Contributing to peace

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Abstract

Contest theory analyses an anarchic economy where agents use resources in acquisitive conflict as well as for consumption, and explores condition for peace or conflict to prevail in equilibrium. History indicates that peacekeepers in the shape of kings, dictators or states often endogenously arise in such circumstances. I analyse an extended version of the canonical Tullock contest in which each of the potential contestants first has the option of contributing some resources to a neutral peacekeeper, and then allocates her remaining resources between arms and consumption. In the subsequent subgame, if one of the contestants attacks the other, then the peacekeeper joins its resources with the agent that is attacked. I show that, for less unequal resource distributions, contribution to peacekeeping is positive and subsequently leads to peace. The deterrence equilibria are pareto-superior to the corresponding equilibria of the pure Tullock contest except in a narrow range. However, no contributions are made when the endowment distribution is sufficiently unequal, and conflict occurs in equilibrium.

Keywords: Tullock contest, peacekeeping, voluntary contribution.

1 Introduction

Rulers throughout history have derived their legitimacy and authority from the ability to perform two functions, those of peacekeeper and protector. The primary legitimacy of the contemporary nation-state is also founded on these roles. Citizens, even individually powerful citizens, pay taxes to the state to enable it to curb their natural tendencies to encroach on each other, and of outsiders to encroach upon them. Conversely, a mechanism that keeps these tendencies in check, both internally and externally, for a multiplicity of agents enhances the well-being of those agents and satisfies the rudimentary definition of a state.

In this paper we focus on the function of peacekeeping; protecting property rights and maintaining peace between agents who might otherwise engage in conflict in a condition of anarchy. In order to maintain peace in a potentially predatory setting a ruler or state must possess two important properties. First, it must generate adequate resources from its subjects to be able to contain and even dissuade conflict. Further, it must be in the peacekeeper's interest to use the resources to indeed contain conflict, rather than for the purposes of self-aggrandisement. Empires and kingdoms in their prime, as well as successful contemporary states, satisfy the second condition because long-term benefits from preserving their rule overshadow the short-term benefits of pillage and plunder. Conversely there have been pillagers and plunderers that do not rule, and are held at bay by walls painstakingly built by the same kings and emperors. In this paper I explore the first condition—circumstances in which potential contestants voluntarily contribute sufficient resources to maintain a central peacekeeping mechanism.

The phenomenon of the emergence of stable property rights out of anarchy has exercised the imagination of social thinkers from the very earliest times. In the absence of a mechanism to protect property, warring adversaries are likely to engage in destructive conflict, or expend substantial resources to defend against aggression. Thus agents have a common interest in collaborating to develop an enforcement mechanism to keep their mutual aggressive inclinations in check. Enforcement mechanisms with varying properties have been provided by kings, emperors and nation-states that sometimes commanded the love and respect of their subjects, as well as by despots and dictators that elicited resentment and hatred.

Security of property is inseparable from internal order, which has historically been fundamental for the stability of kingdoms. The prosperity of the Roman Empire was founded upon a lasting peace between its subject states, which allowed trade and commerce to flourish. During the feudal period in Europe, potential conflict between great lords was often held in check by a monarch, funded by tribute from those very lords. At a local level, markets and trade fairs extended this peace to merchants and traders. At the other extreme, empires that were built on conquest without concern for peacekeeping, like that of Alexander the Great, crumbled with the passing of the conqueror.

In contemporary economic literature, the rational contest model (Tullock, 1980) is a construct that has been widely used to investigate conflict (as well as lobbying contests and patent races). Competing agents can use resources in their possession to engage in production, or to wrest away resources from other agents. Consequences of various formulations of the nature of contests are explored in papers such as Hirshleifer (1991, 1995), Skaperdas (1992), Grossman and Kim (1995), and several others.

In the literature that has developed around the Tullock contest, the primary concern is to investigate technological conditions under which, in the absence of an external enforcer, the potential contestants will enter into active conflict, and conditions under which they will coexist in peace. In Hirshleifer's (1991) formulation resources that are devoted to conflict can be used both for aggression and defence, thus an investment to dissuade the adversary may also turn out to provide incentive for aggression. Grossman and Kim (1995) consider investments that are earmarked for aggression (e.g., cannons) or defence (e.g., fortification) and obtain equilibria in which peace may sometimes prevail. Baliga and Sjöström (forthcoming) explore more complex interactions between conflict strategy and technology. Skaperdas (1992) analyses a model in which contestants possess resources that are complements in production of a consumption good, but can also be used in conflict over allocation of the jointly produced good. Peace occurs when the productivity of resources in production is sufficiently larger than their effectiveness in conflict. Peaceful outcomes are also often more efficient than outcomes that involve armed conflict.[1] The consequences of specific contest success functions are examined in Skaperdas (1996) and Hwang (2012), among others.

Surprisingly, however, very few contributions in this literature explore the viability of peacekeeping institutions or self-enforcing cooperative arrangements to deter aggression. Meirowitz, Morelli, Ramsay, and Squintani (2019) explore how third-party institutions that resolve or mitigate the consequences of destructive disputes influence the conflict strategies of contestans. McBride, Milante, and Skaperdas (2011), is one of the few papers that explore the possibility that contestants can invest in the establishment of a state, which is able to protect from conflict a fraction of all resources (see also McBride and Skaperdas, 2007). Konrad and Skaperdas (2012) analyze a scenario where a number of producers face the threat of extortion from external aggressors ('bandits'), and compare collective provision of security with provision of security for profit by a private provider. The profitability of private provision (a "king") is also exlored in Grossman (1998). This is far from

[1]Peace obtains under less stringent conditions in dynamic infinite-horizon models, where cooperative equilibria can be supported with elaborate punishment strategies (see, for example, Powell, 1993).

an exhaustive list, but the underrepresentation in the literature of publicly funded peacekeeping arrangements is nevertheless striking, given that such arrangements have held sway across much of human civilization over much of the time.

In the very simple extension of the Tullock contest presented here, the two potential contestants choose to make contributions to enable a peacekeeper, who uses those resources to arm itself. In the subsequent subgame each contestant can allocate their resources between consumption and private arms with which she can attack the other. The fruits of successful aggression is the capture of the adversaries consumption, while failure results in one's own consumption resources being forfeited. If one of the contestants chooses to be an aggressor (and the other does not), then the peacekeeper contributes its resources to the defence of the victim. It follows that the potential aggressor would be dissuaded from attacking if the peacekeeper and the defender together have sufficient arms to render aggression unattractive.

Now suppose that the peacekeeper alone receives sufficient resources such that each contestant finds aggression unattractive, even when his rival devotes no resource to arms. Then it is an equilibrium for neither agent to devote resources to arms, and we must have peace in this equilibrium. Such an equilibrium can obtain if each contestant finds it more profitable to make the corresponding contribution and consume his remaining resources rather than to contribute nothing and arm for a contest. Note that if one agent contributes less than the required amount, then the deterrent effect of the peacekeeper is reduced, and that agent also has more resources available to arm and attack. Thus there are tradeoffs and non-trivial strategic concerns involved.

I find nevertheless that the only peace equilibria are of this class, where the peacekeeper is endowed with strong deterrent capabilities and neither contestant devotes resources to arms. Further, peace prevails except when there is extreme inequality in initial endowments between the agents. In the latter case agents no longer contribute to peacekeeping in equilibrium. With appropriate investments, peace becomes incentive compatible for two reasons; first, resources invested in peacekeeping are no longer available as conflict payoffs to the contestants, making conflict less attractive, and secondly the same defence investment acts as a deterrent against aggression by both contestants.[2]

Contrast this with the case where there is no peacekeeper and the corresponding arms were in the possession of the defender. The aggressor would then similarly have no incentive to attack because this is not a profitable option. But if the aggressor did not arm at all, then he would be vulnerable to an attack from the erstwhile defender, and the roles would be reversed. Thus both

[2]The "peacekeeper" is conceived here as a very rudimentary instance of a participatory government. However, a government is by nature a more complicated construct than a mere peacekeeper, and I have refrained from using the term because it will likely confuse more than it will illuminate.

parties must arm. By endowing the peacekeeper, the defender commits to not use the defensive forces in aggression, and further provides a foil against his own aggressive instincts. Since the same resources can be used to defend against aggression by either party, peace with a peacekeeper is less costly than peace with mutual deterrence, even when the latter is possible.

