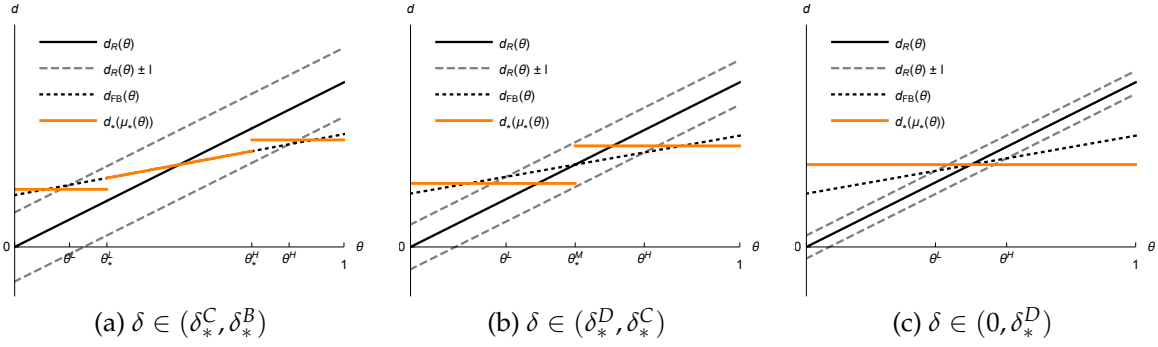


Figure 5: Over-responsive receiver, changing-sign bias

Suppose now that optimal pooling remains incomplete ($\theta_*^L < 1$) when δ reaches δ^B . For $\delta < \delta_B$, the set X consists of two disjoint intervals $[0, \theta^L)$ and $(\theta^H, 1]$. Over-pooling takes the following form. For $\delta \in (\delta_*^C, \delta_*^B)$, each interval of X is separately over-pooled: $P_* = [0, \theta_*^L) \cup (\theta_*^H, 1]$, with $\theta^L < \theta_*^L < \theta_*^H < \theta^H$. As δ decreases towards δ_*^C , the optimal pooling thresholds θ_*^L and θ_*^H move closer together, and the interval $[\theta_*^L, \theta_*^H]$ of fully separated states shrinks. At $\delta = \delta_*^C$, the optimal thresholds θ_*^L and θ_*^H meet at some θ_*^M , so that (almost) all states belong to one of the two pooling intervals $[0, \theta_*^M)$ and $(\theta_*^M, 1]$. The optimal pooling set retains the form $P_* = [0, \theta_*^M) \cup (\theta_*^M, 1]$ over the range $\delta \in (\delta_*^D, \delta_*^C)$.

As δ decreases below δ_*^D , the optimal pooling set changes discontinuously to complete pooling, $P_* = [0, 1]$. Complete pooling remains optimal for all $\delta \in [0, \delta_*^D)$. To build intuition, suppose that $\theta_0 = 1/2$ and θ is uniformly distributed on $[0, 1]$ (see Figure 5). Within the range $\delta \in [0, \delta_*^C)$, the optimal pooling set P_* is either $[0, 1]$ or $[0, \theta_*^M) \cup (\theta_*^M, 1]$, where $\theta_*^M = 1/2$ in this symmetric setting. If $\delta = 0$, then u_* is concave on $[0, 1]$; so complete pooling is uniquely optimal, and, by continuity, it remains uniquely optimal for some range $\delta \in [0, \delta_*^D)$.

5.4 Uniform Distribution

Assume further that the state is uniformly distributed:

Assumption 4. $F(\theta) = \theta$ for all $\theta \in [0, 1]$.

Under this assumption, Propositions 6 and 7 can be restated in a closed form.²⁵

Proposition 8. *Suppose $\alpha < 1/2$, $\alpha/2 + \beta > 1/2$, and $\alpha + \sqrt{1 - 2\alpha}/4 + \beta < 1$. Then the optimal pooling set is*

$$P_* = \begin{cases} \emptyset, & \text{if } \ell \in (\ell_*^A, \infty), \\ [0, \theta_*^L] \text{ where } \theta_*^L = \gamma(\beta - \ell), & \text{if } \ell \in (\ell_*^B, \ell_*^A), \\ [0, \theta_*^L] \cup (\theta_*^H, 1] \text{ where } \theta_*^L = \gamma(\beta - \ell) \text{ and } \theta_*^H = 1 - \gamma(1 - \ell - \alpha - \beta), & \text{if } \ell \in (\ell_*^C, \ell_*^B), \\ [0, \theta_*^M] \cup (\theta_*^M, 1] \text{ where } \theta_*^M = \frac{1 - 2\alpha - 16\ell\beta}{2(1 - 8\ell(1 - \alpha) - 2\alpha)}, & \text{if } \ell \in (\ell_*^D, \ell_*^C), \\ [0, 1], & \text{if } \ell \in [0, \ell_*^D), \end{cases}$$

$$\text{where } \gamma = \frac{2(2 + \sqrt{1 - 2\alpha} - 2\alpha)}{3 - 6\alpha + 4\alpha^2},$$

$$(\ell_*^A, \ell_*^B, \ell_*^C, \ell_*^D) = \left(\beta, 1 - \alpha - \beta, \frac{1 - \sqrt{1 - 2\alpha}}{4} + \frac{\beta\sqrt{1 - 2\alpha} - 1/4}{2(1 - \alpha)}, \frac{1 - 2\alpha}{16(1 - \max\{3/4, \alpha + \beta\})} \right).$$

Assumption 4 also enables a closed-form characterization of the sender's worst equilibrium, as in Proposition 3.

Proposition 9. *Suppose $a/2 + b > 1/2$. There exists an optimal equilibrium in single-period punishment strategies where the sender's penal decision rule is $\underline{\rho}(m) = m - \ell$ for all m and the sender's penal message rule $\underline{\mu}$ is represented by the pooling set $\underline{P} = [0, \underline{\theta}^L]$, with*

$$\underline{\theta}^L = \begin{cases} 0, & \text{if } a(b + \ell) > \frac{3}{32}, \\ \frac{1 + \sqrt{1 - 8a(b + \ell)}}{2a}, & \text{if } a(b + \ell) < \frac{3}{32}. \end{cases} \quad (23)$$

Proposition 9 highlights two points. First, an upwardly-biased (on average) sender is punished with the lowest incentive compatible decision rule. Second, punishing the sender may or may not involve pooling. Pooling hurts the sender by making the decision adapt less to the state, but benefits the sender by reducing his signaling transfers. An increase in $a(b + \ell)$ makes signaling transfers relatively more effective as a punishment, and thus shrinks the optimal penal pooling set $[0, \underline{\theta}^L]$.

²⁵Proposition 8 covers the case $\delta_*^D < \delta_*^C < \delta_*^B = \delta^B$ of Proposition 7. Proposition 6 and the remaining case of Proposition 7 are covered in Appendix D.

6 Separation of Information and Control

In this section, we show how ‘arms-length’ organizational forms that separate information and control enable effective informal communication and decision-making. We consider two changes to the model that reduce the separation of information and control. In Section 6.1, we introduce formal communication processes that mechanically increase transparency. Specifically, we introduce a public signal about the state. In Section 6.2, we allow for delegation of decision rights to informed players.

It turns out that improving public information or delegating decision rights to informed players does not enable better informed decision-making. The availability of transfers as a signaling device implies that better informed decision-making can always be achieved without tightening incentive constraints, so an organizational form that brings information and control together adds no informational benefits for the relationship. On the flip side, such an organizational form tightens incentive constraints in two ways. First, it improves both players’ worst possible equilibrium payoffs, and thus limits the severity of off-path punishments. Second, it prevents information pooling, and thus limits the ability to discipline decision-making in states of extreme conflict.

6.1 Transparency

We augment our model so that at the start of each period, the receiver observes a realization of a state-dependent signal. We maintain Assumption 1 of Section 2, but do not impose Assumptions 2 or 3. Just as with message rules, we assume that the signal rule $\psi(\theta)$ is deterministic and (without loss) identify each signal realization s with the set of states that induce it, $s = \{\theta : \psi(\theta) = s\}$. We also assume that the signal rule ψ is monotone in the sense that each $s \in \psi([0, 1])$ is a convex set.

Since the signal and message rules are deterministic, we can restrict attention to message rules that are *refinements* of the signal rule in that for each realization s of ψ there exists a realization m of μ such that $m \subset s$. In particular, this restriction allows us to consider decision rules ρ that depend on the message m but not the signal realization s , because m incorporates all information contained in s .

The set of equilibrium payoffs V under signal ψ can be computed by applying Proposition 3 separately to each realization of signal ψ . In particular, the optimal and penal message and decision rules are defined for each signal realization $s \in \psi([0, 1])$ as follows: ρ_* is given by (6) and μ_* solves (8) given that the set of states is s rather than $[0, 1]$; $\underline{\rho}_R = \rho_R$ and $\underline{\mu}_R = \psi$; and $\underline{\rho}_S$ and $\underline{\mu}_S$ solve (9) given that the set of states is s .

We say that ψ is *more informative* than $\hat{\psi}$ if ψ is a refinement of $\hat{\psi}$. For monotone signal

rules, this notion coincides with the informativeness criterion of Blackwell (1953). Signal rule ψ is *strictly* more informative than $\hat{\psi}$ if ψ is more informative than $\hat{\psi}$ and the set of states where $\psi(\theta) \neq \hat{\psi}(\theta)$ has strictly positive probability.

Proposition 10. *Suppose that ψ is strictly more informative than $\hat{\psi}$ and let V and \hat{V} be the corresponding equilibrium payoff sets. If $\underline{\rho}_S(\underline{\mu}_S(\theta))$ is nondecreasing in θ under ψ , then $V \subsetneq \hat{V}$.*

By Proposition 3, $\underline{\rho}_S(\underline{\mu}_S(\theta))$ is nondecreasing in θ on each signal realization s but may decrease across signal realizations; thus, the assumption in Proposition 10 that $\underline{\rho}_S(\underline{\mu}_S(\theta))$ is nondecreasing in θ is not innocuous. This assumption holds if Assumptions 3 and 4 hold and $\rho_S(\theta)$ does not cross $\rho_R(\theta)$ from below for $\theta \in [0, 1]$.²⁶ Moreover, this assumption holds if $\delta = 0$ or $\psi(\theta) = \theta$ for all $\theta \in [0, 1]$. In contrast, Proposition 3 implies that $\rho_*(\mu_*(\theta))$ and $\underline{\rho}_R(\underline{\mu}_R(\theta))$ are always nondecreasing in θ .

To build intuition for Proposition 10, we start with the myopic benchmark. We will argue that the set of equilibrium payoffs V expands when moving from a fully informative public signal ($\psi_f(\theta) = \theta$) to a completely uninformative public signal ($\psi_u(\theta) = [0, 1]$). Specifically, \underline{v}_S and \underline{v}_R strictly decrease and \bar{v} weakly increases.

The receiver's worst equilibrium payoff \underline{v}_R is lower under ψ_u than ψ_f . In the receiver's worst equilibrium, the receiver always chooses her preferred decision $\rho_R(\psi(\theta))$ given the public signal ψ and always receives zero transfers. Public information improves the receiver's decision-making and thus her worst equilibrium payoff.

The sender's worst equilibrium payoff \underline{v}_S is lower under ψ_u than ψ_f . The basic idea is that any equilibrium decision outcome implemented under ψ_f (and thus a fully informed receiver) can also be implemented under ψ_u by inducing the sender to fully reveal the state to the receiver. The sender's payoff \underline{v}_S is strictly smaller under ψ_u because inducing full separation requires the sender to make positive interim transfers to the receiver.

The best joint payoff \bar{v} is weakly higher under ψ_u than ψ_f , again because any equilibrium under ψ_f can be implemented under ψ_u . In fact, the best joint payoff may be strictly higher under ψ_u than ψ_f . Under ψ_u , the joint payoff is maximized under complete pooling of the states if the receiver is over-responsive (see Section 5.1). Such pooling, however, is precluded under ψ_f (and thus a fully informed receiver).

In the non-myopic case, these effects are preserved, and are further amplified by the shadow of the future. Moving from ψ_f to ψ_u expands V and thus increases the relational

²⁶This statement follows from Proposition 9. Conversely, if Assumptions 3 and 4 hold, $\ell > 0$, and $\rho_S(\theta)$ crosses $\rho_R(\theta)$ from below at $\theta_0 \in (0, 1)$, then $\rho_S(\mu_S(\theta))$ is *not* nondecreasing in θ on $[0, 1]$ for some ψ . For example, $\rho_S(\mu_S(\theta))$ jumps down from $\theta_0 + \ell$ to $\theta_0 - \ell$ at θ_0 if ψ contains signal realizations $(\theta_0 - \varepsilon, \theta_0)$ and $(\theta_0, \theta_0 + \varepsilon)$ for sufficiently small ε .

leeway ℓ (which increases with \bar{v} and decreases with \underline{v}_S and \underline{v}_R). This in turn relaxes constraints on decision-making and expands the set V even further.

The result that public information hurts the relationship relates to various papers that study the social value of public information. [Hirshleifer \(1971\)](#) argues that welfare may be decreasing in the amount of public information available to agents. [Bergemann and Morris \(2016\)](#) clarifies this point: making more information available to an agent may, by increasing the set of incentive constraints she faces, shrink the set of equilibrium outcomes.²⁷ This relates to the logic of our model, where the availability of public information makes it impossible to pool incentive constraints across states, and thus worsens incentive provision within the relationship. Public information in our model also improves the worst possible equilibrium payoffs for both players; this decreases the surplus and thus worsens intertemporal incentives.²⁸

6.2 Allocation of Authority

In our model of Section 2, decision-making authority always resides with the receiver and is not transferable (*receiver-authority*). Consider a variation of the model where the sender chooses the decision instead of the receiver; call this variation *sender-authority*. For simplicity, assume that the payoffs are quadratic and Assumption 3 holds. Focus on the case $\lambda_S = \lambda_R$, so that the sender has the same temptation to defect from the first-best decision under sender-authority as the receiver has under receiver-authority. In this case, full separation is always optimal under receiver-authority.

It turns out that allocating decision authority to the sender strictly decreases the best joint payoff. This is because the worst equilibrium joint payoff is strictly higher,²⁹ and thus the relational leeway is strictly smaller, under sender-authority. This implies that all our results continue to hold even if decision-making authority could be allocated to either player at the beginning of the game, because the players would always choose

²⁷[Cr mer \(1995\)](#), [Kolotilin \(2015\)](#), and [Fong and Li \(2016\)](#) discuss other settings where public information may be detrimental.

²⁸This point relates to an insight from [Baker, Gibbons and Murphy \(1994\)](#). There, objective performance measures, rather than transparency, improve the players' outside options and make cooperation within the relationship more difficult to sustain.