Two further observations are of interest. First, for a large range of parameter values there are multiple equilibria, but the most efficient equilibrium is always the one in which the richer agent makes the largest contribution. If we interpret contributions as taxes that maintain the state, then the most progressive taxation scheme turns out to be the most efficient. Secondly, we find that there is a range where inequality is high (but not sufficiently extreme for peacekeeping to break down) where a peace equilibrium prevails, but a contest would in fact be more efficient. In this case an inefficiently large peacekeeping force is maintained in equilibrium by contributions from the rich agent, who stands to lose from conflict.

This paper complements many of the papers mentioned earlier, which find that conflict is more likely when there is high inequality between the agents, and that in these cases the poorer agent is more likely to be the aggressor.[3] Some of the intuition in the present paper is close to Beviá and Corchón (2010), who consider the possibility that the richer agent may transfer some of her wealth to the poorer in order to avoid conflict. Such transfers reduce inequality and therefore the likelihood of conflict. It can be shown that if this option were available in the present model, the richer agent would prefer to secure peace by contributing to peacekeeping than by making transfers to the poorer agent when inequality is low, and would be indifferent between the two options when inequality is high (but not extreme).

The next section describes the model and identifies the subgame that constitutes the canonical contest model in its simplest form. Section 3 analyses the subgame that follows after positive contributions are made to peacekeeping. Section 4 characterises the outcomes that follow after investments that ensure mutual deterrence. Section 5 establishes the equilibria and discusses efficiency properties. Section 6 shows that the results are robust to re-specifications of the payoff function that some may find more realistic. Section 7 concludes.

2 The model and preliminaries

There are two agents in the economy, 1 and 2. $N = \{1, 2\}$ is the agent (or player) set. The economy is endowed with a quantity R of resources, which is normalised to unity. The resources are initially

[3] However, in an experimental setting, Prasada and Bose (2018) find that the greatest amount of conflict occurs when the players have only slightly unequal endowments.

distributed between the agents as R_1 and R_2, with $R_1 + R_2 = 1$, and $\mathbf{R} = (R_1, R_2) \gg 0$.

Notation *For any pair of player-indexed quantities* (q_1, q_2)*, we will denote* $\mathbf{q} = (q_1, q_2)$ *and* $q = q_1 + q_2$*. The exception is* $\sigma = (s_1, s_2)$ *which represents a strategy-profile.*

Each agent $i \in N$ can allocate her resources between three uses, (i) contributions to public peacekeeping denoted by g_i, (ii) private arms to attack the other agent or defend against such attacks, denoted by x_i, and (iii) the remaining resources $R_i - (g_i + x_i)$ to consumption goods. The actual consumption enjoyed by an agent is determined by the outcome of the game described below.

Informally, the game proceeds as follows. First, each player chooses a contribution g_i to peacekeeping. The sum of these contributions determine the resources at the disposal of the peacekeeper, which are converted into arms. Then each player may choose to devote some or all of his remaining resources to private arms. Finally, each player that has devoted positive resources to private arms decides whether to attack the other.

The objective of each player is to maximize her final consumption. If neither player attacks the other, then each player consumes her remaining resources and the game ends. If both players attack, then they play a Tullock contest over the remaining resources using their private arms, and the peacekeeper remains neutral. However, if player i attacks and player j does not, then the peacekeeper adds its arms to the private arms of player j, and the same Tullock contest is played for the remaining resources with these arms. The winner of the contest receives the sum of the remaining consumption resources.[4]

It follows that, if neither player makes a contribution to peacekeeping in the first stage, then the remainder of the game reduces to a Tullock contest with the original resource endowments.

2.1 The game

The players play a one-shot, three-stage game with complete information. After each stage, they observe each others' actions and proceed to the next stage.

2.1.1 Game form

Stage 1 (game Γ): Each agent $i \in N$ simultaneously choose the amount $g_i \in [0, R_i]$ she will contribute to peacekeeping. A pair $\mathbf{g} = (g_1, g_2)$ is a *contribution profile*. The sum of contributions $g = g_1 + g_2$ is the aggregate *contribution* to peacekeeping.

[4]The peacekeeper is part of the game form and not a player, it passively follows the rules. Thus this paper does not provide a theory of the state, only a rationale for citizens to willingly fund its peacekeeping function.

Let $w_i = R_i - g_i$; then w_i is agent i's remaining resources after contributions are made. Denote the *post-contribution allocation* by $\mathbf{w} = (w_1, w_2)$.

Stage 2 (subgame $\Gamma_2|\mathbf{g}$): Agents observe $\mathbf{g}$ and simultaneously choose their arms investments $x_i \in [0, w_i]$. A pair $\mathbf{x} = (x_1, x_2)$ is an *arms profile* (or *arms*). The total arms expenditure is $x = x_1 + x_2$.

Stage 3 (subgame $\Gamma_3|(\mathbf{g}, \mathbf{x})$): Agents observe $\mathbf{x}$. Then they simultaneously choose $a_i \in \{0, 1\}$. 0 is "defend", 1 is "attack". A pair $\mathbf{a} = (a_1, a_2)$ is an *attack profile*. An agent i can choose to attack ($a_i = 1$) only if $x_i > 0$.

2.1.2 Payoffs

Given a play z of the game, payoffs $\Pi(z)$ are determined in the following way.

If neither player attacks the other, then each player consumes his remaining resources, and the peacekeeper plays no role:

$$(a_1, a_2) = (0, 0) \quad \Longrightarrow \quad \Pi_i(z) = R_i - g_i - x_i, \quad i \in N.$$

If both players attack, then also the peacekeeper plays no role. Each player wins with a probability equal to the ratio of his arms in the total. The winner captures the sum of their remaining resources.

$$(a_1, a_2) = (1, 1) \quad \Longrightarrow \quad \Pi_i(z) = \tfrac{x_i}{x_i + x_j}[1 - x - g] \quad i \in N.$$

If one player attacks and the other does not, the winner is again determined as in the previous case, except that now the peacekeeper adds its arms to that of the player that did not attack.

$$(a_i = 1,\ a_j = 0) \quad \Longrightarrow \quad \begin{cases} \Pi_i(z) &= \tfrac{x_i}{x_i + x_j + g}[1 - x - g] \\ \Pi_j(z) &= \tfrac{x_j + g}{x_i + x_j + g}[1 - x - g] \end{cases}$$

Comment: The payoffs above imply that, if i attacks j, the peacekeeper's forces join i's forces in defence, and if i wins the contest then he keeps his own consumption resources and also acquires j's resources. This can be significantly relaxed. In Section 6 we show that the results hold even if j's resources are confiscated or destroyed rather than being allocated to i.

2.1.3 Equilibrium

A strategy for player i is therefore a triple $s_i = \{g_i, x_i(\mathbf{g}), a_i(\mathbf{g}, \mathbf{x})\}$. A strategy profile is represented by $\sigma = (s_1, s_2)$, and the resulting choices by $\mathbf{g}(\sigma)$ or $g_i(\sigma)$ etc. We use $z = [\mathbf{g}, \mathbf{x}, \mathbf{a}]$ to denote an arbitrary play of the game without reference to a strategy, and $z(\sigma) = [\mathbf{g}(\sigma), \mathbf{x}(\sigma), \mathbf{a}(\sigma)]$ to denote a play of the game resulting from σ. We will sometimes drop the argument (σ) when no confusion will arise. The restriction of σ to subgames Γ_2 and Γ_3 are denoted $\sigma|\mathbf{g}$ and $\sigma|(\mathbf{g}, \mathbf{x})$, and similarly for the restrictions of $z(\sigma)$.

Note that, given $\mathbf{R}$, the contribution profile $\mathbf{g}$ determines the post-contribution endowments $\mathbf{w}$. Thus the subgame Γ_2 is completely specified by the sum of contributions g and the post contribution allocation $\mathbf{w}$.

Each agent tries to maximize his payoff which is his final consumption. $\sigma = (s_1, s_2)$ is an *equilibrium* if it is a subgame-perfect Nash equilibrium of the game Γ. The corresponding play $z(\sigma)$ is an *equilibrium outcome.*

2.2 Peace, war and deterrence

We are particularly interested in equilibria in which neither player attacks the other. As Section 2.4 shows, such equilibria generically do not exist in the pure Tullock contest. If they exist in the present game, then it is a consequence of the presence of the peacekeeper and the size of the player contributions. Sufficient peacekeeping resources have a deterrent effect on potentially aggressive players. This section defines important concepts and establishes preliminary results, which we use in the next section to identify a critical level of public contributions that must be made to pre-empt armed conflict.

Definition 1 (Peace and War Equilibria.) *An equilibrium σ is a* peace equilibrium *if, in the associated outcome $z(\sigma)$ we have $\mathbf{a} = (0, 0)$. It is a* war equilibrium *if it is not a peace equilibrium.*[5]

Definition 2 (Deterrence.) *A player i is* deterred *by a contribution profile $\mathbf{g} = (g_1, g_2)$ if, in the subgame $\Gamma_2|\mathbf{g}$, $x_i = 0$, i.e., not devoting any resources to arms, is a best response for i to $x_j = 0$, i.e, when $j \neq i$ chooses not to arm.*

Correspondingly $\mathbf{g}$ is full deterrent *(fd) if both players are deterred in $\Gamma_2|\mathbf{g}$. $\mathbf{g}$ is* minimal full deterrent *(mfd) if there does not exist $\mathbf{g}' \lneqq \mathbf{g}$ which is also full-deterrent. $\mathbf{g}$ is* not full deterrent *(nfd) if at least one player is not deterred in $\Gamma_2|\mathbf{g}$.*

2.3 Preliminary observations

Consider the subgame Γ_3 after an arbitrary history $(\mathbf{g}, \mathbf{x})$, where $g_i + x_i \leq R_i$ for $i \in N$. We first observe that:

[5] Since we are restricting to pure strategies, an equilibrium must be a war equilibrium or a peace equilibrium.

Lemma 1 *Let* $\mathbf{g} \neq 0$*, and let* $z|\mathbf{g} = (\mathbf{x}, \mathbf{a})$ *be an equilibrium of the subgame* $\Gamma_2|\mathbf{g}$*. Then we must have* $\mathbf{a} \neq (1, 1)$*.*

[All proofs are in the Appendix.]

Lemma 1 is self-evident. It says that if a positive peacekeeping contribution has been made then both players will not attack in equilibrium. If one player attacks then the other is better off not attacking since he gets the benefit of the public defence. Further, for tie-breaking reasons we assume:

Assumption 1 *Given* $(\mathbf{g}, \mathbf{x}, a_j)$*, if player* i *is indifferent between setting* $a_i = 0$ *and* $a_i = 1$ *(i.e., between attack and not attack) then he sets* $a_i = 0$*.*

The assumption posits that agents are not inherently warlike. An agent engages in conflict only when it is strictly beneficial, the expected prize is strictly larger than her consumption when she does not attack.

This translates to:

Lemma 2 *Given* $(\mathbf{g}, \mathbf{x})$*, player* $i \in N$ *will attack if and only if* $\frac{x_i}{w_i} > g + x$*.*

If in an equilibrium one player attacks in the last stage, then both will choose their arms optimally in the second stage. These best responses are catalogued in the following lemma.

Lemma 3 (Best war responses) *Let* $\mathbf{g} < \mathbf{R}$ *be arbitrary, and suppose* $\sigma|\mathbf{g}$ *is an equilibrium of* $\Gamma_2|\mathbf{g}$ *with* $a_i(\sigma|\mathbf{g}) = 1$*. Then (dropping the argument* σ *for convenience),*

$$
\begin{aligned}
x_i &= min\{\sqrt{x_j + g} - [x_j + g],\ w_i\} \\
x_j &= max\{\sqrt{x_i} - [x_i + g],\ 0\}
\end{aligned}
$$

These expressions are derived by choosing each agent's arms expenditure to maximize her payoff, and noting that it is i that attacks. Of course, in order to be consistent, the choices must be such that i satisfies the condition in Lemma 2.

2.4 The Tullock contest: $\Gamma_2|(\mathbf{g} = 0)$

The canonical Tullock contest consists of the subgame Γ_2 that follows the contribution profile $\mathbf{g} = 0$, i.e., zero contributions to peacekeeping by both players. In the subgame the players first simultaneously choose private arms $x_i \leq R_i$. They observe $\mathbf{x}$ and then make attack decisions $a_i \in \{0, 1\}$. Note that here it is inconsequential if one or both agents choose to attack, since there are no peacekeeping resources to aid a non-attacker that is attacked. This section summarizes the equilibrium in that game for future reference and for benchmarking purposes.

Proposition 1 *Let σ be an equilibrium in $\Gamma_2|(\mathbf{g}=0)$. Then the following are true.*

(i) $\mathbf{x}(\sigma) \gg 0$.

(ii) If $R_1 \neq R_2$ then exactly one player attacks.

(iii) σ is a peace equilibrium if and only if $R_1 = R_2 = \frac{1}{2}$. In this equilibrium, each player invests $x_i = \frac{1}{4}$ in arms, and is subsequently indifferent between peace and war.

(iv) The payoffs to the two players are as follows:
If $\min\{R_1, R_2\} \geq \frac{1}{4}$, then $\Pi_i(\sigma|(\mathbf{g}=0)) = \frac{1}{4}, \quad i = 1, 2$.
If (wlog) $R_1 = \min\{R_1, R_2\} < \frac{1}{4}$, then $\Pi_1(\sigma|(\mathbf{g}=0)) = \sqrt{R_1}(1-\sqrt{R_1})$, $\Pi_2(\sigma|(\mathbf{g}=0) = (1-\sqrt{R_1})^2$

Notation *The pure contest payoffs listed in Proposition 1 part (iv) will be denoted $\Pi^{contest}$ below.*

Note that when $R_1 = R_2$, the players invest as in the best response war efforts (Lemma 3), and are subsequently indifferent between war and peace. The peace equilibrium is unique only as a consequence of our Assumption 1.

The pure contest payoffs of each player as a function of the player's initial endowment, described in Proposition 1 part (iv) above, will be useful later, and are plotted in Figure 1 below. Note that for an endowment $R_i \in (0, \frac{1}{4})$, player i is better off with a contest than if he were able to consume his entire initial endowment, but this is reversed for $R_i > \frac{1}{4}$.

3 The subgame Γ_2 with positive contributions ($g > 0$)

In this section we turn to the characterization of equilibrium outcomes of the subgame Γ_2 when positive contributions to peacekeeping are made in the first stage. The significance of positive contributions $g > 0$ is that it may dissuade one or both players from investing in private arms, and therefore pre-empt the possibility of war. We establish two primary propositions. First, for each initial endowment vector there is a locus of contribution profiles that are minimal full-deterrent. Secondly, if a full-deterrent contribution is made, then in the ensuing subgame there is a unique peace equilibrium with no investment in private arms by either agent. Finally, if the contribution profile is not full-deterrent, then in the subsequent subgame there are no peace equilibria.

Consider an arbitrary contribution profile $\mathbf{g} = (g_1, g_2)$ that results in a total contribution g. Let

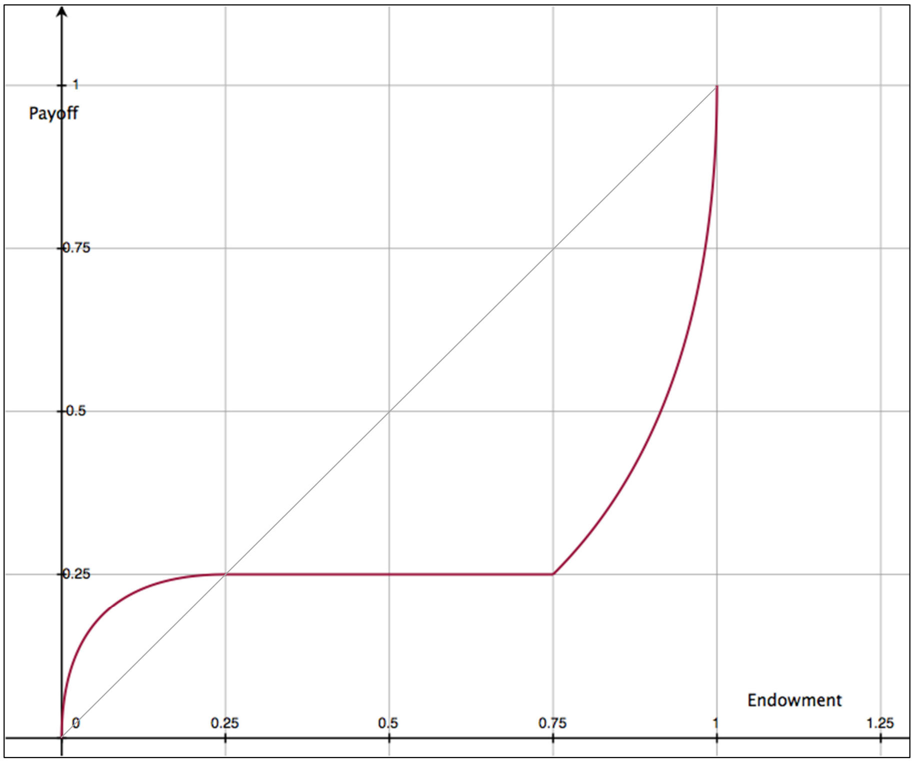

Figure 1: Pure contest payoffs plotted against endowment

the remaining endowments be $\mathbf{w} = (w_1, w_2)$. In the subgame $\Gamma_2|\mathbf{g}$, the players' optimal strategies depend only on $(g, \mathbf{w})$, and hence this is the only information that is relevant to the analysis in the present section.[6]