²⁹Denote $\lambda_R = \lambda_S = \lambda$. Under sender-authority, $\underline{v}_S = \lambda \mathbb{E}[(a\theta + b)^2]/2$ because the sender can always choose his preferred decision in each state, whereas $\underline{v}_R = \lambda \mathbb{E}[\theta^2 - (a\theta + b + \ell - \theta)^2]/2$ because $a\theta + b + \ell$ is the worst possible decision for the receiver that is enforceable for the (upwardly biased) sender. On the other hand, under receiver-authority, $\underline{v}_R = \lambda \mathbb{E}[\theta^2 - (\mathbb{E}[\theta] - \theta)^2]/2 < \lambda \mathbb{E}[\theta^2]/2$ because the receiver can always choose the uninformed decision $d = \mathbb{E}[\theta]$, whereas $\underline{v}_S < \lambda \mathbb{E}[(a\theta + b)^2 - (\theta - \ell - (a\theta + b))^2]/2$ because full separation, non-negative interim transfers by the sender, and decision $\theta - \ell$ can be achieved in equilibrium. Thus, the worst equilibrium joint payoff is strictly higher under sender-authority.

receiver-authority over sender-authority.

When $\lambda_S \neq \lambda_R$, two additional effects make the comparison between sender- and receiver-authority more nuanced. The first effect favours giving authority to the player who cares more about the decision. Under i -authority where $i \in \{S, R\}$, the temptation to defect from the first-best decision is $\lambda_i (\rho_i(\theta) - \rho_{FB}(\theta))^2 / 2 = \lambda_i \lambda_{-i}^2 (\rho_i(\theta) - \rho_{-i}(\theta))^2 / 2$, which is higher than the corresponding temptation under $(-i)$ -authority when $\lambda_i < \lambda_{-i}$. The second effect weakly favours receiver-authority. Under receiver-authority, when $\lambda_S > \lambda_R$, the optimal equilibrium may involve pooling to discipline decision-making; this tool is unavailable under sender-authority.

Consider another variation where decision-making authority is allocated at the beginning of each period (*short-term-authority*). Specifically, following [Baker, Gibbons and Murphy \(2011\)](#), suppose that at the beginning of each period, the receiver has decision-making authority by default, and can make a take-it-or-leave-it offer to transfer authority to the sender for that period in exchange for a transfer. As above, focus on the case $\lambda_S = \lambda_R$. We know from above that the best (worst) equilibrium joint payoff is higher (lower) under receiver-authority than under sender-authority. This implies that relative to receiver-authority, short-term-authority does not improve on the best equilibrium joint payoff (because the players cannot do better than to allocate authority to the receiver in each period), but increases the worst equilibrium joint payoff (because the players always have the option to allocate authority to the sender in each period). This then implies that the relational leeway, and thus the best equilibrium joint payoff, is strictly lower under short-term-authority than under receiver-authority.

The standard rationale for delegation is that the better-informed player can more effectively adapt the decision. For example, [Dessein \(2002\)](#), [Alonso, Dessein and Matouschek \(2008\)](#), and [Rantakari \(2008\)](#) explore the tradeoff between allocating authority to an uninformed receiver versus an informed but biased sender.³⁰ The standard rationale for delegation no longer applies in our setting because interim transfers can credibly achieve arbitrary communication outcomes at no welfare cost.

³⁰Relatedly, [Holmstrom \(1984\)](#), [Melumad and Shibano \(1991\)](#), [Martimort and Semenov \(2006\)](#), [Alonso and Matouschek \(2008\)](#), [Goltsman et al. \(2009\)](#), [Kováč and Mylovanov \(2009\)](#), and [Amador and Bagwell \(2013\)](#) study the optimal delegation problem. [Krähmer \(2006\)](#) and [Lim \(2012\)](#) allow authority to be allocated after the sender observes the state.

7 Discussion of Model Assumptions

We discuss some of our modelling assumptions and highlight the extent to which our results depend on these assumptions.

7.1 Exogenous Outside Options

We follow [Abreu \(1988\)](#) and [Abreu, Pearce and Stacchetti \(1990\)](#) in characterizing the entire set of equilibrium payoffs. In particular, optimal equilibria utilize the worst possible equilibria as off-path punishments. One alternative modeling approach taken by [Levin \(2003\)](#) is to specify exogenous outside option payoffs \underline{u}_S and \underline{u}_R for both players; so that players are punished for deviations by receiving their outside option payoffs thereafter. In this approach, at the beginning of each period, the receiver makes an offer to the sender consisting of a contractible commitment to an ex-ante transfer. If the sender rejects this offer, the players receive their outside option payoffs, and time moves on to the next period. Another alternative modeling approach taken by [Baker, Gibbons and Murphy \(1994, 2002\)](#) is to restrict attention to *trigger strategy* equilibria where off-path punishments correspond to some static equilibria of the stage game. Our results continue to hold in these settings, with the worst equilibrium payoffs equal to either the outside option payoffs or the static equilibrium payoffs.

7.2 Imperfect Monitoring

We have assumed perfect monitoring in that all actions of the sender and the receiver are immediately publicly observed. Consider a variation in which the receiver's decision is imperfectly monitored. Specifically, suppose that the receiver's (private) decision d stochastically determines an *output* $y = d + \varepsilon$ which is publicly observed and replaces d as an argument in the players' payoff functions: $u_R(y, \theta) = \lambda_R(\theta y - y^2/2)$ and $u_S(y, \theta) = \lambda_S((a\theta + b)y - y^2/2)$. Assume that $\mathbb{E}[\varepsilon] = 0$ and that the density g of ε satisfies the appropriate Mirrlees-Rogerson conditions ([Rogerson, 1985](#)), ensuring that the receiver's decision choice can be represented by a first-order condition.

Focusing on the case where the sender is upwardly biased, consider the highest decision $\bar{\rho}(m)$ that can be supported in equilibrium. Parallel to Theorem 6 of [Levin \(2003\)](#), $\bar{\rho}(m)$ can be implemented by the strongest 'one-step' incentive scheme that satisfies the self-enforcement constraint: this scheme may take the form of stationary continuation

payoffs $v_S(\theta) = \underline{v}_S$, $v_R(\theta) = \bar{v} - \underline{v}_S$ and the ex-post transfer rule

$$T_R(y) = -T_S(y) = \begin{cases} 0 & \text{if } y \geq \bar{\rho}(m) + \varepsilon_*, \\ \frac{\delta}{1-\delta}(\bar{v} - \underline{v}_S - \underline{v}_R) & \text{if } y < \bar{\rho}(m) + \varepsilon_* \end{cases}$$

where ε_* is the point where $g'(\varepsilon)$ switches from negative to positive. The receiver's (unobserved) decision then satisfies the first-order condition

$$\frac{\partial}{\partial d} \mathbb{E} \left[\lambda_R(m(d + \varepsilon) - (d + \varepsilon)^2/2) - T_R(d + \varepsilon) \right] \Big|_{d=\bar{\rho}(m)} = 0,$$

which simplifies to $\bar{\rho}(m) = m + \ell$ for some $\ell > 0$. In other words, the self-enforcement constraint effectively specifies that the equilibrium decision cannot exceed the receiver's preferred decision by more than the leeway ℓ . (As before, the sender's incentive problem does not contribute to the self-enforcement constraint.) Consequently, retracing the steps of our analysis, our results continue to hold in this variation.

7.3 Correlated States

We have assumed that the state θ_t is i.i.d. across periods. Consider a variation where θ_t is correlated across periods. Specifically, introduce a finite-valued random variable ω_t that is publicly observed at the beginning of each period t (before ex-ante transfers are made), where ω_t is a Markov chain. The realization of ω_t fully determines the (time-independent) distribution $F(\theta_t|\omega_t)$ of the state θ_t . Crucially, given ω_t , θ_t contains no further information about ω_{t+1} (and thus about θ_{t+1}); so the sender and the receiver are always symmetrically informed about the future distribution of states. Note that this property would no longer hold if we did not introduce ω_t , but simply assumed that θ_t was a Markov chain.

With this modification, without loss of generality, we can restrict attention to equilibria that are stationary conditional on ω_t . Consequently, our results continue to hold, except that the key objects such as V , ℓ , and θ_* are now functions of ω_t .

8 Conclusion

In our model, incomplete information transmission does not reflect communication failure, but instead is an instrument for managing decision-making. This finding relies on the capacity of voluntary transfers to credibly support any monotone message rule at no welfare cost. It suggests that when modeling strategic communication in applied settings, it is crucial to understand whether monetary or non-monetary transfers (such as wages or favours) are available, because our implications differ significantly from those of the standard literature on strategic communication without transfers. In fact, one interpretation of our model is that voluntary transfers endogenously endow the privately-informed sender with the ability to commit to any monotone message rule, even with impatient players. Such commitment is the premise of the literature on Bayesian persuasion (Kamenica and Gentzkow 2011). So, our analysis extends the applicability of the Bayesian persuasion framework to settings without commitment but with transfers.

Our model is remarkably tractable and thus allows for a thorough treatment of repeated interactions. This analysis produces a rich and intuitive set of results. In particular, incomplete information transmission is implemented only for states of extreme conflict, and only if the receiver's decision-making is too responsive to information. One implication is that with constant bias, pooling does not occur. In contrast, in the standard constant-bias cheap-talk game (Crawford and Sobel 1982), information transmission is always incomplete, and this is generally exacerbated in high (low) states if the sender is upwardly (downwardly) biased.

In our model, an 'arms-length' approach with separation of information and control benefits the relationship. This provides a rationale for opaque organizations which put information in the hands of superiors and prevent subordinates from acquiring information elsewhere. A related implication is that mediators who control the flow of information from the sender to the receiver cannot improve the relationship. This is because it is optimal to give the sender as much control over the release of information as possible.

We hope that future work will use our tractable framework to study other challenging problems in strategic communication. For example, one might examine the case with multiple senders and receivers, possibly connected by a communication network. Another promising avenue would be to allow for costly information acquisition by the sender and the receiver.

Appendix A Stationarity

This appendix specifies necessary and sufficient conditions for equilibrium, and proves Lemma 1.

To show that the set of equilibrium payoffs is compact, restrict decisions and transfers to compact sets $d \in [-\bar{d}, \bar{d}]$ and $\tau_i, t_i, T_i \in [-\bar{t}, \bar{t}]$ for $i \in \{S, R\}$. Under this restriction and under Assumption 1, it can be shown that the set of equilibrium payoffs is compact (see, for example, Mailath and Samuelson 2006). Now, observe that this restriction is without loss of generality if the bounds \bar{d} and \bar{t} are chosen to be large enough that (in any equilibrium) decisions and transfers are interior. Indeed, under Assumption 1, we can show that such bounds exist.

We now show that the set of equilibrium payoffs is the simplex V defined by (1). Consider an optimal equilibrium payoff vector (v_S^*, v_R^*) with $v_S^* + v_R^* = \bar{v}$, and let σ_* be an equilibrium supporting (v_S^*, v_R^*) . Let (v_S, v_R) be any point in the simplex V . Notice that we can modify σ_* to produce (v_S, v_R) by changing only the ex-ante transfers in the first period from τ_i^* to $\tau_i = \tau_i^* + (v_i^* - v_i)/(1 - \delta)$ for each $i \in \{S, R\}$. The modified ex-ante transfers remain feasible, $\tau_S + \tau_R \geq 0$, because $v_S + v_R \leq v_S^* + v_R^*$ by definition of V . Further, this modification affects the players' incentives only at the ex-ante round of the first period. Each player is willing to make the ex-ante transfer τ_i because $v_S \geq \underline{v}_S$ and $v_R \geq \underline{v}_R$ by definition of V . Thus, the modified strategy profile is an equilibrium. Conversely, it is easy to see that any $(v_S, v_R) \notin V$ cannot be supported in equilibrium. We conclude that V is the set of equilibrium payoffs.

A message rule $\mu(\theta)$, a decision rule $\rho(m)$, transfer rules $\tau_i, t_i(m), T_i(m)$, continuation payoff function $v_i(m)$, for each $i \in \{S, R\}$, and punishment decision d^p and message m^p constitute an equilibrium if and only if the following seven conditions hold (see, for example, Mailath and Samuelson 2006):

C1. Each player i is willing to make ex-ante transfer τ_i :

$$\begin{aligned} v_S &= (1 - \delta)[- \tau_S + \mathbb{E}[u_S(\rho(\mu(\theta)), \theta) - t_S(\mu(\theta)) - T_S(\mu(\theta))] + \delta \mathbb{E}[v_S(\mu(\theta))] \geq \underline{v}_S; \\ v_R &= (1 - \delta)[- \tau_R + \mathbb{E}[u_R(\rho(\mu(\theta)), \theta) - t_R(\mu(\theta)) - T_R(\mu(\theta))] + \delta \mathbb{E}[v_R(\mu(\theta))] \geq \underline{v}_R. \end{aligned}$$

C2. For each state θ , the sender is willing to send message $\mu(\theta)$ and to make interim transfer $t_S(\mu(\theta))$.

(a) There is no profitable deviation to another message – interim-transfer pair

$(\mu(\hat{\theta}), t_S(\mu(\hat{\theta})))$ that is observed on the equilibrium path:

$$\begin{aligned} & (1 - \delta)[u_S(\rho(\mu(\theta)), \theta) - t_S(\mu(\theta)) - T_S(\mu(\theta))] + \delta v_S(\mu(\theta)) \\ \geq & (1 - \delta)[u_S(\rho(\mu(\hat{\theta})), \theta) - t_S(\mu(\hat{\theta})) - T_S(\mu(\hat{\theta}))] + \delta v_S(\mu(\theta)) \text{ for all } \theta, \hat{\theta} \in [0, 1]. \end{aligned}$$

(It is without loss of generality to let t_S depend on $\mu(\theta)$ but not directly on θ ; since the sender makes his interim transfer choice before the receiver, we can always modify $\mu(\theta)$ to incorporate any additional information contained in t_S without changing the receiver's information set.)

(b) There is no profitable deviation to some pair (\hat{m}, \hat{t}_S) that is never observed on the equilibrium path:

$$\begin{aligned} & (1 - \delta)[u_S(\rho(\mu(\theta)), \theta) - t_S(\mu(\theta)) - T_S(\mu(\theta))] + \delta v_S(\mu(\theta)) \\ \geq & (1 - \delta)u_S(d^p, \theta) + \delta \underline{v}_S \text{ for all } \theta \in [0, 1]. \end{aligned}$$

Here, we specify that following any such deviation, the receiver chooses punishment decision d^p .