Recall from Lemma 2 that player i will only attack if $\frac{x_i}{w_i} > g + x$, and this cannot be simultaneously satisfied for both players (see the proof of the lemma). Now suppose that $g > 0$ and player 2 chooses $x_2 = 0$, hence she cannot choose to attack. Player 1 will then compare his current resources w_1 with his maximized war payoff if he arms and attacks. We know that his war payoff is maximized by setting $x_1 = \min\{\sqrt{g} - g, w_1\}$. He attacks if this payoff is greater than w_1, otherwise he, too, sets $x_1 = 0$ and we have peace. This defines a threshold level of g, which we call $\hat{g}(w_1)$, such that 1 will not attack if 2 does not. A similar condition holds for 2. These are the contribution levels that *deter* players 1 and 2 respectively (see Definition 2). Clearly a contribution larger than $\hat{g}_i$ will also deter i (given w_i). Thus the larger of the two thresholds define the lower bound for contributions that are *full deterrent* (Definition 2), written $\hat{g}(\mathbf{w})$.

[6]The subgame Γ_2 is defined fully by the triple $(g, w_1, w_2) \geq 0$ where $w_1 + w_2 = 1 - g$. Note that the same subgame can result from different initial endowments, with appropriate contribution profiles.

Lemma 4 *A contribution profile* $\mathbf{g} = (g_1, g_2)$ *is full deterrent if* $g \equiv g_1 + g_2 \geq \hat{g}(\mathbf{w})$*, where* $w_i = R_i - g_i$*, and*

$$\hat{g}(\mathbf{w}) = \begin{cases} (1 - \sqrt{\min\{w_1, w_2\}})^2 & \text{if } \min\{w_1, w_2\} \geq \frac{1}{4} \\ \frac{1}{2} - \min\{w_1, w_2\} & \text{if } \min\{w_1, w_2\} < \frac{1}{4} \end{cases}$$

A player that retains more resources after contributions are made is deterred by a smaller peacekeeping contribution, since the player has more to lose if he loses the contest, and less to win. In order to ensure full deterrence it is therefore sufficient to deter the player who has the smaller remaining resource endowment $\min\{w_1, w_2\}$ after contributions. Further, the minimum contribution needed for full deterrence increases as $\min\{w_1, w_2\}$ falls. Hence it is intuitive that full-deterrence is attained with the smallest peacekeeping force when both players retain equal resources after contributions. The specific configuration that yields this outcome is $w_1 = w_2 = \frac{4}{9}, g = \frac{1}{9}$. This is catalogued for later reference.

Observation 1 *If* $\mathbf{g}$ *is full-deterrent and* $(w_1, w_2) \gg 0$*, then* $g \geq \frac{1}{9}$*.*

In Assumption 1 we asserted that agents are not inherently inclined to war. In the same spirit we make the further tie-breaking assumption that if $\mathbf{g}$ is full-deterrent and agent j sets $x_j = 0$, then $i \neq j$ chooses not to arm. This is non-trivial only when $\mathbf{g}$ is minimal full deterrent ($g = \hat{g}(w)$) and $w_i \leq w_j$, so i is indifferent between attacking optimally and not arming. When this is the case, we assume that she will indeed choose the peaceful option.

Assumption 2 *If* $g \geq \hat{g}(w)$ *and* $x_i = 0$*, then* j *chooses* $x_j = 0$*; where* $i, j = 1, 2;\ i \neq j$*.*

Lemma 4 and Assumption 2 lead to the following results.

Lemma 5 *If* $\mathbf{g}$ *is full deterrent, then in the subgame* $\Gamma_2|\mathbf{g}$ *there is a peace equilibrium with* $x = 0,\ a = 0$*.*

Lemma 6 *If* $\mathbf{g}$ *is full deterrent, then in the subgame* $\Gamma_2|\mathbf{g}$ *there is no peace equilibrium with* $x \neq 0,\ a = 0$*.*

Lemma 7 *If* $\mathbf{g}$ *is full deterrent, then in the subgame* $\Gamma_2|\mathbf{g}$ *there is no war equilibrium.*

What if a full-deterrence contribution is not made in the first stage? It must then be true that at least one of the players is not deterred by the contribution, and would attack if the opponent did not acquire further arms. However, *a priori* it is possible that the other agent may invest enough in arms such that, together with the public contribution, his total defence is sufficient to deter the first player. Hence it is possible that we could have a peace equilibrium with $g < \hat{g}(\mathbf{w})$ and $\mathbf{x} \gneqq 0$.

The following lemma assures us that such an outcome is not an equilibrium.

Lemma 8 *If g is nfd, then there is no peace equilibrium in the subgame $\Gamma_2|\mathbf{g}$.*

Together these imply that a full deterrent contribution is necessary and sufficient to ensure that there is a unique peace equilibrium in the subsequent subgame.

Proposition 2 *There is a peace equilibrium in the subgame $\Gamma_2|\mathbf{g}$ if and only if $\mathbf{g}$ is full-deterrent. Further, this peace equilibrium is unique and has $(\mathbf{x}, \mathbf{a}) = (\mathbf{0}, \mathbf{0})$.*

It therefore follows that, in equilibrium, either a full-deterrence contribution is made and peace obtains with no further investment in arms, or there is war.[7] In the latter case, it is rational for a player to invest in peacekeeping only if she will not subsequently attack, and even then her contribution could not exceed the amount she would invest in private defense in the absence of public resources. The following proposition formalises this intuition.

Proposition 3 *If σ is an equilibrium of Γ, then either (i) $\mathbf{g}(\sigma)$ is minimal full deterrent and $(\mathbf{x}, \mathbf{a}) = (\mathbf{0}, \mathbf{0})$, or (ii) $\Pi(\sigma)$ is identical to the pure contest payoffs $\Pi^{contest}$.*

4 Deterrence outcomes

Proposition 3 tells us that two kinds of outcomes are possible in equilibrium: either the players receive their pure (Tullock) contest payoffs, or they contribute enough to peacekeeping to achieve a minimal full-deterrence outcome. We already know how endowments map into payoffs under pure contest. In this section we map endowments to feasible full-deterrence payoffs. The next section will identify the endowment distributions for which peace and war prevail, respectively, in equilibrium.

Pick[8] $w_1 \leq \frac{4}{9}$ and consider configurations $(g; w_1, w_2)$ such that $w_1 = \min\{w_1, w_2\}$. Then w_2 lies between w_1 and $(1 - w_1)$, with g correspondingly ranging between $1 - 2w_1$ and 0. Let us evaluate player 1's incentive to arm and attack for different configurations in this range, assuming that player 2 invests no resources in arms. Clearly, when w_2 is at its highest feasible value $1 - w_1$ and g is correspondingly 0, player 1 has the strongest incentive to arm and attack, both because there is much to gain from a victory, and the probability of a victory is large. As w_2 declines and g correspondingly increases, the incentive to arm and attack declines on both counts. By Lemma 4 we know that player 1 is indifferent between attacking and not arming at all when $g = \hat{g}(\mathbf{w})$,

[7] It is possible that for some not-full-deterrent contributions there is no equilibrium in pure strategies in the subgame.

[8] We know from Observation 1 that $\min\{w_1, w_2\} > \frac{4}{9}$ is incompatible with full-deterrence.

and prefers to not arm at all when g increases beyond this value. Thus we can graph the set of pairs $\mathbf{w}$ that correspond to $g \geq \hat{g}(\mathbf{w})$, which are the pairs consistent with full-deterrence. This is summarised in the following proposition.

Proposition 4 *Consider post-contribution allocations* (w_1, w_2) *and associated total peacekeeping contributions* $g = 1 - (w_1 + w_2)$*. W.l.o.g. let* $w_1 = \min\{w_1, w_2\}$*. Then* g *is full-deterrent if and only if*

$$w_2 \leq \begin{cases} \frac{1}{2} & \text{if} \quad w_1 < \frac{1}{4} \\ 2(\sqrt{w_1} - w_1) & \text{if} \quad w_1 \in [\frac{1}{4}, \frac{4}{9}] \end{cases} \tag{1}$$

If $\min\{w_1, w_2\} > \frac{4}{9}$*, then* g *cannot be full-deterrent.*

Replacing the inequalities in equation (1) with strict equalities yields the *full-deterrence frontier*, which is the locus of the maximal post-contribution allocation pairs that are consistent with peace outcomes in the subgame Γ_2. Of course, these correspond to minimal full deterrence contributions. This is graphed in Figure 2.

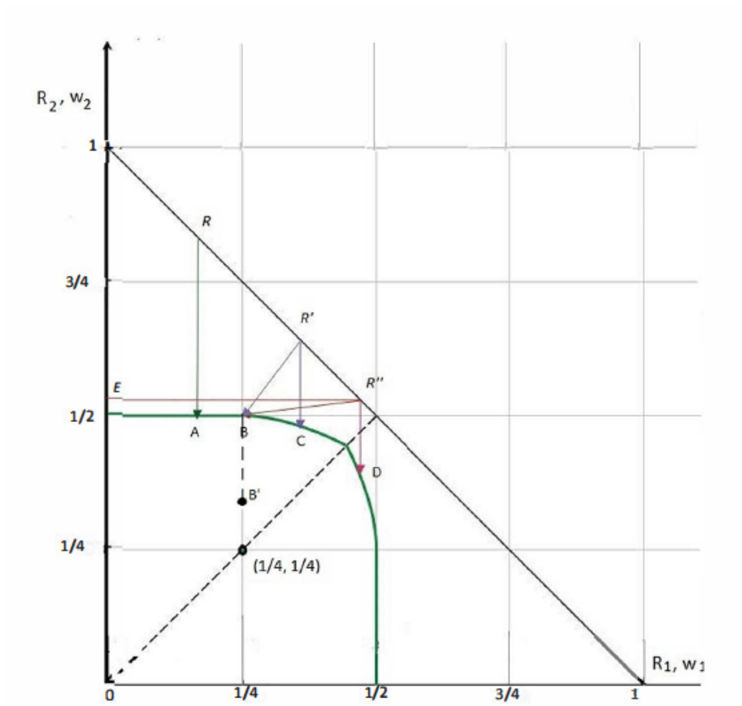

Figure 2: Payoff frontier with full deterrence

The straight line from $(0, 1)$ to $(1, 0)$ in Figure 2 shows the feasible initial endowments. The curved locus from $(0, \frac{1}{2})$ to $(\frac{1}{2}, 0)$ (through the points A, B, C, D) is the upper bound ("frontier") of allocations (w_1, w_2) that result from full-deterrence contributions.[9] To see that allocations below the frontier also induce full-deterrence, note that in an allocation such as B', the public contribution is larger than in B, but $\min\{w_1, w_2\} = w_1$ is the same as in B. Thus since B is compatible with full-deterrence so is B'. We restrict attention to the section of the frontier lying above the 45-degree line, where $R_1 \leq R_2$. The analysis of the complementary segment is symmetrical.

Agents are restricted to non-negative contributions, and transfers between agents are not allowed.[10] Therefore, starting from any initial endowment point, the attainable post-contribution allocations lie in the southwest quadrant relative to that point. The allocations that result from *minimal* full-deterrence are those attainable allocations that lie on the frontier, and are not dominated by another attainable allocation also on the frontier.

For example, starting from the initial distribution R'', all post-contribution allocations in the quadrant $ER''D$ are attainable. However, allocations outside the frontier are not compatible with full-deterrence. Allocations on the frontier that lie on the horizontal section to the left of B are pareto-dominated by B, and each point strictly inside the frontier is dominated by one or more points on the frontier. Thus the minimal full-deterrence outcomes starting from an initial endowment vector R'' are those on the segment BD on the frontier. If the initial endowment is R, however, then there is only a unique attainable mfd allocation A. A corresponding statement is true for any initial endowment with $\min\{w_1, w_2\} \leq \frac{1}{4}$.[11]

Let $G(\mathbf{R})$ denote the set of minimal full deterrence contributions for a given initial endowment $\mathbf{R}$, and let $W(\mathbf{R})$ be the set of allocations that satisfy $\mathbf{w} = \mathbf{R} - \mathbf{g}$ such that $\mathbf{g} \in G(\mathbf{R})$. Then each $\mathbf{w} \in W(\mathbf{R})$ is a consumption pair on the full deterrence frontier. Note that no vector in $W(\mathbf{R})$ (weakly) dominates any other vector in $W(\mathbf{R})$.

In Figure 2, if $\mathbf{R} \gg (\frac{1}{4}, \frac{1}{4})$, then $W(\mathbf{R})$ is the segment of the full deterrence frontier contained in the rectangle defined by $\mathbf{R}$ and $(\frac{1}{4}, \frac{1}{4})$. If $R_1 \leq \frac{1}{4}$, then $W(\mathbf{R}) = \{(R_1, \frac{1}{2})\}$, and if $R_2 \leq \frac{1}{4}$ then $W(\mathbf{R}) = \{(\frac{1}{2}, R_2)\}$. Thus when R_1 or R_2 is $\leq \frac{1}{4}$ the full deterrence consumption vector is unique, but when $R_i \in (\frac{1}{4}, \frac{3}{4}), i = 1, 2$, there is a continuum of such vectors. Using Proposition 4, we can define the range of feasible mfd allocations for player i that correspond to his initial endowment R_i.

Proposition 5 *Let $W(\mathbf{R})$ be the set of minimal full-deterrence allocations corresponding to initial*

[9] Since full-deterrence implies $\mathbf{x} = 0$ in the equilibrium of the subgame, w_i is also the equilibrium consumption of i in the subgame that follows after (w_1, w_2).

[10] See Section 6 for a different formulation.

[11] If transfers between agents were allowed, then starting from R it would have been possible to attain allocation B, which pareto-dominates the (unique) minimal full-deterrence allocation A.

resource endowment $\mathbf{R}$*. Then the set of attainable allocations for Player i in allocation* $\mathbf{w} \in W(\mathbf{R})$ *are (where* $j \neq i$*):*

$$w_i \begin{cases} = R_i & \textit{if} \quad R_i \leq \frac{1}{4} \\ \in [\frac{1}{4}, R_i] & \textit{if} \quad R_i \in (\frac{1}{4}, \frac{1}{2}] \\ \in [\frac{1}{2}\{1 + \sqrt{(2R_j - 1)}\}, \frac{1}{2}] & \textit{if} \quad R_i \in (\frac{1}{2}, \frac{5}{9}] \\ \in [2\{\sqrt{1 - R_j} - (1 - R_j)\}, \frac{1}{2}] & \textit{if} \quad R_i \in (\frac{5}{9}, \frac{3}{4}] \\ = \frac{1}{2} & \textit{if} \quad R_i > \frac{3}{4} \end{cases} \tag{2}$$

For $R_i \in (\frac{1}{4}, \frac{3}{4})$, the set of feasible full-deterrence contributions constitute a non-degenerate compact interval. To attain full-deterrence, a corresponding well-defined contribution must be made by player j. In figure 3 we plot the lower and upper bounds of the payoffs of player i that correspond to this range of contributions, corresponding to i's initial resource allocation R_i. Note that the curvature of the full deterrence frontier in the range $R_1 \in (\frac{1}{4}, \frac{1}{2})$ implies that the contributions of the two players are imperfect substitutes; a reduction in g_2 must be compensated by a more than equal increase in g_1. The reverse is true in the range $R_1 \in (\frac{1}{2}, \frac{3}{4})$, where a reduction in g_2 must be compensated by a less than equal increase in g_1.