C3. The receiver is willing to make interim transfer $t_R(m)$:

$$\begin{aligned} & (1 - \delta)[u_R(\rho(m), m) - t_R(m) - T_R(m)] + \delta v_R(m) \\ \geq & (1 - \delta)u_R(\rho_R(m), m) + \delta \underline{v}_R \text{ for all } m \in \mu([0, 1]). \end{aligned}$$

C4. The receiver is willing to choose decision $\rho(m)$ on-path and d^p off-path.

(a) After an on-path message – interim-transfer pair, the receiver is willing to choose decision $\rho(m)$:

$$\begin{aligned} & (1 - \delta)[u_R(\rho(m), m) - T_R(m)] + \delta v_R(m) \\ \geq & (1 - \delta)u_R(\rho_R(m), m) + \delta \underline{v}_R \text{ for all } m \in \mu([0, 1]). \end{aligned}$$

(b) After an off-path message – interim-transfer pair, the receiver is willing to choose decision d^p :

$$(1 - \delta)u_R(d^p, m^p) + \delta(\bar{v} - \underline{v}_S) \geq (1 - \delta)u_R(\rho_R(m^p), m^p) + \delta \underline{v}_R.$$

Here, we specify that following any deviation by the sender, the receiver believes that the state is in $m^p \subset [0, 1]$.

C5. Each player i is willing to make ex-post transfer $T_i(m)$:

$$\begin{aligned} -(1 - \delta)T_S(m) + \delta v_S(m) &\geq \delta \underline{v}_S \text{ for all } m \in \mu([0, 1]); \\ -(1 - \delta)T_R(m) + \delta v_R(m) &\geq \delta \underline{v}_R \text{ for all } m \in \mu([0, 1]). \end{aligned}$$

C6. The continuation payoffs can be supported in equilibrium:

$$(v_S(m), v_R(m)) \in V \text{ for all } m \in \mu([0, 1]).$$

C7. There is no creation of money:

$$\begin{aligned} \tau_S + \tau_R &\geq 0; \\ t_S(m) + t_R(m) &\geq 0 \text{ for all } m \in \mu([0, 1]); \\ T_S(m) + T_R(m) &\geq 0 \text{ for all } m \in \mu([0, 1]). \end{aligned}$$

Proof of Lemma 1. We have already shown that the set of equilibrium payoffs is the simplex V defined by (1). In any optimal equilibrium, continuation is optimal: (i) $v_S(m) + v_R(m) = \bar{v}$ for all θ , and (ii) money is not burned, that is, the constraints of Condition C7 hold with equality. Otherwise, one could (i) increase $v_R(m)$ without violating Condition C6, and (ii) decrease transfers τ_R , $t_R(m)$, and $T_R(m)$, thereby relaxing the constraints of Conditions C1–C5 and increasing joint payoff $v_S + v_R$.

An optimal equilibrium σ with zero first-period ex-ante transfers clearly exists. Let (v_S, v_R) be the payoff vector under σ . We will modify σ to construct an optimal stationary equilibrium with the same payoff vector. Let $\mu(\theta)$, $\rho(m)$, $t_i(m)$, $T_i(m)$, and $v_i(m)$, for each $i \in \{S, R\}$, be the message rule, decision rule, transfer rules, and continuation payoff function in the first period on the equilibrium path of σ . Define $T_i^*(m)$ by

$$-(1 - \delta)T_i^*(m) + \delta v_i = -(1 - \delta)T_i(m) + \delta v_i(m).$$

Consider the following stationary strategy profile σ_* . On the equilibrium path, $\mu(\theta)$, $\rho(m)$, $\tau_i = 0$, $t_i(m)$, and $T_i^*(m)$ are played in each period. Following any deviation, except for an undetectable deviation by the sender as in Condition C2(a), play proceeds according to σ . By construction, the sender's and receiver's expected payoffs under σ_* are the same as under σ .

We now show that σ_* constitutes an equilibrium. In each period the constraints of Conditions C1 – C5 continue to hold under σ_* because they are identical to the first-period

constraints under σ , as $-(1 - \delta)T_i^*(m) + \delta v_i$ replaces $-(1 - \delta)T_i(m) + \delta v_i(m)$. Condition C6 holds because (v_S, v_R) belongs to V by supposition. Further, since $v_S + v_R = v_S(m) + v_R(m) = \bar{v}$ and $T_S(m) + T_R(m) = 0$ by optimality of σ , we have $T_S^*(m) + T_R^*(m) = 0$, so Condition C7 holds.

Finally, by modifying the first-period ex-ante transfer in σ_* from 0 to $\tau_i = (v_i - \hat{v}_i)/(1 - \delta)$ for $i \in \{S, R\}$, we can support any equilibrium payoff vector $(\hat{v}_S, \hat{v}_R) \in V$. \square

Lemma 3. *If $0 \leq \delta < \hat{\delta} < 1$, then the corresponding equilibrium payoff sets satisfy $V \subset \hat{V}$.*

Proof. Given $\delta \in [0, 1)$, consider a stationary optimal equilibrium σ_* with zero ex-ante transfers. Let this equilibrium produce an equilibrium payoff vector (v_S^*, v_R^*) , with $v_S^* + v_R^* = \bar{v}$. We can support any equilibrium payoff vector $(v_S, v_R) \in V$ by modifying the first-period ex-ante transfer in σ_* from 0 to $\tau_i = (v_i^* - v_i)/(1 - \delta)$ for each $i \in \{S, R\}$. Notice that Conditions C1 – C7 continue to hold under $\hat{\delta} \in (\delta, 1)$, after replacing $\tau_i = (v_i^* - v_i)/(1 - \delta)$ with $\hat{\tau}_i = (v_i^* - v_i)/(1 - \hat{\delta})$, because

$$\frac{\hat{\delta}}{1 - \hat{\delta}}(v_i^* - v_i) \geq \frac{\delta}{1 - \delta}(v_i^* - v_i) \text{ for each } i \in \{S, R\}.$$

Therefore, the set V is self-generating under $\hat{\delta}$, which proves that $V \subset \hat{V}$ (see, for example, Mailath and Samuelson 2006). \square

Appendix B Equilibrium

Proof of Proposition 1. Consider a stationary equilibrium σ that produces a joint payoff v . Let $\mu(\theta)$, $\rho(m)$, τ_i , $t_i(m)$, and $T_i(m)$, for $i \in \{S, R\}$, be the message rule, decision rule, and transfer rules on the equilibrium path of σ . Define $U_S(\theta)$ as the one-period payoff of the sender if the state is θ ,

$$U_S(\theta) = u_S(\rho(\mu(\theta)), \theta) - p(\theta),$$

where $p(\theta)$ is the net one-period transfer of the sender if the state is θ ,

$$p(\theta) = \tau_S + t_S(\mu(\theta)) + T_S(\mu(\theta)).$$

Condition C2 (a) requires that

$$U_S(\theta) \geq u_S(\rho(\mu(\hat{\theta})), \theta) - p(\hat{\theta}) \text{ for all } \theta, \hat{\theta} \in [0, 1].$$

Since $\partial^2 u_S(d, \theta) / \partial d \partial \theta > 0$ by Assumption 1, this inequality holds if and only if $\rho(\mu(\theta))$ is nondecreasing in θ and

$$U_S(\theta) = U_S(0) + \int_0^\theta \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \text{ for all } \theta \in [0, 1], \quad (24)$$

by Proposition 1 of Rochet (1987) and Corollary 1 of Milgrom and Segal (2002).

Adding the constraint of Condition C4 (a) and the sender's constraint of Condition C5, and taking into account that $T_S(m) + T_R(m) \geq 0$ and $v_S + v_R = v$, gives (3).

Conversely, suppose that $\mu(\theta)$ and $\rho(m)$ are such that $\rho(\mu(\theta))$ is nondecreasing in θ and (3) holds. We construct transfer rules and punishment variables that satisfy Conditions C1 – C7, and thus constitute a stationary equilibrium. We consider the case $\delta > 0$; the case $\delta = 0$ is simpler but slightly different. Let $T_S(m) = T_R(m) = 0$ and $\tau_S = -\tau_R = \mathbb{E}[u_S(\rho(\mu(\theta)), \theta) - t_S(\mu(\theta))] - \underline{v}_S$. Moreover, let $t_S(m)$ and m^p be defined by (4) and (5), and let $d^p = \rho(m^p)$.

Equation (5) assumes that $t_S(m^p) = \inf_{m \in \mu([0,1])} t_S(m)$ for some message $m^p \in \mu([0,1])$. If this assumption does not hold, then we specify m^p and d^p as follows. By the Bolzano-Weierstrass theorem, there exists a sequence $\{m_k\} \in \mu([0,1])$ such that as $k \rightarrow \infty$, $t_S(m_k) \rightarrow \inf_{m \in \mu([0,1])} t_S(m)$, $\theta(m_k) \rightarrow \theta_*$, and $\rho(m_k) \rightarrow d_*$ for some $\theta(m_k) \in m_k$, $\theta_* \in [0,1]$, and $d_* \in \mathbb{R}$. Set $m^p = \theta_*$ and $d^p = d_*$. Since $u_R(d, \theta)$ is continuous, and (3) holds for all $(m_k, \rho(m_k))$, it also holds for (m^p, d^p) .

Notice that the lefthand side of (3) is nonnegative, so $v \geq \underline{v}_S + \underline{v}_R$. The constraints of Condition C7 hold with equality. Condition C6 holds because the continuation payoffs are $v_S = \underline{v}_S$ and $v_R = v - \underline{v}_S$. Condition C5 holds because Condition C6 holds and $T_S(m) = T_R(m) = 0$. The sender's constraint of Condition C1 holds with equality. The receiver's constraint of Condition C1 holds because it can be simplified to $v \geq \underline{v}_S + \underline{v}_R$. Condition C2 (a) holds because $\rho(\mu(\theta))$ is nondecreasing in θ and (24) holds. Condition C2 (b) holds because by deviating to a message-transfer pair (\hat{m}, \hat{t}_S) that is not observed on the equilibrium path, the sender induces $d^p = \rho(m^p)$, which he can induce more cheaply on the equilibrium path with message m^p and zero interim transfer $t_S(m^p) = 0$. This argument assumes that there exists m^p such that $t_S(m^p) = \inf_{m \in \mu([0,1])} t_S(m)$. Condition C2 (b) still holds even if such m^p does not exist. This is because Condition C2 (a) holds for each $\hat{\theta} = \theta(m_k)$, and thus in the limit $k \rightarrow \infty$. But in this limit, Condition C2 (a) coincides with Condition C2 (b). Condition C4 is a restatement of (3). Note that, as for Condition C2 (b), a limiting argument needs to be made for Condition C4 (b) if $\inf_{m \in \mu([0,1])} t_S(m)$ is not attained by any m^p . Condition C3 holds because Condition C4 holds and $t_R(m)$ is nonpositive. \square

Proof of Proposition 2. By Lemma 1 and Proposition 1, in an optimal equilibrium, the decision and message rules solve

$$\bar{v} = \max_{\mu, \rho} \mathbb{E}[u(\rho(\mu(\theta)), \theta)] \quad (25)$$

$$\text{subject to } \rho(\mu(\theta)) \text{ is nondecreasing in } \theta, \quad (26)$$

$$w(\rho(m), m) \leq L(\bar{v}) \text{ for all } m \in \mu([0, 1]). \quad (27)$$

Without loss of generality, we can restrict attention to monotone message rules. The argument is similar to the revelation principle. To this end, consider any μ and ρ that satisfy (26) and (27). Define new rules $\tilde{\mu}$ and $\tilde{\rho}$ as $\tilde{\mu}(\tilde{\theta}) = \{\theta : \rho(\mu(\theta)) = \rho(\mu(\tilde{\theta}))\}$ for all $\tilde{\theta} \in [0, 1]$ and $\tilde{\rho}(\tilde{m}) = \rho(\mu(\tilde{\theta}(\tilde{m})))$ for all $\tilde{m} \in \tilde{\mu}([0, 1])$, where $\tilde{\theta}(\tilde{m})$ is an arbitrary state $\tilde{\theta} \in \tilde{m}$. It is easy to see that $\tilde{d}(\tilde{m})$ is independent of the choice of a representative state $\tilde{\theta} \in \tilde{m}$ and that $\tilde{\rho}(\tilde{\mu}(\theta)) = \rho(\mu(\theta))$ for all $\theta \in [0, 1]$. Since $\rho(\mu(\theta))$ is nondecreasing in θ by (26), $\tilde{\rho}(\tilde{\mu}(\theta))$ is also nondecreasing in θ and $\tilde{\mu}$ is monotone. Moreover, since each set $\tilde{m} \in \tilde{\mu}([0, 1])$ is the union of some disjoint sets $m \in \mu([0, 1])$ and the constraint (27) holds for $\rho(m)$ for each $m \in \mu([0, 1])$, the constraint (27) also holds for $\tilde{\rho}(\tilde{m})$ for each $\tilde{m} \in \tilde{\mu}([0, 1])$.

Consider a relaxed problem

$$\bar{v} = \max_{\mu, \rho} \mathbb{E}[u(\rho(\mu(\theta)), \theta)]$$

$$\text{subject to } \mu \text{ is monotone,}$$

$$w(\rho(m), m) \leq L(\bar{v}) \text{ for all } m \in \mu([0, 1]).$$

We can solve this relaxed problem in two steps. First, for a given monotone message rule μ , the optimal decision rule is given by ρ_* defined by (6). Second, given the optimal decision rule ρ_* , the optimal message rule is clearly μ_* defined by (8). To prove that the solution ρ_* and μ_* to the relaxed problem are the actual optimal decision and message rules that solve the problem (25), it remains to show that $\rho_*(m)$ is nondecreasing in m .

We first rewrite the constraint of the problem (6) as $d \in D(m)$ where $D(m)$ is nondecreasing in m in the strong set order. Since $u_R(d, \theta)$ is strictly concave in d and has a unique maximum, $w(d, m)$ is strictly convex in d and has a unique minimum. Taking into account that $w(\rho_R(m), m) = 0$ and $L(\bar{v}) \geq 0$, we have that the set of decisions d that satisfy the constraint of the problem (6) is a nonempty closed convex set and thus can be written as $D(m) = [\rho_-(m), \rho_+(m)]$, where $\rho_-(m)$ and $\rho_+(m)$ satisfy the constraint with equality. Moreover, since $u_R(d, \theta)$ is concave in d and is supermodular, $w(d, m)$ is nonin-

creasing in d and nondecreasing in m if $d < \rho_R(m)$, and $w(d, m)$ is nondecreasing in d and nonincreasing in m if $d > \rho_R(m)$. This implies that $\rho_-(m)$ and $\rho_+(m)$ are nondecreasing in m , and thus $D(m)$ is nondecreasing in m . Taking into account that $u(d, m)$ is strictly concave and has increasing differences, $\rho_*(m) = \arg \max_{d \in D(m)} u(d, m)$ is nondecreasing in m , as follows, for example, from Theorem 4 of [Milgrom and Shannon \(1994\)](#). \square

Proof of Proposition 3. By Proposition 2, $\bar{v} = \mathbb{E}[u_*(\rho_*(\mu_*(\theta)), \theta)]$.

A receiver's worst equilibrium with zero first-period ex-ante transfers ($\tau_S = \tau_R = 0$) clearly exists. Let $\mu(\theta)$, $t_R(m)$, $\rho(m)$, $T_R(m)$, and $v_R(m)$ be used in the first period of such an equilibrium. Thus,

$$\begin{aligned} \underline{v}_R &= \mathbb{E}[(1 - \delta)[u_R(\rho(\mu(\theta)), \mu(\theta)) - t_R(\mu(\theta)) - T_R(\mu(\theta))] + \delta v_R(\mu(\theta))] \\ &\geq \mathbb{E}[(1 - \delta)u_R(\rho_R(\mu(\theta)), \mu(\theta)) + \delta \underline{v}_R] \\ &\geq (1 - \delta)u_R(\rho_R([0, 1]), [0, 1]) + \delta \underline{v}_R, \end{aligned}$$

where the equality follows from $\tau_R = 0$, the first inequality follows from Condition C3, and the last inequality follows from the definition of ρ_R . Rearranging gives

$$\underline{v}_R \geq u_R(\rho_R([0, 1]), [0, 1]).$$

Similarly, a sender's worst equilibrium with zero first-period ex-ante transfers exists. Let $\mu(\theta)$, $t_S(m)$, $\rho(m)$, $T_S(m)$, $v_S(m)$, d^p , and m^p be used in the first period of such an equilibrium. Define $V_S(\theta)$ as the expected payoff of the sender if the first-period state is θ ,

$$\begin{aligned} V_S(\theta) &= (1 - \delta)u_S(\rho(\mu(\theta)), \theta) - p(\theta), \\ \text{where } p(\theta) &= (1 - \delta)[t_S(\mu(\theta)) + T_S(\mu(\theta))] - \delta v_S(\mu(\theta)). \end{aligned}$$

Condition C2 (a) requires that

$$V_S(\theta) \geq (1 - \delta)u_S(\rho(\mu(\hat{\theta})), \theta) - p(\hat{\theta}) \text{ for all } \theta, \hat{\theta} \in [0, 1]. \quad (28)$$

As explained in the proof of Proposition 1, this inequality holds if and only if

$$\rho(\mu(\theta)) \text{ is nondecreasing in } \theta, \quad (29)$$

$$V_S(\theta) = V_S(0) + (1 - \delta) \int_0^\theta \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \text{ for all } \theta \in [0, 1]. \quad (30)$$

Condition C2 (b), the constraint of Condition C4 (a) and the sender's constraint of Condi-

tion C5, and the constraint of Condition C4 (b), respectively imply that

$$V_S(\theta) \geq (1 - \delta)u_S(d^p, \theta) + \delta\underline{v}_S \text{ for all } \theta \in [0, 1], \quad (31)$$

$$w(\rho(m), m) \leq L(\bar{v}) \text{ for all } m \in \mu([0, 1]), \quad (32)$$

$$w(d^p, m^p) \leq L(\bar{v}). \quad (33)$$

Thus, \underline{v}_S is greater or equal than the value of the following problem

$$\begin{aligned} & \min_{\mu, \rho, V_S, m^p, d^p} \mathbb{E}[V_S(\theta)] \\ & \text{subject to (29) – (33)}. \end{aligned} \quad (34)$$

Claim 1. *There exists an optimal solution to the problem (34) such that $m^p \in \mu([0, 1])$, $d^p = \rho(m^p)$, and (31) holds with equality for $\theta \in m^p$.*

Proof. Given ρ and μ that satisfy (29) and (32), define the function

$$h(m) = u_S(\rho(m), \theta(m)) - \int_0^{\theta(m)} \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \quad (35)$$

where $\theta(m) \in m$. Define

$$m_\star \in \arg \min_{m \in \mu([0, 1])} h(m) \text{ and } \theta_\star \in m_\star.$$

Hereafter, we assume that the infimum of h is attained. If the infimum is not attained by any m_\star , a limiting argument, as in the proof of Proposition 1, needs to be made. It is easy to see that $\mu, \rho, m^p = m_\star, \theta^p = \theta_\star, d^p = \rho(m^p)$, and

$$V_S(\theta) = (1 - \delta) \left(u_S(\rho(m^p), \theta^p) + \int_{\theta^p}^{\theta} \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \right) + \delta\underline{v}_S \quad (36)$$

constitute a feasible solution to the problem (34). In particular, (36) clearly satisfies (30), and (31) holds because

$$\begin{aligned} V_S(\theta) &= (1 - \delta) (u_S(\rho(\mu(\theta)), \theta) - (h(\mu(\theta)) - h(m^p))) + \delta\underline{v}_S \\ &\geq (1 - \delta)u_S(\rho(m^p), \theta) + \delta\underline{v}_S, \end{aligned}$$

where the equality follows from (35) and (36), and the inequality follows from (28) evaluated at $\hat{\theta} = \theta_\star$, where (28) holds because (29) and (30) hold.

Suppose for contradiction that there does not exist an optimal solution to (34) with the stated properties. Thus, in an optimal solution, $d^p \notin \rho(\mu([0, 1]))$ and

$$u_S(\rho(\mu(\theta)), \theta) - (h(\mu(\theta)) - h(m_*)) > u_S(d^p, \theta) \text{ for all } \theta \in [0, 1]. \quad (37)$$

There are two cases to consider: $d^p \in (\rho(\mu(0)), \rho(\mu(1)) \setminus \rho(\mu([0, 1])))$ and $d^p < \rho(\mu(0))$ (the case $d^p > \rho(\mu(1))$ is analogous).

Suppose that $d^p \in (\rho(\mu(0)), \rho(\mu(1)) \setminus \rho(\mu([0, 1])))$. Then there exists $\hat{\theta} \in (0, 1)$ such that $d^p \in (\rho(\mu(\hat{\theta}-)), \rho(\mu(\hat{\theta}+)))$. By continuity of u_S and V_S , we have

$$u_S(\rho(\mu(\hat{\theta}-)), \hat{\theta}) - (h(\mu(\hat{\theta}-)) - h(m_*)) = u_S(\rho(\mu(\hat{\theta}+)), \hat{\theta}) - (h(\mu(\hat{\theta}+)) - h(m_*)).$$

Since $u_S(d, \theta)$ is strictly concave in d by Assumption 1 and $h(m)$ is minimized at m_* , this equality is incompatible with (37) evaluated at $\hat{\theta}-$, leading to a contradiction.

Suppose that $d^p < \rho(\mu(0))$. The optimal V_S is such that (31) holds with equality for some θ ,

$$\min_{\theta \in [0, 1]} (V_S(\theta) - (1 - \delta)u_S(d^p, \theta)) = \delta \underline{v}_S,$$

which can be rewritten using (30) as

$$(1 - \delta) \min_{\theta \in [0, 1]} \int_0^\theta \left(\frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) - \frac{\partial u_S}{\partial \theta}(d^p, \tilde{\theta}) \right) d\tilde{\theta} = (1 - \delta)u_S(d^p, 0) + \delta \underline{v}_S - V_S(0).$$

Since $\partial^2 u_S(d, \theta) / \partial d \partial \theta > 0$ and $\rho(\mu(\theta)) > d^p$, the minimum is achieved at $\theta = 0$. Moreover, (37) implies that $u_S(d^p, 0) < u_S(\rho(\mu(0)), 0)$. Therefore, $u_S(\rho_-(0), 0) \leq u_S(d^p, 0)$ because $\rho_-(0) \leq d^p$ by (33), $d^p < \rho(\mu(0))$ by supposition, and u_S is strictly concave in d . So an optimal $d^p < \rho(\mu(0))$ must be given by $\rho_-(0)$ to minimize $V_S(0)$, and thus V_S . But then we can modify μ and ρ only in that μ separates $\theta = 0$ and $\rho(\mu(0))$ is replaced with $\rho_-(0)$. Under this modification, we can support the same V_S given by (30) with $V_S(0) = (1 - \delta)u_S(\rho_-(0), 0) + \delta \underline{v}_S$, leading to a contradiction. \square

Claim 1, together with (36), implies that \underline{v}_S is greater or equal than

$$\begin{aligned} \min_{\mu, \rho, \theta^p} \left\{ u_S(\rho(\mu(\theta^p)), \theta^p) + \mathbb{E} \left[\int_{\theta^p}^\theta \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta} \right] \right\} \\ \text{subject to } \rho(\mu(\theta)) \text{ is nondecreasing in } \theta, \\ w(\rho(m), m) \leq L(\bar{v}) \text{ for all } m \in \mu([0, 1]). \end{aligned} \quad (38)$$

Claim 2. *There exists an optimal solution to the problem (38) that solves the problem (9).*

Proof. Consider an optimal solution (μ, ρ, θ^p) to (38). Without loss of generality,

$$m^p = \mu(\theta^p) = \{\theta : \rho(\mu(\theta)) = \rho(\mu(\theta^p))\},$$

otherwise we can modify the message and decision rules such that all states in $\{\theta : \rho(\mu(\theta)) = \rho(\mu(\theta^p))\}$ are pooled, the same decision $\rho(\mu(\theta))$ is induced for all θ , the constraints of (38) hold, and the value of (38) remains the same. Moreover,

$$\rho(\mu(\theta)) = \begin{cases} \rho_-(\mu(\theta)), & \text{if } \mu(\theta) > m^p, \\ \rho_+(\mu(\theta)), & \text{if } \mu(\theta) < m^p, \end{cases} \quad (39)$$

otherwise we can decrease the value of (38) without violating the constraints either by decreasing $\rho(\mu(\theta))$ for $\mu(\theta) > m^p$ or by increasing $\rho(\mu(\theta))$ for $\mu(\theta) < m^p$.

Finally, suppose for contradiction that $\rho(m^p) \in (\rho_-(m^p), \rho_+(m^p))$. If $\bar{\theta}_* = \sup m^p < 1$, then some states adjacent to m^p from above, say $(\bar{\theta}_*, \bar{\theta}_* + \varepsilon)$, must be pooled, otherwise we can decrease the value of (38) by pooling states $(\bar{\theta}_*, \bar{\theta}_* + \varepsilon)$ and m^p and inducing the same decision $\rho(m^p)$. Similarly, if $\underline{\theta}_* = \inf m^p > 0$, then some states adjacent to m^p from below, say $(\underline{\theta}_* - \varepsilon, \underline{\theta}_*)$, must be pooled. Notice that the objective function in (38) is concave in $\rho(m^p)$; so we can decrease the value of (38) by either decreasing or increasing $\rho(m^p)$, leading to a contradiction. \square

It remains to show that a single-period punishment strategy profile from Proposition 3 can be supported in an equilibrium using the ex-ante transfers $\underline{\mathcal{I}}_0, \underline{\mathcal{I}}_S, \underline{\mathcal{I}}_R$ given by

$$\begin{aligned} \underline{\mathcal{I}}_0 &= \underline{\mathcal{I}}_S = \mathbb{E}[u_S(\rho_*(\mu_*(\theta)), \theta) - \underline{t}_0(\mu_*(\theta))] - \underline{v}_S, \\ (1 - \delta)[\underline{\mathcal{I}}_R + \mathbb{E}[u_R(\rho_*(\mu_*(\theta)), \theta) + \underline{t}_0(\mu_*(\theta))]] + \delta(\bar{v} - \underline{v}_S) &= \underline{v}_R. \end{aligned}$$

The constraints of Condition C7 hold with equality. Condition C6 holds because the continuation payoffs are $v_S(m) = \underline{v}_S$ and $v_R(m) = \bar{v} - \underline{v}_S$. Condition C5 holds because Condition C6 holds and $T_S(m) = T_R(m) = 0$. The sender's (receiver's) constraint of Condition C1 holds with equality for $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}_S$ (for $\underline{\mathcal{I}}_R$). The receiver's (sender's) constraint of Condition C1 holds for $\underline{\mathcal{I}}_0 = \underline{\mathcal{I}}_S$ (for $\underline{\mathcal{I}}_R$) because it can be simplified to $\bar{v} \geq \underline{v}_S + \underline{v}_R$. Condition C2 (a) holds because $\rho_{-j}(\underline{\mu}_{-j}(\theta))$ is nondecreasing in θ and \underline{t}_j satisfies (4). Condition C2 (b) holds because by deviating to a message-transfer pair (\hat{m}, \hat{t}) that is not observed on the equilibrium path, the sender induces $\underline{d}_j^p = \rho_{-j}(\underline{m}_j^p)$, which he can induce more cheaply

on the equilibrium path with message \underline{m}_j^p and zero interim transfer $\underline{t}_j(\underline{m}_j^p) = 0$, as required by (5). Condition C4 (a) holds because $w(\underline{\rho}_j(m), m) \leq L(\bar{v})$ for all $m \in \underline{\mu}_j([0, 1])$. Condition C4 (b) holds because Condition C4 (a) holds and $\underline{d}_j^p = \underline{\rho}_j(\underline{m}_j^p)$. Condition C3 holds because Condition C4 holds and $\underline{t}_j(m)$ is nonpositive. \square

Appendix C Monotone Persuasion

Proof of Lemma 2. Since $u_*(\theta)$ is twice continuously differentiable in θ almost everywhere, we can integrate by parts twice and write the expected joint payoff as

$$\begin{aligned} \int_0^1 u_*(\theta) dG_P(\theta) &= u_*(\theta) G_P(\theta) \Big|_0^1 - \int_0^1 u'_*(\theta) G_P(\theta) d\theta \\ &= u_*(\theta) G_P(\theta) \Big|_0^1 - u'_*(\theta) \Gamma_P(\theta) \Big|_0^1 + \int_0^1 u''_*(\theta) \Gamma_P(\theta) d\theta \\ &= u_*(1) - u'_*(1)(1 - \mathbb{E}[\theta]) + \int_0^1 u''_*(\theta) \Gamma_P(\theta) d\theta, \end{aligned} \quad (40)$$

where the last equality follows from

$$\Gamma_P(1) = \int_0^1 G_P(\theta) d\theta = \theta G_P(\theta) \Big|_0^1 - \int_0^1 \theta dG_P(\theta) = 1 - \mathbb{E}[\theta].$$

Since only the last term of (40) depends on P , the proposition follows. \square

Lemma 4. For all open $P \subset [0, 1]$,

1. $\Gamma_P(\theta)$ is convex in θ .
2. $\Gamma_{[0,1]}(\theta) \leq \Gamma_P(\theta) \leq \Gamma_\emptyset(\theta)$ for all $\theta \in [0, 1]$.
3. $\Gamma_P(\theta) = \Gamma_\emptyset(\theta)$ if and only if $\theta \notin P$.