5 Equilibria and efficiency

Proposition 5 describes the contribution profiles and consequent allocations that correspond to minimal full-deterrence outcomes. By Proposition 2, if these contributions are made, then a peace equilibrium will prevail in the subgame, and players will receive these allocations as payoffs. Observe that the richer player must always contribute to a full deterrence outcome. The poorer player may not contribute, and will indeed not contribute at all when his initial resource endowment is less than $\frac{1}{4}$. Conversely, if these contributions are not made then in equilibrium players will receive their pure contest payoffs.

Starting with an arbitrary endowment $\mathbf{R}$, there are therefore two classes of candidates for equilibrium. Any or all of the full-deterrence outcomes are potential equilibria, as is the pure contest outcome starting from $\mathbf{R}$. There are no other possible equilibrium payoffs. A full-deterrence outcome will obtain in equilibrium only if each player that is required to make a positive contribution in that equilibrium prefers the outcome to that resulting from pure contest.

Consider an individual player i. Figure 4 superimposes the full deterrence payoffs (Figure 3) to i on the pure contest payoffs (Figure 1) for each level of the initial endowment. The pure contest

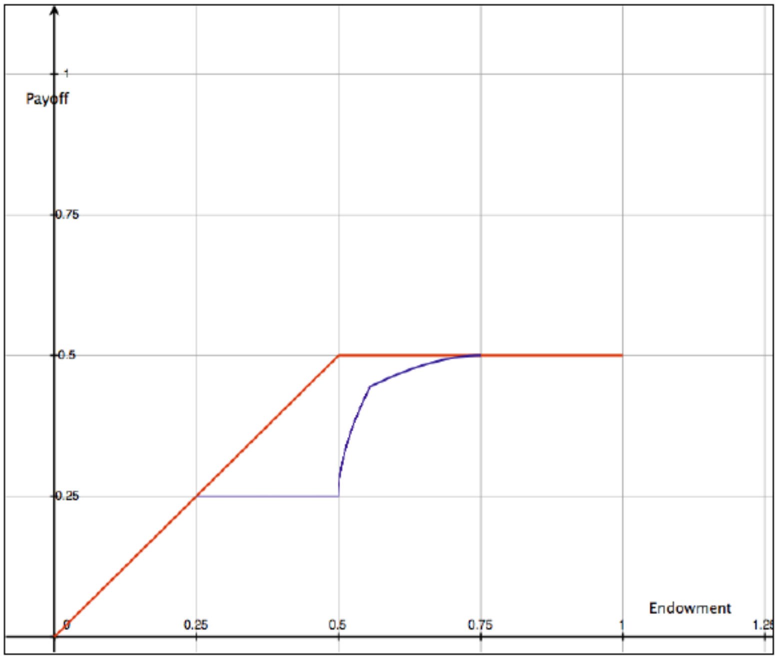

Figure 3: Upper and lower bounds of full-deterrence payoffs plotted against agent's endowment

payoffs are strictly greater than full deterrence payoffs for $R_i \in (0, \frac{1}{4})$, and in $R_i \in (q, 1]$, where $q = \sqrt{2} - \frac{1}{2}$. Thus, in these two regions, i prefers the pure contest outcome to a deterrence outcome.

When $R_i \in (q, 1]$, player j has an endowment $R_j \leq 1 - q = \frac{3}{2} - \sqrt{2} < \frac{1}{4}$. Thus both players prefer the contest outcome to deterrence. Hence each player will offer a zero contribution, and contest is the unique equilibrium.

Note that $q > \frac{3}{4}$. When $R_i \in (\frac{3}{4}, q]$ Player i prefers deterrence even though he has to make the entire deterrence contribution. Player j's endowment is still $R_j \leq \frac{1}{4}$, so j prefers the contest outcome to deterrence. Thus in equilibrium i makes the required contribution, j responds optimally with no contribution, and there is full deterrence in equilibrium.

Finally, for $\max\{R_1, R_2\} \in (\frac{1}{2}, q]$, deterrence is weakly preferred by both players if the richer player makes the maximum contribution, and strictly preferred if the poorer player makes any contribution at all, hence full deterrence is the equilibrium outcome.

This establishes the equilibria corresponding to the different resource endowments, summarised in the following result.

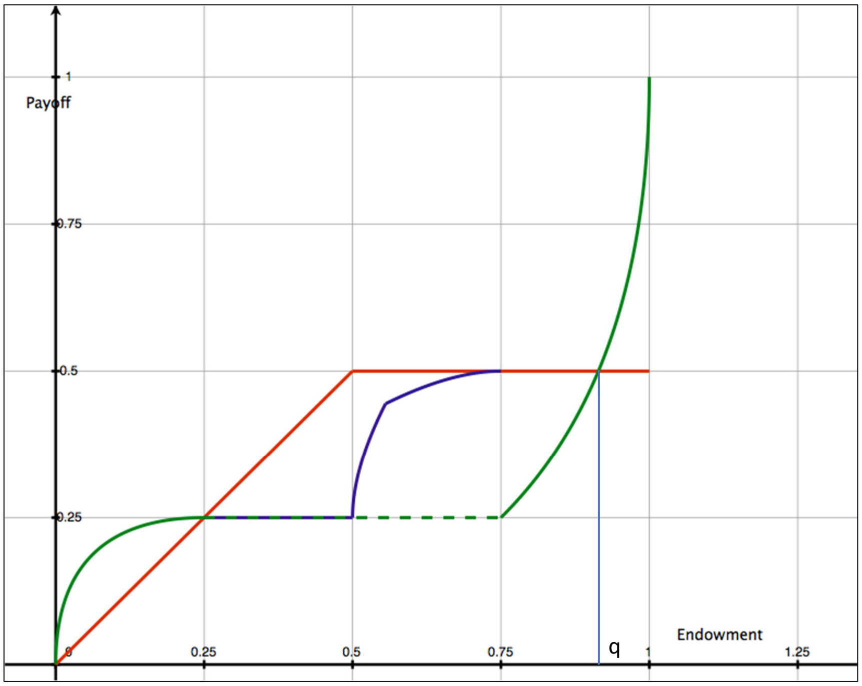

Figure 4: Comparison of payoffs under pure contest and full deterrence

Theorem 1 (Equilibria) *Let $q = \sqrt{2} - \frac{1}{2}$. If $R_1, R_2 \in [1-q, q]$ and $R_1 \neq R_2$, then all equilibria are full-deterrence. If initial endowments are outside these limits then in the equilibrium outcome there is war, and payoffs are equal to the pure contest payoffs for those endowments. If $R_1 = R_2 = \frac{1}{2}$, then there are both full-deterrence equilibria and a war equilibrium, and all of the full-deterrence equilibria pareto-dominate the war equilibrium.*

One should note that either both endowments are within the interval stated in Theorem 1, or both are outside the interval. Further, both endowments are outside the interval, with one within $1 - q$ of zero and the other within that distance of unity, if and only if $|R_1 - R_2| > 2q - 1 = 2[\sqrt{2} - 1]$.

For a given initial distribution of resources within the appropriate interval, each deterrence equilibrium is efficient, since under minimal full deterrence either only one player contributes (when $\min\{R_1, R_2\} \leq \frac{1}{4}$), or the contributions of the two players are (imperfect) substitutes for each other. However, owing to imperfect substitutability, the total consumption in the economy in an equilibrium differs with the allocation of contributions between the two players. A possible measure of aggregate efficiency is total consumption in the economy:

$$c = 1 - g - x.$$

We can compute c in the pure conflict outcome corresponding to each distribution of resources. In full deterrence equilibria $x = 0$, so $c = 1 - g$, hence the equilibrium that maximizes aggregate consumption is the one that minimizes g. But since g is strictly negatively related to $(\min\{w_1, w_2\})$, this is equivalent to maximizing the smaller of the two incomes $\min\{w_1, w_2\}$ subject to full-deterrence. This can be restated as:

Theorem 2 (Rawlsian condition) *For resource distributions that accommodate multiple full deterrence equilibria, the contribution profile that maximizes total consumption in the economy is also the contribution profile that maximises* $\min\{w_1, w_2\}$*, the income of the poorer player, conditional on full-deterrence.*

The proof follows directly from Lemma 4, and is omitted. Theorem 2 says that, for efficient full deterrence, the richer agent must make the maximum contribution consistent with full deterrence. If contributions were levied as taxes by a public authority, then the proposition indicates that the most efficient taxation scheme is one that is most progressive subject to incentive-compatibility.