Proof. Part 1 holds because $\Gamma_P(\theta) = \int_0^\theta G_P(\tilde{\theta}) d\tilde{\theta}$ and $G_P(\theta)$ is a (non-decreasing) distribution function. For Parts 2 and 3, we first show that

$$\int_0^\theta G_P(\tilde{\theta}) d\tilde{\theta} = \Gamma_P(\theta) \leq \Gamma_\emptyset(\theta) = \int_0^\theta F(\tilde{\theta}) d\tilde{\theta} \text{ for all } \theta \in [0, 1], \quad (41)$$

with equality if and only if $\theta \notin P$. It is sufficient to observe that for each disjoint interval

(ξ_i, ζ_i) of P , we have

$$\begin{aligned} \int_{\xi_i}^{\theta} G_P(\tilde{\theta}) d\tilde{\theta} &= F(\xi_i)(\theta - \xi_i) < \int_{\xi_i}^{\theta} F(\tilde{\theta}) d\tilde{\theta} \text{ for } \theta \in (\xi_i, \mathbb{E}[\theta | (\xi_i, \zeta_i)]), \\ \int_{\theta}^{\zeta_i} G_P(\tilde{\theta}) d\tilde{\theta} &= F(\zeta_i)(\zeta_i - \theta) > \int_{\theta}^{\zeta_i} F(\tilde{\theta}) d\tilde{\theta} \text{ for } \theta \in [\mathbb{E}[\theta | (\xi_i, \zeta_i)], \zeta_i), \\ \int_{\xi_i}^{\zeta_i} G_P(\tilde{\theta}) d\tilde{\theta} &= F(\xi_i)(\mathbb{E}[\theta | (\xi_i, \zeta_i)] - \xi_i) + F(\zeta_i)(\zeta_i - \mathbb{E}[\theta | (\xi_i, \zeta_i)]) = \int_{\xi_i}^{\zeta_i} F(\tilde{\theta}) d\tilde{\theta}, \end{aligned}$$

where each line holds, respectively, because

$$\begin{aligned} F(\xi_i) &< F(\theta) \text{ for } \theta \in (\xi_i, \mathbb{E}[\theta | (\xi_i, \zeta_i)]), \\ F(\zeta_i) &> F(\theta) \text{ for } \theta \in [\mathbb{E}[\theta | (\xi_i, \zeta_i)], \zeta_i), \\ \int_{\xi_i}^{\zeta_i} F(\tilde{\theta}) d\tilde{\theta} &= F(\theta)\theta \Big|_{\xi_i}^{\zeta_i} - \int_{\xi_i}^{\zeta_i} \tilde{\theta} dF(\tilde{\theta}) = F(\zeta_i)\zeta_i - F(\xi_i)\xi_i - (F(\zeta_i) - F(\xi_i))\mathbb{E}[\theta | (\xi_i, \zeta_i)]. \end{aligned}$$

Similarly, the remainder of part 2 that $\Gamma_{[0,1]}(\theta) \leq \Gamma_P(\theta)$ for all $\theta \in [0, 1]$ follows from

$$\begin{aligned} \int_0^{\theta} G_{[0,1]}(\tilde{\theta}) d\tilde{\theta} &\leq \int_0^{\theta} G_P(\tilde{\theta}) d\tilde{\theta} \text{ for } \theta \in (0, \mathbb{E}[\theta]), \\ \int_{\theta}^1 G_{[0,1]}(\tilde{\theta}) d\tilde{\theta} &\geq \int_{\theta}^1 G_P(\tilde{\theta}) d\tilde{\theta} \text{ for } \theta \in [\mathbb{E}[\theta], 1), \\ \int_0^1 G_{[0,1]}(\tilde{\theta}) d\tilde{\theta} &= \int_0^1 G_P(\tilde{\theta}) d\tilde{\theta}, \end{aligned}$$

where each line holds, respectively, because

$$\begin{aligned} G_{[0,1]}(\theta) &= 0 \leq G_P(\theta) \text{ for } \theta \in (0, \mathbb{E}[\theta]), \\ G_{[0,1]}(\theta) &= 1 \geq G_P(\theta) \text{ for } \theta \in [\mathbb{E}[\theta], 1), \\ \int_0^1 G_P(\theta) d\theta &= \theta G_P(\theta) \Big|_0^1 - \int_0^1 \theta dG_P(\theta) = 1 - \mathbb{E}[\theta]. \end{aligned}$$

□

Lemma 5. $P_* = \emptyset$ if and only if $u''_*(\theta) \geq 0$ for almost all $\theta \in [0, 1]$.

Proof. The lemma follows from (12) and Lemma 4. In particular, if $u''_*(\theta) < 0$ for θ in some nonempty interval (ξ, ζ) , then $\Gamma_{(\xi, \zeta)}(\theta) < \Gamma_{\emptyset}(\theta)$ for $\theta \in (\xi, \zeta)$ and $\Gamma_{(\xi, \zeta)}(\theta) = \Gamma_{\emptyset}(\theta)$ for $\theta \notin (\xi, \zeta)$; so $P_* \neq \emptyset$. □

Proof of Proposition 4. By Lemma 4, for any open $P \subset [0, 1]$, we have $\Gamma_{[0,1]}(\theta^L) \leq \Gamma_P(\theta^L) \leq$

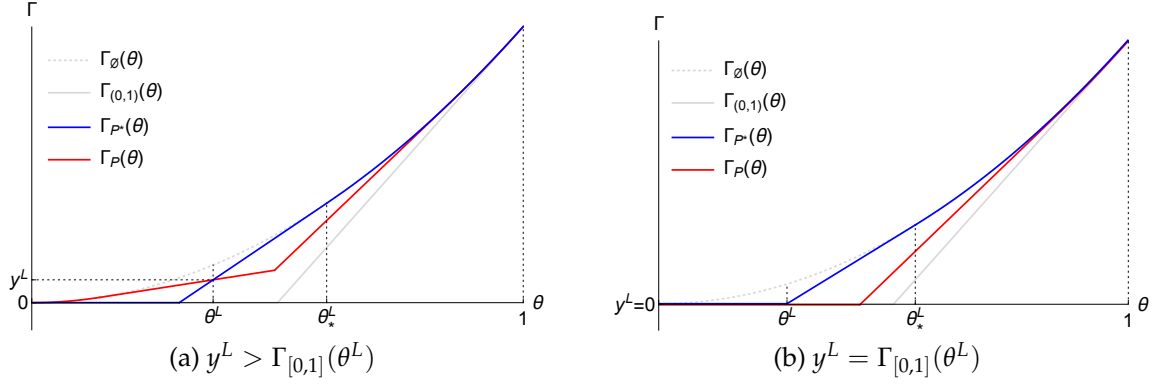


Figure 6: Optimal pooling set $P_* = [0, \theta_*^L]$

$\Gamma_{\emptyset}(\theta^L)$. Fix a value $y^L \in [\Gamma_{[0,1]}(\theta^L), \Gamma_{\emptyset}(\theta^L)]$. Define (see Figure 6)

$$\theta_*^L = \min\{\theta \in [\theta^L, 1] : \Gamma_{[0,\theta]}(\theta^L) = y^L\}.$$

We first show that $P_* = [0, \theta_*^L]$ uniquely solves a constrained problem (12) subject to the additional constraint that $\Gamma_P(\theta^L) = y^L$.

By Lemma 4, for any open $P \subset [0, 1]$ such that $\Gamma_P(\theta^L) = y^L$, we have $\Gamma_P(\theta)$ is convex in θ and $\Gamma_{[0,1]}(\theta) \leq \Gamma_P(\theta) \leq \Gamma_{\emptyset}(\theta)$ for all $\theta \in [0, 1]$. It is easy to verify (see Figure 6) that for any such Γ_P , we have $\Gamma_{P_*}(\theta) \leq \Gamma_P(\theta)$ for $\theta < \theta^L$ and $\Gamma_{P_*}(\theta) \geq \Gamma_P(\theta)$ for $\theta > \theta^L$. Moreover, at least one of the two inequalities is strict for an open interval of θ if $P \neq P_*$. Since $u_*''(\theta) < 0$ for $\theta < \theta^L$ and $u_*''(\theta) > 0$ for $\theta > \theta^L$, the set P_* uniquely solves the constrained problem.

The expected payoff under P_* is

$$v_* = \int_0^1 u_*(\theta) dG_{P_*}(\theta) = F(\theta_*^L) u_*(m_*^L) + \int_{\theta_*^L}^1 u_*(\theta) dF(\theta).$$

Thus, taking into account that

$$\frac{dm_*^L}{d\theta_*^L} = \frac{f(\theta_*^L)}{F(\theta_*^L)} (\theta_*^L - m_*^L),$$

we have

$$\begin{aligned} \frac{dv_*}{d\theta_*^L} &= F(\theta_*^L) u_*'(m_*^L) \frac{dm_*^L}{d\theta_*^L} + f(\theta_*^L) u_*(m_*^L) - f(\theta_*^L) u_*(\theta_*^L) \\ &= f(\theta_*^L) \left(u_*'(m_*^L) (\theta_*^L - m_*^L) + u_*(m_*^L) - u_*(\theta_*^L) \right). \end{aligned} \quad (42)$$

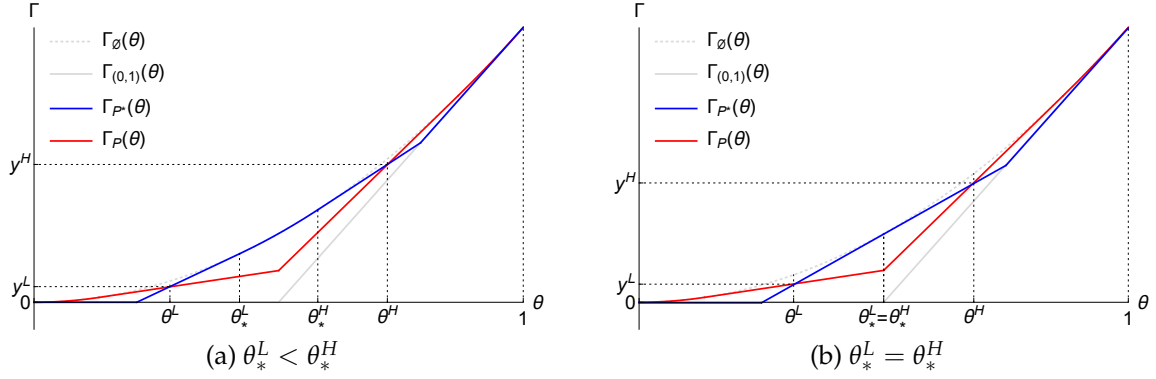


Figure 7: Optimal pooling set P_* given $\theta_*^L \leq \theta_*^H$

Since $u_*(\theta)$ is strictly concave in θ on $[0, \theta^L]$, we have $dv_*/d\theta_*^L|_{\theta_*^L=\theta^L} > 0$ implying that $\theta_*^L > \theta^L$. Further, the necessary first-order condition is $dv_*/d\theta_*^L = 0$ if $\theta_*^L < 1$, which simplifies to (14). By Proposition 3 of Kolotilin (2018), this condition is also sufficient. Also, if (14) holds, then $m_*^L < \theta^L$ because u_* is convex on $(\theta^L, 1]$. If (14) does not hold for any $\theta_*^L \leq 1$, then $dv_*/d\theta_*^L|_{\theta_*^L=1} > 0$; so $\theta_*^L = 1$ and $m_*^L = \mathbb{E}[\theta] \in (0, \theta^L)$. \square

Proof of Proposition 5. Define Y as the set of pairs $(y_L, y_H) \in \mathbb{R}_+^2$ such that $\Gamma_P(\theta^L) = y^L$ and $\Gamma_P(\theta^H) = y^H$ for some open $P \subset [0, 1]$. Fix $(y^L, y^H) \in Y$. We first consider a constrained problem (12) subject to the two additional constraints that $\Gamma_P(\theta^L) = y^L$ and $\Gamma_P(\theta^H) = y^H$.

$$\begin{aligned}
 P_* \in \arg \max_P \int_0^1 u''(\theta) \Gamma_P(\theta) d\theta \\
 \text{subject to } P \text{ is an open subset of } [0, 1], \\
 \Gamma_P(\theta^L) = y^L \text{ and } \Gamma_P(\theta^H) = y^H.
 \end{aligned} \tag{43}$$

Define (see Figure 7)

$$\begin{aligned}
 \theta_*^L &= \min\{\theta \in [\theta^L, 1] : \Gamma_{[0,\theta]}(\theta^L) = y^L\}, \\
 \theta_*^H &= \max\{\theta \in [0, \theta^H] : \Gamma_{(\theta,1]}(\theta^H) = y^H\}.
 \end{aligned}$$

Claim 3. If $\theta_*^L \leq \theta_*^H$, then $P_* = [0, \theta_*^L] \cup (\theta_*^H, 1]$ uniquely solves (43).

Proof. By Lemma 4, for any open $P \subset [0, 1]$ such that $\Gamma_P(\theta^L) = y^L$ and $\Gamma_P(\theta^H) = y^H$, we have $\Gamma_P(\theta)$ is convex in θ and $\Gamma_{[0,1]}(\theta) \leq \Gamma_P(\theta) \leq \Gamma_\emptyset(\theta)$ for all $\theta \in [0, 1]$. It is easy to verify (see Figure 7) that for any such Γ_P , we have $\Gamma_{P_*}(\theta) \leq \Gamma_P(\theta)$ for $\theta \in [0, \theta^L] \cup (\theta^H, 1]$ and $\Gamma_{P_*}(\theta) \geq \Gamma_P(\theta)$ for $\theta \in (\theta^L, \theta^H)$. Moreover, at least one of the two inequalities is strict

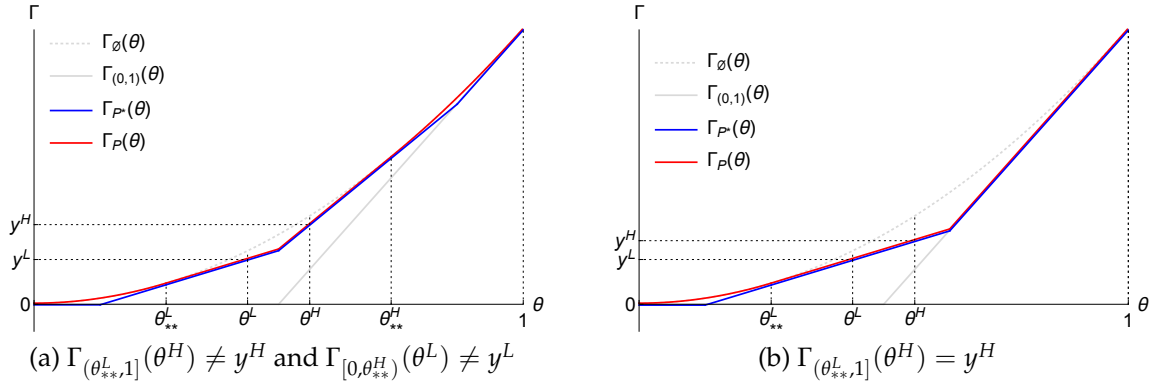


Figure 8: Optimal pooling set P_* given $\theta_*^L > \theta_*^H$

for an open interval of θ if $P \neq P_*$. Since $u''_*(\theta) < 0$ for $\theta \in [0, \theta^L) \cup (\theta^H, 1]$ and $u''_*(\theta) > 0$ for $\theta \in (\theta^L, \theta^H)$, the set P_* uniquely solves (43). \square

Define (see Figure 8a)

$$\begin{aligned} \theta_{**}^L &= \min\{\theta \in [0, 1] : \Gamma_{(\theta, 1]}(\theta^L) = y^L\}, \\ \theta_{**}^H &= \max\{\theta \in [0, 1] : \Gamma_{[0, \theta)}(\theta^H) = y^H\}. \end{aligned} \quad (44)$$

Note that $\theta_{**}^L \leq \theta^L < \theta^H \leq \theta_{**}^H$.

Claim 4. Suppose $\theta_*^L > \theta_*^H$.

1. If $\Gamma_{(\theta_{**}^L, 1]}(\theta^H) = y^H$, then $P_* = [0, \theta_{**}^L) \cup (\theta_{**}^L, 1]$ uniquely solves (43).
2. If $\Gamma_{[0, \theta_{**}^H)}(\theta^L) = y^L$, then $P_* = [0, \theta_{**}^H) \cup (\theta_{**}^H, 1]$ uniquely solves (43).
3. Otherwise, $P_* = [0, \theta_{**}^L) \cup (\theta_{**}^L, \theta_{**}^H) \cup (\theta_{**}^H, 1]$ uniquely solves (43).

Proof. The proof of parts 1 and 2 is analogous to the proof of Claim 3 (see Figure 8b).

We now outline the proof of part 3, omitting tedious details. The reader may refer to Figure 8a for guidance. If $\theta_*^L > \theta_*^H$ with $(y_L, y_H) \in Y$, then

$$y_L + \frac{y^H - y^L}{\theta^H - \theta^L}(\theta - \theta^L) < \Gamma_{\emptyset}(\theta) \text{ for } \theta \in [\theta^L, \theta^H]. \quad (45)$$

Taking into account (45), if $\Gamma_{(\theta_{**}^L, 1]}(\theta^H) \neq y^H$ and $\Gamma_{[0, \theta_{**}^H)}(\theta^L) \neq y^L$ with $(y_L, y_H) \in Y$, then $\theta_{**}^L \in [0, \theta^L)$ and $\theta_{**}^H \in (\theta^H, 1]$. We can then show, using the definitions (10) and (11) of G_P and Γ_P , that $\Gamma_P(\theta^L) = y^L$ and $\Gamma_P(\theta^H) = y^H$ with $(y_L, y_H) \in Y$ if and only if $(\theta_{**}^L, \theta_{**}^H)$ is a disjoint interval in P . By Lemma 4, for any such P , we have $\Gamma_{P^*}(\theta) = \Gamma_P(\theta)$ for $\theta \in$

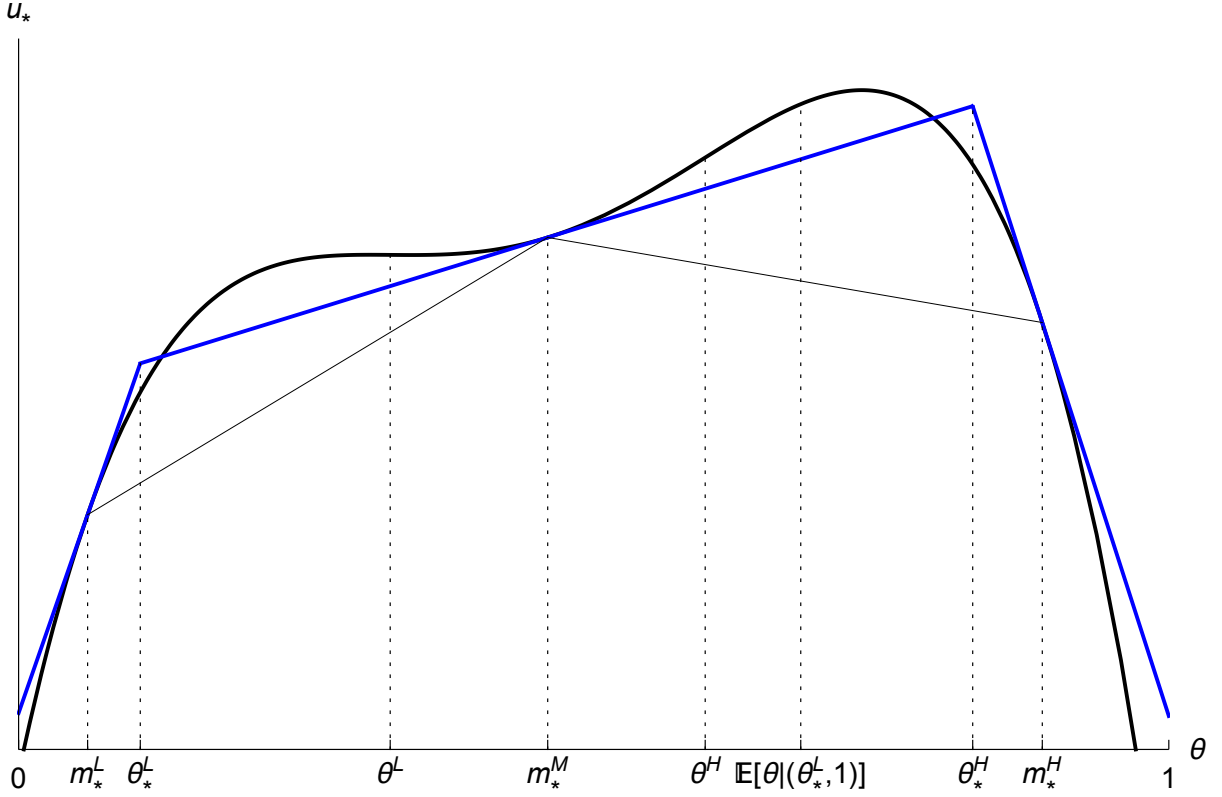


Figure 9: Optimal pooling set $P_* = [0, \theta_*^L] \cup (\theta_*^L, \theta_*^H) \cup (\theta_*^H, 1]$

$[\theta_{**}^L, \theta_{**}^H]$ and $\Gamma_{P_*}(\theta) \leq \Gamma_P(\theta)$ for $\theta \in [0, \theta_{**}^L] \cup (\theta_{**}^H, 1]$. Moreover, the inequality is strict for an open interval of θ if $P \neq P_*$. Since $u_*''(\theta) < 0$ for $\theta \in [0, \theta_{**}^L] \cup (\theta_{**}^H, 1] \subset [0, \theta^L] \cup (\theta^H, 1]$, the set P_* uniquely solves (43). \square

We now consider the original problem (12), without the constraints that $\Gamma_P(\theta^L) = y^L$ and $\Gamma_P(\theta^H) = y^H$.

Claim 5. *If $P_* = [0, \theta_*^L] \cup (\theta_*^L, \theta_*^H) \cup (\theta_*^H, 1]$ with $\theta_*^L < \theta_*^H < 1$ solves (12), then $m_*^M < \theta^L$ and $m_*^H > \theta^H$, where $m_*^M = \mathbb{E}[\theta | (\theta_*^L, \theta_*^H)]$ and $m_*^H = \mathbb{E}[\theta | (\theta_*^H, 1)]$.*

Proof. To prove this claim, we eliminate one by one the cases (i) $m_*^M \geq \theta^L$ and $m_*^H \geq \theta^H$, (ii) $m_*^M \leq \theta^L$ and $m_*^H \leq \theta^H$, and (iii) $m_*^M > \theta^L$ and $m_*^H > \theta^H$.

First, suppose for contradiction that $m_*^M \geq \theta^L$ and $m_*^H \geq \theta^H$. The expected payoff under P_* is

$$v_* = \int_0^1 u_*(\theta) dG_{P_*}(\theta) = u_*(m_*^L)F(\theta_*^L) + u_*(m_*^M)(F(\theta_*^H) - F(\theta_*^L)) + u_*(m_*^H)(1 - F(\theta_*^H)),$$

where $m_*^L = \mathbb{E}[\theta | (0, \theta_*^L)]$. Since θ_*^H is interior, it satisfies the following first-order condition. Taking into account that

$$\frac{dm_*^M}{d\theta_*^H} = \frac{f(\theta_*^H)}{F(\theta_*^H) - F(\theta_*^L)}(\theta_*^H - m_*^M) \quad \text{and} \quad \frac{dm_*^H}{d\theta_*^H} = \frac{f(\theta_*^H)}{1 - F(\theta_*^H)}(m_*^H - \theta_*^H),$$

we have

$$\frac{dv_*}{d\theta_*^H} = f(\theta_*^H) \left(u_*(m_*^M) + u'_*(m_*^M)(\theta_*^H - m_*^M) - u_*(m_*^H) - u'_*(m_*^H)(\theta_*^H - m_*^H) \right) = 0,$$

which can be rewritten as

$$u_*(m_*^M) + u'_*(m_*^M)(\theta_*^H - m_*^M) = u_*(m_*^H) + u'_*(m_*^H)(\theta_*^H - m_*^H). \quad (46)$$

Combining (46) with the fact that u_* is either strictly concave on $(m_*^M, 1]$ or strictly convex on (m_*^M, θ_*^H) and strictly concave on $(\theta_*^H, 1]$, we can show (see Figure 9) that

$$u_*(\theta) > \frac{m_*^H - \theta}{m_*^H - m_*^M} u_*(m_*^M) + \frac{\theta - m_*^M}{m_*^H - m_*^M} u_*(m_*^H) \quad \text{for } \theta \in (m_*^M, m_*^H),$$

which, given $\mathbb{E}[\theta | (\theta_*^L, 1]] \in (m_*^M, m_*^H)$, implies that

$$\begin{aligned} \int_0^1 u_*(\theta) dG_{[0, \theta_*^L] \cup (\theta_*^L, 1]}(\theta) &= u_*(m^L) F(\theta_*^L) + u_*(\mathbb{E}[\theta | (\theta_*^L, 1]]) (1 - F(\theta_*^L)) \\ &> u_*(m^L) F(\theta_*^L) + u_*(m_*^M) (F(\theta_*^H) - F(\theta_*^L)) + u_*(m_*^H) (1 - F(\theta_*^H)) \\ &= \int_0^1 u_*(\theta) dG_{[0, \theta_*^L] \cup (\theta_*^L, \theta_*^H) \cup (\theta_*^H, 1]}(\theta). \end{aligned}$$

This leads to the desired contradiction. A symmetric argument eliminates the case $m_*^M \leq \theta^L$ and $m_*^H \leq \theta^H$.

Finally, suppose for contradiction that $m_*^M > \theta^L$ and $m_*^H > \theta^H$. In the subcase $\theta_*^L \geq \theta^L$, so that $u_*''(\theta) > 0$ for $\theta \in (\theta_*^L, \theta_*^H)$, let $P = [0, \theta_*^L) \cup (\theta_*^H, 1]$. Then $\Gamma_P(\theta) > \Gamma_{P_*}(\theta)$ on the interval (θ_*^L, θ_*^H) and $\Gamma_P(\theta) = \Gamma_{P_*}(\theta)$ everywhere else; so P_* cannot be optimal. In the subcase $\theta_*^L < \theta^L$, so that u_* is strictly concave on (θ_*^L, θ^L) and strictly convex on (θ^L, θ_*^H) , an application of Proposition 4 to the interval (θ_*^L, θ_*^H) implies, for some $\theta_*^M \in (\theta^L, \theta_*^H)$, that the pooling set (θ_*^L, θ_*^M) uniquely solves (12) on the interval (θ_*^L, θ_*^H) . Thus, $P = [0, \theta_*^L) \cup (\theta_*^L, \theta_*^M) \cup (\theta_*^H, 1]$ produces a strictly higher expected payoff than P_* . Both subcases thus lead to contradiction. \square

Claim 6. If $P_* = [0, \theta_{**}^L) \cup (\theta_{**}^L, \theta_{**}^H) \cup (\theta_{**}^H, 1]$ solves (12), and if (44) holds with $y^L = \Gamma_{P_*}(\theta^L)$

and $y^H = \Gamma_{P_*}(\theta^H)$, then $\theta_{**}^L = 0$ and $\theta_{**}^H = 1$.

Proof. Suppose for contradiction that $\theta_{**}^H < 1$. Define $m_{**}^M = \mathbb{E}[\theta | (\theta_{**}^L, \theta_{**}^H)]$ and $m_{**}^H = \mathbb{E}[\theta | (\theta_{**}^H, 1)]$. Then $\theta^H \leq \theta_{**}^H < m_{**}^H < 1$. It is also easy to verify (see Figure 9) that $m_{**}^M \in [\theta^L, \theta^H]$ by definition of θ_{**}^L and θ_{**}^H and given that $\theta_{**}^H < 1$. But this contradicts Claim 5; thus $\theta_{**}^H = 1$. A symmetric argument shows that $\theta_{**}^L = 0$. \square

Combining Claims 3–6, we conclude that P^* takes one of three forms: $[0, \theta_*^L) \cup (\theta_*^H, 1]$ with $\theta^L < \theta_*^L < \theta_*^H < \theta^H$, or $[0, \theta_*^M) \cup (\theta_*^M, 1]$ with $\theta_*^M \in (0, 1)$, or $[0, 1]$.