Finally we note that full deterrence is not efficient over the entire range in which it is an equilibrium. Full-deterrence is more efficient than contest only when the endowment vector is in a sub-interval of the range where full-deterrence is an equilibrium. Thus there is a range of values for $\max\{R_1, R_2\}$ to the left of q (and a corresponding range for $\min\{R_1, R_2\}$ to the right of $1-q$) where the pure contest outcome is more efficient than the equilibrium outcome, but the equilibrium is full deterrence.

Theorem 3 (Inefficient deterrence equilibria) *In the range* $\min\{R_1, R_2\} \in (1-q, 1-\frac{\sqrt{3}}{2}) \Leftrightarrow \max\{R_1, R_2\} \in (\frac{\sqrt{3}}{2}, q)$*, the equilibrium is full-deterrence where pure conflict would yield a more efficient outcome.*

The intuition is that in this range the richer player unilaterally pays for deterrence in equilibrium, because for her the deterrence payoff is larger than the conflict payoff. However, the poorer player would gain relative to his deterrence payoff in a pure contest, and this gain is larger than the loss that the richer player would suffer if pure contest replaced the equilibrium deterrence outcome. Thus contest is more efficient in aggregate, but peace is enforced in equilibrium by the richer player.

Note that the nature of equilibria and the resulting degree of efficiency are determined by the initial inequality in the distribution of endowments. Observation 2 summarises this aspect of theorems 1 and 3 in economically relevant terms.

Definition 3 (Unequal distributions) *The endowment distribution is* extremely unequal *if* $\min\{R_1 - R_2\} < 1 - q$. *The endowment distribution is* highly unequal *if* $\min\{R_1 - R_2\} \in [1 - q,\ 1 - \frac{\sqrt{3}}{2})$. *If the distribution is neither extremely unequal nor highly unequal, then it is* moderate.

Observation 2 (Inequality and conflict) *(i) When the endowment distribution in the economy is moderate, all equilibria in the economy are peace equilibria, and every equilibrium generates greater total consumption than does pure contest.*

(ii) When the endowment distribution is extremely unequal, the unique equilibrium is a pure contest (with no resources devoted to peacekeeping), and this generates greater total consumption than any feasible peace outcome.

(iii) However, when the endowment distribution is highly unequal but not extremely unequal, in equilibrium there is peace, but this generates less total consumption than a pure contest (war) outcome.

6 Two variations

6.1 A confiscating peacekeeper

Throughout this paper I assumed that if there is conflict the peacekeeper sides with the defender against the aggressor, and if the aggressor is defeated the spoils of war are awarded to the defender. However, peacekeepers or protectors in reality, be they governments or international organizations, do not often act in this way. Some or all of the aggressor's wealth may be confiscated, and he may be subjected to penalties or sanctions. Proceeds may sometimes be used to make reparations to the other party. The peacekeeper does not simply reduce to a mercenary army at the service of the agent that has been attacked.[12] Fortunately the equilibrium outcomes in our model are preserved if we modify the payoff function to accommodate this consideration. Consider the following alternative specification of payoffs:

(i) If there is no conflict then each agent consumes any resources that remain after contributions and arms expenditures as before.

(ii) If there is conflict and there is a single aggressor (i.e., $a_1 + a_2 = 1$), then any peacekeeping contributions are pooled with the arms of the agent that is attacked. If the aggressor loses, then his remaining wealth is confiscated by the peacekeeper and destroyed. In particular, the defender is not awarded the aggressor's remaining resources.

[12] For a related concern see Meirowitz, Morelli, Ramsay, and Squintani (2019).

Define the game Γ' as the game form described in Section 2 with these payoffs, which translates to the payoff function Π' described below:

If $(a_1, a_2) = (0, 0)$, then

$$\Pi'_i(z) = R_i - g_i - x_i, \quad i = 1, 2.$$

If $(a_1, a_2) = (1, 1)$ then

$$\Pi'_i(z) = \frac{x_i}{x_i + x_j}[1 - x - g]$$

If $a_i = 1$ and $a_j = 0$, then

$$\begin{aligned} \Pi'_i(z) &= \frac{x_i}{x_i + x_j + g}[1 - x - g] \\ \Pi'_j(z) &= \frac{x_j + g}{x_i + x_j + g}[R_j - g_j - x_j] \end{aligned}$$

Note that the only difference between the two payoff functions is in j's payoff when $a_i = 1$ and $a_j = 0$, where j now only retains her remaining resources after contributions and arms, but does not acquire the consumable resources of the attacker. Then there is a correspondence between the equilibria in Γ and Γ' as follows:

Proposition 6 (The game with confiscation) *: A strategy profile σ^* is a peace equilibrium of Γ' if and only if it is a peace equilibrium of Γ.*
A strategy profile σ^* is a war equilibrium of Γ' if and only if it is a war equilibrium of Γ, and $\mathbf{g}(\sigma^*) = 0$.

We omit the proof, but provide the intuition here. The change in payoffs between Γ and Γ' does not affect the players' equilibrium payoffs in a full-deterrence peace equilibrium. The only payoffs that change in such an equilibrium are off the equilibrium path, where one agent does attack. In this case the defending agent receives a smaller payoff under Γ' than under Γ. A reduction in a player's payoff off the equilibrium path cannot destroy the equilibrium. Thus a peace equilibrium under the old rules must remain an equilibrium under the new rules. The converse is also true, since an attacker's payoff remains the same under both rules, so a change in the defender's payoff cannot induce an agent to change his strategy and attack.

In a war equilibrium, this change from Γ and Γ' does not affect the aggressor's incentives and best responses, since if he is defeated then he loses his remaining resources under this amended payoff function just as he did under the original one, and his payoff is no different if he wins. We therefore need to examine how the change affects the defender. For the defender, it is now less attractive to contribute to peacekeeping. Recall from Proposition 3 that in a war equilibrium under the original payoff rules if there is a positive peacekeeping contribution at all then it comes only from the defender, who is indifferent between putting his defensive resources into the peacekeeper and into his private arms. With the amended payoff function he is no longer indifferent, and puts his entire defensive resources into private arms. Conversely, in a war equilibrium in Γ', neither player invests in the peacekeeper. Thus all war equilibria are pure contests that are not dominated by a peacekeeping outcome, and it is easy to see that these would continue to be equilibria in Γ.

Thus the set of payoffs that correspond to equilibrium outcomes do not change if the peacekeeper confiscates the aggressor's resources rather than channeling them to the defender.[13] With the reformulation of the payoffs in Γ', we only lose those war equilibria in which the defender deployed part of her defensive forces as peacekeeping contributions (see Proposition 3), when they would serve exactly the same purpose if deployed as private arms.[14]

6.2 Transfers between agents

We saw earlier that the amount of resources that must be devoted to deterrence increases as the smaller of the post-contribution endowments declines. Further, we know that in a deterrence equilibrium, the richer agent must always contribute to peacekeeping. It is therefore natural to ask if it may not be more attractive for the richer agent to transfer some resources to the poorer one, instead of contributing those resources to peacekeeping. This question is in the spirit of Beviá and Corchón (2010), who explore the effectiveness of inter-player transfers for avoiding war in a contest economy. In the main specification of their model, for a range of parameters the (richer) potential defender transfers resources to the (poorer) potential attacker until the latter is indifferent between war and peace, which leaves the defender better off.

In the present model, there is a range of endowments in which the rich defender is indifferent between transferring resources to the peacekeeper or the attacker, and if he does transfer to the attacker then the latter is better off. In the rest of endowment range the defender prefers to make the same contributions to the peacekeeper, or not transfer to either the peacekeeper or to his rival. In both cases the equilibria that we identified remain unaffected.

Proposition 7 (Transfers between players) *: Suppose that, before contributions are made to*

[13] This of course generates additional incentives for the peacekeeper, but they are not too relevant here since no attack ever happens when the peacekeeper is endowed with resources.