By Proposition 3 of Kolotilin (2018), $P = [0, \theta_*^L) \cup (\theta_*^H, 1]$ with $\theta^L < \theta_*^L < \theta_*^H < \theta^H$ is optimal if and only if the first-order conditions (16) and (17) hold. Also, if (16) and (17) hold, then $m_*^L < \theta^L$ and $m_*^H > \theta^H$ because u_* is convex on (θ^L, θ^H) ; so part 1 of the proposition follows. If such θ_*^L and θ_*^H do not exist, then P_* takes one of the two remaining forms: either $[0, 1]$ or $[0, \theta_*^M) \cup (\theta_*^M, 1]$ with $\theta_*^M \in (0, 1)$. Clearly, $P = [0, 1]$ is not optimal if and only if (19) holds for some $\theta_*^M \in (0, 1)$. Moreover, $P = [0, \theta_*^M) \cup (\theta_*^M, 1]$ is optimal only if the first-order condition (18) holds. Finally, if $P = [0, \theta_*^M) \cup (\theta_*^M, 1]$ is optimal, then $m_*^L < \theta^L$ and $m_*^H > \theta^H$ by Claim 5. So, parts 2 and 3 of the proposition follow. \square

Appendix D Quadratic Payoffs

Lemma 6. *Under Assumption 3,*

1. $u_*(\theta)$ is continuously differentiable in θ for all $\theta \in [0, 1]$;
2. $u_*(\theta)$ is twice continuously differentiable in θ for almost all $\theta \in [0, 1]$.

Proof. From (20), we have

$$\rho_*(\theta) = \begin{cases} \rho_R(\theta) + \ell, & \text{if } \rho_{FB}(\theta) - \rho_R(\theta) > \ell, \\ \rho_{FB}(\theta), & \text{if } |\rho_{FB}(\theta) - \rho_R(\theta)| \leq \ell, \\ \rho_R(\theta) - \ell, & \text{if } \rho_{FB}(\theta) - \rho_R(\theta) < -\ell, \end{cases} \quad (47)$$

and thus

$$\rho'_*(\theta) = \begin{cases} \rho'_R(\theta), & \text{if } |\rho_{FB}(\theta) - \rho_R(\theta)| > \ell, \\ \rho'_{FB}(\theta), & \text{if } |\rho_{FB}(\theta) - \rho_R(\theta)| < \ell. \end{cases} \quad (48)$$

Further, $u_*(\theta)$ is continuously differentiable in θ for all $\theta \in [0, 1]$, with

$$u'_*(\theta) = \begin{cases} \rho'_{FB}(\theta)\rho_*(\theta) + (\rho_{FB}(\theta) - \rho_*(\theta))\rho'_*(\theta), & \text{if } |\rho_{FB}(\theta) - \rho_R(\theta)| \neq \ell, \\ \rho'_{FB}(\theta)\rho_*(\theta), & \text{if } |\rho_{FB}(\theta) - \rho_R(\theta)| = \ell. \end{cases} \quad (49)$$

Finally, since $\rho_*(\theta)$ is twice continuously differentiable in θ everywhere except at most two states where $|\rho_{FB}(\theta) - \rho_R(\theta)| = \ell$, it follows that $u_*(\theta)$ is twice continuously differentiable everywhere except at most these two states, with

$$u''_*(\theta) = (2\rho'_{FB}(\theta) - \rho'_*(\theta))\rho'_*(\theta) \text{ if } |\rho_{FB}(\theta) - \rho_R(\theta)| \neq \ell. \quad (50)$$

□

Proof of Corollary 2. From (48) and (50), $u''_*(\theta) < 0$ for some $\theta \in [0, 1]$ if and only if $|\rho_{FB}(\theta) - \rho_R(\theta)| > \ell$ and $\rho'_R(\theta) > 2\rho'_{FB}(\theta)$. In this case, $u''_*(\theta) < 0$ in some open interval, because $u''_*(\theta)$ is continuous in θ almost everywhere. Lemma 5 completes the proof. □

Lemma 7. *If the receiver is over-responsive, ℓ is strictly increasing in δ .*

Proof. If $\delta = 0$, then $u''_*(\theta) < 0$ for all $\theta \in [0, 1]$. By Lemmas 2 and 4, the expected joint payoff is strictly higher under $P = [0, 1]$ than under $P = \emptyset$. Thus, Lemma 3 implies that $\bar{v} - \underline{v}_S - \underline{v}_R > 0$ and ℓ is strictly increasing in δ for all $\delta \in [0, 1)$. □

Proof of Proposition 6. By Lemma 5, $P_* = \emptyset$ if $\delta \in (\delta^A, 1)$, and $P_* = [0, 1]$ if $\delta \in (0, \delta^B)$. By Proposition 4, $P_* = [0, \theta_*^L]$ for some $\theta_*^L \in (\theta^L, 1]$ if $\delta \in (\delta^B, \delta^A)$.

Differentiating (42) with respect to ℓ yields

$$\begin{aligned} \frac{d^2 v_*}{d\ell d\theta_*^L} &= f(\theta_*^L) \left(\frac{du'_*(m_*^L)}{d\ell} (\theta_*^L - m_*^L) + \frac{du_*(m_*^L)}{d\ell} \right) \\ &= f(\theta_*^L) \left((\rho'_{FB}(m_*^L) - \rho'_*(m_*^L)) (\theta_*^L - m_*^L) + (\rho_{FB}(m_*^L) - \rho_*(m_*^L)) \right) \\ &< f(\theta_*^L) \left((\rho'_{FB}(m_*^L) - \rho'_*(m_*^L)) (\theta^L - m_*^L) + (\rho_{FB}(m_*^L) - \rho_*(m_*^L)) \right) = 0, \end{aligned}$$

where the inequality holds because $\theta_*^L > \theta^L$ and $\rho'_{FB}(m_*^L) < \rho'_*(m_*^L)$, and the last equality holds because $\rho_{FB}(\theta^L) = \rho_*(\theta^L)$ and $\rho_{FB}(\theta) - \rho_*(\theta)$ is linear in θ for $\theta \in (0, \theta^L)$. So θ_*^L is nonincreasing in ℓ , and $d\theta_*^L/d\ell < 0$ if $\theta_*^L < 1$, as follows, for example, from Theorem 1 of Edlin and Shannon (1998). Further, at $\delta = \delta^B$, we have $\theta^L = 1$ and (42) implies that $dv_*/d\theta_*^L|_{\theta_*^L=1} > 0$. Therefore, θ_*^L reaches 1 at $\delta_*^B > \delta^B$. □

Corollary 3. Suppose the receiver is over-responsive, and $X = [0, \theta^L)$ for some $\theta^L \in (0, 1)$. Keeping θ^L , ℓ , and ρ_R constant, θ_*^L is strictly decreasing in $\alpha = \rho'_{FB}(\theta)$ if $\theta_*^L < 1$. Moreover, $\theta_*^L \rightarrow \theta^L$ as $\alpha \rightarrow 1/2$.

Proof. Notice that $\rho_{FB}(\theta) = \alpha(\theta - \theta^L) + \rho_R(\theta^L)$ for all $\theta \in [0, 1]$. By the same argument as in the proof of Proposition 6, we have $d\theta_*^L/d\alpha < 0$ if $\theta_*^L < 1$ because

$$\begin{aligned} \frac{d^2 v_*}{d\alpha d\theta_*^L} &= f(\theta_*^L) \left(\frac{du'_*(m_*^L)}{d\alpha} (\theta_*^L - m_*^L) - \frac{du_*(\theta_*^L)}{d\alpha} + \frac{du_*(m_*^L)}{d\alpha} \right) \\ &= f(\theta_*^L) \left((\rho_*(m_*^L) + (m_*^L - \theta^L))(\theta_*^L - m_*^L) - (\theta_*^L - \theta^L)\rho_*(\theta_*^L) + (m_*^L - \theta^L)\rho_*(m_*^L) \right) \\ &= -f(\theta_*^L) \left((\rho_*(\theta_*^L) - \rho_*(m_*^L))(\theta_*^L - \theta^L) + (\theta^L - m_*^L)(\theta_*^L - m_*^L) \right) < 0, \end{aligned}$$

where the inequality holds because each term in the parentheses is positive. Finally, if $\alpha \rightarrow 1/2$, then $u''(\theta) \rightarrow 0$ for $\theta \in (0, \theta^L)$ and thus $dv_*/d\theta_*^L|_{\theta_*^L=\theta^L} \rightarrow 0$ by (42), implying that $\theta_*^L \rightarrow \theta^L$. \square

Proof of Proposition 7. By Lemma 5, $P_* = \emptyset$ if $\delta \in (\delta^A, 1)$. By Proposition 4, $P_* = [0, \theta_*^L)$ for some $\theta_*^L \in (\theta^L, 1]$ if $\delta \in (\delta^B, \delta^A)$. Further, by Proposition 6, $d\theta_*^L/d\delta < 0$ if $\theta_*^L < 1$. If θ_*^L reaches 1 at $\delta_*^B > \delta^B$, then $P_* = [0, 1]$ remains optimal for $\delta \in [0, \delta_*^B)$ as follows from the following claim.

Claim 7. If $P_* = [0, 1]$ solves (12) at $\delta \in (0, 1)$, then $\hat{P}_* = [0, 1]$ solves (12) at $\hat{\delta} \in [0, \delta)$.

Proof. Using (12), (48), and (50), we obtain that, for any open set $P \subset [0, 1]$,

$$\begin{aligned} & - \int_{\theta \in \hat{X}} (1 - 2\rho'_{FB}(\theta))(\Gamma_P(\theta) - \Gamma_{[0,1]}(\theta))d\theta + \int_{\theta \notin \hat{X}} (\rho'_{FB}(\theta))^2(\Gamma_P(\theta) - \Gamma_{[0,1]}(\theta))d\theta \\ & \leq - \int_{\theta \in X} (1 - 2\rho'_{FB}(\theta))(\Gamma_P(\theta) - \Gamma_{[0,1]}(\theta))d\theta + \int_{\theta \notin X} (\rho'_{FB}(\theta))^2(\Gamma_P(\theta) - \Gamma_{[0,1]}(\theta))d\theta \leq 0, \end{aligned}$$

where the first inequality holds because $1 > 2\rho'_{FB}(\theta) = 2\alpha$ for an over-responsive receiver, $\Gamma_P(\theta) \geq \Gamma_{[0,1]}(\theta)$ for all $\theta \in [0, 1]$, and $X \subset \hat{X}$ for $\delta > \hat{\delta}$, and the second inequality holds because $P_* = [0, 1]$ solves (12) at δ . \square

Suppose now that $\theta_*^L < 1$ at $\delta = \delta^B$. For $\delta < \delta_B$, the set X consists of two disjoint intervals $[0, \theta^L)$ and $(\theta^H, 1]$. At $\delta = \delta^B$, we have that $\theta_*^L < 1$ satisfies (16) and $\theta_*^H = \theta^H = 1$ satisfies (17). By continuity, there exist $\theta_*^L < \theta_*^H$ that satisfy (16) and (17) for δ in some left neighbourhood (δ_*^C, δ^B) of δ^B ; so, by Proposition 5, $P_* = [0, \theta_*^L) \cup (\theta_*^H, 1]$ with $\theta_*^L < \theta_*^H$ is optimal for $\delta \in (\delta_*^C, \delta^B)$. Moreover, by the same argument as in the proof of Proposition 6, $d\theta_*^L/d\delta < 0$ and $d\theta_*^H/d\delta > 0$ if $\theta_*^L < \theta_*^H$. Thus, at $\delta = \delta_*^C$, the optimal thresholds θ_*^L and

θ_*^H coincide. Notice that $\delta_*^C > 0$, because at $\delta = 0$ the left hand sides of (16) and (17) evaluated at $\theta_*^L = \theta_*^H = \theta_0$ are strictly higher than the right hand sides of (16) and (17), by strict concavity of $u_*(\theta)$ in θ on $X = [0, \theta_0) \cup (\theta_0, 1]$.

For $\delta < \delta_*^C$, there do not exist thresholds $\theta_*^L < \theta_*^H$ that satisfy (16) and (17); so, by Proposition 5, P_* is either $[0, 1]$ or $[0, \theta_M^*] \cup (\theta_M^*, 1]$. By Claim 7, there exists $\delta_*^D \in [0, \delta_*^C]$ such that $P_* = [0, \theta_M^*] \cup (\theta_M^*, 1]$ for $\delta \in (\delta_*^D, \delta_*^C)$ and $P_* = [0, 1]$ for $\delta \in (0, \delta_*^D)$. It remains to show that $\delta_*^D \in (0, \delta_*^C)$. Notice that $\delta_*^D < \delta_*^C$, because at $\delta = \delta_*^C$, the set $P_* = [0, \theta_*^L] \cup (\theta_*^H, 1]$ with $\theta_*^L = \theta_*^H$, which satisfy (16) and (17), yields a strictly higher expected joint payoff than $P = [0, 1]$, as follows from

$$\begin{aligned} \int_0^1 u_*(\theta) dG_{[0, \theta_*^L] \cup (\theta_*^H, 1]}(\theta) &= u_*(m_*^L) F(\theta_*^L) + u_*(m_*^H) (1 - F(\theta_*^H)) \\ &> u_*(\mathbb{E}[\theta]) = \int_0^1 u_*(\theta) dG_{[0, 1]}(\theta), \end{aligned}$$

where the inequality follows from $\mathbb{E}[\theta] \in (m_*^L, m_*^H)$, (16) and (17).

It remains to show that $\delta_*^D > 0$. By Proposition 5, if $[0, \theta_M^*] \cup (\theta_M^*, 1]$ with $\theta_M^* \in (0, 1)$ is optimal at δ , then θ_M^* satisfies (18). It is easy to see that there is no solution θ_*^M to (18) such that $\theta_*^M \rightarrow 0$ or $\theta_*^M \rightarrow 1$ as $\delta \rightarrow 0$, because $u'_*(0)$ and $u'_*(1)$ are finite and $u_*(\theta)$ is strictly concave in θ on $[0, 1]$ at $\delta = 0$. Moreover, by Lemmas 2 and 4, $P_* = [0, 1]$ uniquely solves (12) at $\delta = 0$. It follows by continuity that $\delta_*^D > 0$. \square

Proof of Proposition 8. In the case $\ell > \ell_*^A$, there are no extreme states; so, by Corollary 2, the optimal pooling set P_* is empty. Thus, this case holds.

In the case $\ell \in (\ell_*^B, \ell_*^A)$, there are low extreme states ($\theta^L > 0$) but there are no high extreme states. It is straightforward to show that (14) of Proposition 4 is equivalent to $\theta_*^L = \gamma(\beta - \ell)$. Further, we can show that θ_*^L is decreasing in ℓ , and (given our assumption that $\alpha + \sqrt{1 - 2\alpha}/4 + \beta < 1$) that $\theta_*^L < 1$ when $\ell = \ell_*^B$. Thus, this case holds.

In the case $\ell < \ell_*^B$, there are low and high extreme states ($0 < \theta^L < \theta^H < 1$). It is straightforward to show that (16) and (17) of Proposition 5 are respectively equivalent to $\theta_*^L = \gamma(\beta - \ell)$ and $\theta_*^H = 1 - \gamma(1 - \ell - \alpha - \beta)$. The conditions from Part 1 of Proposition 5 are satisfied if $\theta_*^L < \theta_*^H$, which is equivalent to $\ell > \ell_*^C$. Thus, the case $\ell \in (\ell_*^C, \ell_*^B)$ holds.

Now, consider the case $\ell \leq \ell_*^C$. By Proposition 7 and Lemma 7, there exists $\ell_*^D \in (0, \ell_*^C)$ such that $P_* = [0, 1]$ if $\ell \in [0, \ell_*^D)$ and $P_* = [0, \theta_*^M] \cup (\theta_*^M, 1]$ for some $\theta_*^M \in (0, 1)$ if $\ell \in (\ell_*^D, \ell_*^C)$. If $P_* = [0, \theta_*^M] \cup (\theta_*^M, 1]$, then Proposition 5 implies that $\theta_*^M/2 \in (0, \theta^L)$ and $(\theta_*^M + 1)/2 \in (\theta^H, 1)$, and thus that the expected joint payoff under such P_* is

$$\int_0^{\theta_*^M} u_* \left(\frac{\theta_*^M}{2} + \ell, \theta \right) d\theta + \int_{\theta_*^M}^1 u_* \left(\frac{\theta_*^M+1}{2} - \ell, \theta \right) d\theta.$$

Maximization of this expression yields

$$\theta_*^M = \frac{1 - 2\alpha - 16\ell\beta}{2(1 - 8\ell(1 - \alpha) - 2\alpha)}.$$

Further, at the threshold $\ell = \ell_*^D$, the pooling sets $[0, \theta_*^M] \cup (\theta_*^M, 1]$ and $[0, 1]$ yield equal joint expected payoffs:

$$\int_0^{\theta_*^M} u_* \left(\frac{\theta_*^M}{2} + \ell_*^D, \theta \right) d\theta + \int_{\theta_*^M}^1 u_* \left(\frac{\theta_*^M+1}{2} - \ell_*^D, \theta \right) d\theta = \int_0^1 u_* \left(\rho_* \left(\frac{1}{2} \right), \theta \right) d\theta, \quad (51)$$

with $\theta_*^M = \frac{1 - 2\alpha - 16\ell_*^D\beta}{2(1 - 8\ell_*^D(1 - \alpha) - 2\alpha)}$ and $\rho_* \left(\frac{1}{2} \right) = \min \left\{ \frac{\alpha}{2} + \beta, \frac{1}{2} + \ell_*^D \right\}$.

Given $\alpha < 1/2$ and $\alpha + \sqrt{1 - 2\alpha}/4 + \beta < 1$, we can calculate using (51) that

$$\ell_*^D = \begin{cases} \frac{1-2\alpha}{4} & \text{if } \alpha + \beta < 3/4, \\ \frac{1-2\alpha}{16(1-\alpha-\beta)} & \text{if } \alpha + \beta > 3/4; \end{cases}$$

The cases $\ell \in (\ell_*^D, \ell_*^C)$ and $\ell \in [0, \ell_*^D)$ follow immediately. \square

For completeness, we also state the optimal pooling set for values of α and β where separation in the middle with pooling on both sides never occurs.

Proposition 8'. *Suppose $\alpha < 1/2$, $\alpha/2 + \beta > 1/2$, and $\alpha + \sqrt{1 - 2\alpha}/4 + \beta > 1$. Then the optimal pooling set is*

$$P_* = \begin{cases} \emptyset, & \text{if } \ell \in (\ell_*^A, \infty) \\ [0, \theta_*^L] \text{ where } \theta_*^L = \gamma(\beta - \ell), & \text{if } \ell \in (\ell_*^B, \ell_*^A), \\ [0, 1], & \text{if } \ell \in [0, \ell_*^B). \end{cases}$$

and where

$$\gamma = \frac{2(2 + \sqrt{1 - 2\alpha} - 2\alpha)}{3 - 6\alpha + 4\alpha^2} \text{ and } \ell_*^A = \beta \text{ and } \ell_*^B = \frac{\sqrt{1 - 2\alpha}}{2} - (1 - \alpha - \beta).$$

Proof. The cases $\ell \in (\ell_*^A, \infty)$ and $\ell \in (\ell_*^B, \ell_*^A)$ follow a similar argument to that of the corresponding cases from Proposition 8. At $\ell = \ell_*^B$, it is straightforward to calculate that

$\theta_*^L = 1$, and that there are low extreme states but there are no high extreme states. Thus, by Proposition 7 and Lemma 7, $P_* = [0, 1]$ for all $\ell < \ell_*^B$. \square

Two cases of the uniform-distribution setting are not covered by Propositions 8 and 8'. First, if the receiver is not over-responsive, $\alpha > 1/2$, then complete pooling is optimal by Corollary 2. Second, if the sender is downwardly-biased (on average), $\alpha/2 + \beta < 1/2$, then the solution is symmetric to that of the upwardly-biased (on average) sender. Specifically, if the optimal pooling set is P_* for parameter values (α, β) , then the optimal pooling set is $\hat{P}_* = \{\theta : 1 - \theta \in P_*\}$ for parameter values $(\hat{\alpha}, \hat{\beta}) = (\alpha, 1 - \beta - \alpha)$.

Proof of Proposition 9. Define $\bar{\theta}_* = \sup m^p$ and $\underline{\theta}_* = \inf m^p$, where m^p solves (9). We first show by contradiction in each case that if $\bar{\theta}_* < 1$, then $\underline{\rho}(m^p) = m^p - \ell$ and all states $\theta > \bar{\theta}_*$ are separated by $\underline{\mu}$. Suppose that $\underline{\rho}(m^p) \in (m^p - \ell, \bar{\theta}_* - \ell]$. If some states above $\bar{\theta}_*$ are pooled, say (θ_1, θ_2) , we can decrease the value of (9) by separating these states, as follows from

$$a \int_{\theta_1}^{\theta} \left(\frac{\theta_1 + \theta_2}{2} - \ell \right) d\tilde{\theta} > a \int_{\theta_1}^{\theta} (\tilde{\theta} - \ell) d\tilde{\theta} \text{ for } \theta \in (\theta_1, \theta_2).$$

If all states above $\bar{\theta}_*$ are separated, we can decrease the value of (9) by pooling states $(\bar{\theta}_*, \bar{\theta}_* + \varepsilon)$ and m^p , and inducing the same decision $\rho(m^p)$, leading to a contradiction. Next suppose that $\underline{\rho}(m^p) > \bar{\theta}_* - \ell$. Then some states adjacent to $\bar{\theta}_*$ from above, say $(\bar{\theta}_*, \hat{\theta})$, must be pooled, such that $\underline{\rho}(m^p) < (\bar{\theta}_* + \hat{\theta})/2 - \ell$. But then we can decrease the value of (9) by separating states $(\hat{\theta} - \varepsilon, \hat{\theta})$, as follows from

$$a \int_{\bar{\theta}_*}^{\theta} \left(\frac{\bar{\theta}_* + \hat{\theta}}{2} - \ell \right) d\tilde{\theta} > a \int_{\bar{\theta}_*}^{\theta} \left(\frac{\bar{\theta}_* + \hat{\theta} - \varepsilon}{2} - \ell \right) d\tilde{\theta} \text{ for } \theta \in (\bar{\theta}_*, \hat{\theta} - \varepsilon),$$

$$a \int_{\bar{\theta}_*}^{\theta} \left(\frac{\bar{\theta}_* + \hat{\theta}}{2} - \ell \right) d\tilde{\theta} > a \int_{\bar{\theta}_*}^{\hat{\theta} - \varepsilon} \left(\frac{\bar{\theta}_* + \hat{\theta} - \varepsilon}{2} - \ell \right) d\tilde{\theta} + a \int_{\hat{\theta} - \varepsilon}^{\theta} (\tilde{\theta} - \ell) d\tilde{\theta} \text{ for } \theta \in (\hat{\theta} - \varepsilon, \hat{\theta}).$$

Analogously, we can show that if $\underline{\theta}_* > 0$, then $\underline{\rho}(m^p) = m^p + \ell$ and all states $\theta < \underline{\theta}_*$ are separated by $\underline{\mu}$. This implies that either $\underline{\theta}_* = 0$ or $\bar{\theta}_* = 1$.

Thus, the sender's worst equilibrium payoff \underline{v}_S is achieved either by pooling set $[0, \underline{\theta}^L)$ and decision rule $\underline{\rho}(m) = m - \ell$, or by pooling set $(\underline{\theta}^H, 1]$ and decision rule $\underline{\rho}(m) = m + \ell$. Computation reveals that the value of (9) under $[0, \underline{\theta})$ and $\underline{\rho}(m) = m - \ell$ is

$$\underline{v}_S = \frac{\lambda_S}{24} \left(-3(2\ell - \underline{\theta})(4b + 2\ell - \underline{\theta}) + 2a(2 - 6\ell + \underline{\theta}^3) \right), \quad (52)$$

which is smaller than the value of (9) under $(1 - \underline{\theta}, 1]$ and $\underline{\rho}(m) = m + \ell$ for all $\underline{\theta} \in [0, 1]$ if $a/2 + b > 1/2$. Moreover, the value of (52) is minimized for $\underline{\theta} \in [0, 1]$ at either $\underline{\theta} = 0$ or

$$\underline{\theta} = \frac{1 + \sqrt{1 - 8a(b + \ell)}}{2a} < 1,$$

where the inequality follows from the assumption $a/2 + b > 1/2$. Further computation then produces (23). \square

Substituting $\underline{\theta}^L$ in (52) gives an explicit form expression for \underline{v}_S ,

$$\underline{v}_S = \frac{\lambda_S}{6} \left(a - 3a\ell - 6b\ell - 3\ell^2 \right) + \begin{cases} 0, & \text{if } a(b + \ell) > \frac{3}{32}, \\ \frac{\lambda_S}{48} \left(1 + \sqrt{1 - 8a(b + \ell)} - 4a(b + \ell)(3 + 2\sqrt{1 - 8a(b + \ell)}) \right), & \text{if } a(b + \ell) < \frac{3}{32}. \end{cases}$$

The case $a/2 + b < 1/2$, which is not covered by Proposition 9, follows a symmetric argument. In this case, the sender's worst equilibrium payoff is achieved by the decision rule $\underline{\rho}(m) = m + \ell$ and the pooling set $(\underline{\theta}^H, 1]$ where

$$\underline{\theta}^H = \begin{cases} 1, & \text{if } a(1 - a - b + \ell) > \frac{3}{32}, \\ 1 - \frac{1 + \sqrt{1 - 8a(1 - a - b + \ell)}}{2a}, & \text{if } a(1 - a - b + \ell) < \frac{3}{32}. \end{cases}$$

Appendix E Transparency

Proof of Proposition 10. Suppose, for the sake of argument, that $L(\bar{v})$ defined by (1) takes the same value under ψ and $\hat{\psi}$. We will show that the best equilibrium joint payoff is higher and the worst monotone equilibrium payoffs are smaller under $\hat{\psi}$ than under ψ . Specifically, $\hat{v} \geq \bar{v}$, $\hat{v}_R < \underline{v}_R$, and $\hat{v}_S \leq \underline{v}_S$. From (1) it follows that $\hat{L}(\hat{v}) \geq L(\bar{v})$, with strict inequality if $\delta > 0$. The proposition follows easily from this observation.

The best equilibrium joint payoff \bar{v} under ψ can be supported by an equilibrium in single-period punishment strategies such that $\rho_*(\mu_*(\theta))$ is induced in each period on the equilibrium path, by application of Proposition 3 to each realization of signal ψ . Since $\rho_*(\mu_*(\theta))$ is nondecreasing in θ on $[0, 1]$, it can be supported in an equilibrium, without money burning, under less informative signal $\hat{\psi}$ by application of an analogue of Proposition 1 to each realization of signal ψ ; so, $\hat{v} \geq \bar{v}$.

By Proposition 3, the receiver's worst equilibrium payoffs under $\hat{\psi}$ and ψ are

$$\hat{v}_R = \mathbb{E}[u_R(\rho_R(\hat{\psi}(\theta)), \theta)] < \mathbb{E}[u_R(\rho_R(\psi(\theta)), \theta)] = \underline{v}_R,$$

where the inequality holds because ψ is strictly more informative than $\hat{\psi}$.

By a similar argument to the proof of Proposition 3, the sender's worst equilibrium payoff under ψ can be supported by $\tau_S = 0$, $T_S(m) = 0$, and $v_S(m) = \underline{v}_S$; that is, the sender may refuse to make any ex-ante or ex-post transfers, and the worst punishment for him would involve zero transfers from the receiver and the worst continuation value. Let $\mu(\theta)$ and $\rho(m)$ be penal message and decision rules that support this equilibrium. By assumption $\rho(\mu(\theta))$ is nondecreasing in θ . Then the interim transfer $t_S(\mu(\theta))$ is defined by (4) and (5) given that the set of states is $\psi(\theta) \subset [0, 1]$ rather than $[0, 1]$:

$$\begin{aligned} t_S(m) &= h(m) - \min_{m \in \mu(\psi(\theta))} h(m), \\ h(m) &= u_S(\rho(m), \theta(m)) - \int_0^{\theta(m)} \frac{\partial u_S}{\partial \theta}(\rho(\mu(\tilde{\theta})), \tilde{\theta}) d\tilde{\theta}, \end{aligned} \tag{53}$$

where $\theta(m) \in m$. The message and decision rules $\mu(\theta)$ and $\rho(m)$ such that $\rho(\mu(\theta))$ is nondecreasing in θ can be supported in equilibrium under $\hat{\psi}$ using the interim transfer rule $\hat{t}_S(m)$ that differs from $t_S(m)$ given by (53) only in that the minimum of h is taken over $m \in \mu(\hat{\psi}(\theta))$ rather than over $m \in \mu(\psi(\theta))$. Since $\psi(\theta) \subset \hat{\psi}(\theta)$ for all $\theta \in [0, 1]$ by the definition of more informative signals, we have $\hat{t}_S(\mu(\theta)) \geq t_S(\mu(\theta))$ for all $\theta \in [0, 1]$, and thus

$$\hat{v}_S \leq \mathbb{E}[u_S(\rho(\mu(\theta)), \theta) - \hat{t}_S(\mu(\theta))] \leq \mathbb{E}[u_S(\rho(\mu(\theta)), \theta) - t_S(\mu(\theta))] = \underline{v}_S, \tag{54}$$

which completes the proof. □

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