[14] An alternative formulation would have the defender winning a fraction θ of the aggressor's resources if the defence prevailed. $\theta = 0$ and 1 would correspond to games Γ' and Γ respectively. Then we lose all of the war equilibria in which the defender also endows the peacekeeper as soon as θ falls below unity, which underlines that only the full-deterrence and pure contest equilibria are robust.

the peacekeeper, each player i is allowed to make a transfer t_{ij} to the other player j. Then the equilibria corresponding to the different endowment distributions are as follows (where $q = \sqrt{2} - \frac{1}{2}$, and $R_i = 1 - R_j$):

- *(i) For $R_j \in (0, 1-q)$, player i will make no transfer to player j, and the equilibria in the remainder of the game are unaffected.*
- *(ii) For $R_j \in [1-q, \frac{1}{4})$, player i is indifferent between transferring nothing to player j, and transferring any amount up to $\frac{1}{4} - R_j$. In the subsequent subgame, player i reduces his peacekeeping contribution by an amount equal to the transfer. Player j utility increases by the amount of the transfer, and player i is indifferent.*
- *(iii) For $R_j \in [\frac{1}{4}, \frac{1}{2})$, player i transfers nothing to player j and the remainder of the game remains unaffected.*

We outline the (straightforward) intuition rather than provide a formal proof. For $R_j < 1-q$, we know that i prefers a contest than to peacekeeping. In the contest j is the one that has the incentive to attack, putting all his resources into arms. A transfer to j that still leaves him in this range would only increase j's arms and hence reduce i's expected payoff. A transfer that lifts j into the range $R_j + t_{ij} > 1 - q$, on the other hand, would produce a peace equilibrium in the subgame, with i receiving a payoff of $\frac{1}{2}$ (see Proposition 5 and Theorem 1). From Figure 4, it is clear that i's payoff is larger than $\frac{1}{2}$ in the pure contest with the original endowments. Thus i prefers not to make a transfer.

In the range $1 - q \leq R_j \leq \frac{1}{2}$, on the other hand, the equilibrium without transfers is a peace equilibrium, where i contributes $\frac{1}{2} - R_j$ to the peacekeeper, retaining a consumption of $\frac{1}{2}$. Thus he is indifferent between transferring any amount up to $\frac{1}{4} - R_j$ to j, and contributing the remainder to the peacekeeper.

Once R_j (or $R_j + t_{ij}$) rises above $\frac{1}{4}$, however, there is no longer perfect substitutability between w_j and the peacekeeper's resources. As the curvature of the full-deterrence frontier (see Figure 2) shows, an increase of Δ in $\min\{w_1, w_2\}$ reduces the full-deterrence contribution by less than Δ. This can also be seen from Lemma 4, which shows that in this range

$$\frac{\partial \hat{g}(\mathbf{w})}{\partial \min\{w_1, w_2\}} = 1 - \frac{1}{\sqrt{\min\{w_1, w_2\}}}$$

which must be negative and smaller in magnitude than unity since $\min\{w_1, w_2\} > \frac{1}{4}$.

Recall that in this range there are multiple equilibria, all of them full-deterrence, with the two players contributing different amount of the total which is determined by the smaller of the remaining resource endowments. Let us fix an equilibrium of the original game, and ask if transfers can produce a pareto-improvement. It follows immediately from the observations above that this is not possible.

Thus when inequality is extreme, the possibility of transfers does not rescue the economy from anarchy. Voluntary inter-agent transfers can reduce inequality to produce a pareto-superior outcome in economies that are just this side of anarchy, where inequality is somewhat less than extreme. However, these possibilities disappear as the gap between the incomes narrows, even though it still remains considerable with the poorer agent receiving no more than a quarter of the total endowment.

7 Conclusion

This paper re-examines a standard model of contest in anarchy, where two agents possess resources that can be devoted to consumption or to acquisitive warfare. In the simplest version of that economy, the equilibrium necessarily involves conflict. However, since war is wasteful, it is likely that one or both agents would be willing to precommit some resources to avoid conflict, even if such precommitment is somewhat costly.

This is a context that is intuitively conducive to the genesis of a peacekeeping state. I use a simple model of an exogenous peacekeeper which, to be effective, must be voluntarily endowed with resources provided by the potential contestants. I find that, when inequality is low to moderate (in a sense made precise), all agents find that the existence of a public peacekeeper is in their interest. Hence agents voluntarily commit sufficient resources to the peacekeeper, and the resultant equilibrium is characterised by the absence of conflict. For higher inequality, the poorer agent finds such enforcement contrary to his interest, but the richer agent unilaterally endows the peacekeeper. In part of this range there is peace, but this is less efficient than pure conflict. At very high levels of inequality a peacekeeper is incompatible with the interests of either agent. The peacekeeper is not endowed with resources in equilibrium, and there is conflict.We should thus expect to see the least conflict in more equal societies and the most in very unequal ones. Some of these observations, summarised in Observation 2, may arguably reflect the shapes of some troubled states, where law and order is either absent, or enforced by militias that are maintained by wealthy overlords at great cost to contain the dissent of a discontented population.

It is important to underline the relevant questions that this paper does not address. I have assumed throughout that the peacekeeper acts impartially, even though it may be funded entirely or largely by the richer agent. If instead the agent that contributes more to the peacekeeper can bend the latter to his own purposes, then to that extent the peacekeeper is less impartial, and the nature of the equilibria must be affected. The analysis here needs to be supported by a much more sophisticated political theory of the nature of the state. A concern that naturally springs to mind is that, in order to gain legitimacy, even a government funded entirely by the elite must distance itself to some extent from that elite and assume some semblance of impartiality. Another relevant concern is the number of contestants and the weight of each in the pool of resources yielded up to the peacekeeper. In the introduction I have also alluded to the long-term interests of the peacekeeper, whose long-term revenue streams may well be dependent on the security of property among the subjects and the resulting commercial activity that is generated. Finally, the peacekeeper may also act as a "true" state, providing productive public goods or effecting redistribution using the contributed resources, once it is ascertained that the players have not invested in arms. We leave these concerns for future consideration.

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Appendix: Proofs

Proof of Lemma 1:

(i) If $a_i = 1$, then $j \neq i$ improves his payoff by setting $a_j = 0$ since then in the subsequent contest the public defence is added to j's arms. ■

Proof of Lemma 2*:*

Player i will attack if his payoff from doing so is strictly greater than otherwise, which implies

(i) $\frac{x_i}{g+x}[1-(g+x)] \; > \; w_i - x_i$, and

(ii) $a_j = 0$.

The first condition says that the player's expected consumption from a contest is greater than that when there is no contest. The second follows from Lemma 1 if $g > 0$, and from Assumption 1 otherwise.

The inequality in the Lemma is a rearrangement of Condition (i) above. Condition (ii) follows from the fact that the inequality cannot be simultaneously satisfied for both players. Otherwise suppose that it is; rearrange and add for $i = 1, 2$ to get

$$\begin{aligned} x_1 + x_2 &> (w_1 + w_2)(g + x) \\ \Rightarrow \quad x &> wx + w(1-w) \quad \text{since } g = 1 - w \\ \Rightarrow \quad x &> \begin{cases} x & \text{if } g = 0 \\ w & \text{if } g > 0 \end{cases} \end{aligned}$$

which cannot be true. Thus if i satisfies (i), then we must have $\frac{x_j}{g+x}[1-(g+x)] \; < \; w_j - x_j$, so $j \neq i$ strictly prefers not to attack and (ii) is also satisfied. ■

Proof of Lemma 3*:*

Since i attacks, he uses arms x_i, while j defends with $x_j + g$, and the prize is $1 - g - x$. Let $\Pi_i^A(x_i, x_j; g)$ denote i payoff when he attacks and the arms are as specified. Then i solves:

$$\max_{x_i \leq w_i} \Pi_i^A(x_i, x_j; g) \; \frac{x_i}{g + x}[1 - (g + x)]$$

which yields $x_i = \sqrt{x_j + g} - [x_j + g]$ if the constraint is not binding. The payoff Π is strictly increasing for values of x_i to the left of this, so $x_i = w_i$ when the constrain is binding.

Similarly, let j maximize $\Pi_j^D(x_i, x_j; g)$ which denotes his payoff when defending. This yields $x_j + g = \sqrt{x_i} - x_i$ when the maximum is interior. Note that the resource constraint cannot be binding for j, for if she were to devote all her resources to defence then i would not have an incentive to attack. However, if the peacekeeping contribution g already (weakly) exceeds the optimum defence $\sqrt{x_i} - x_i$

then $x_j = 0$. ■

Proof of Proposition 1*:*

(i) Otherwise suppose $x_i(\sigma) = 0$, then $j \neq i$ can invest a small amount $0 < \epsilon < R_i$ and attack, thereby capturing all of the remaining resources and obtaining consumption in excess of R_j. But then i consumes nothing, and can improve her position by choosing $x_i > 0$.

(ii) Otherwise suppose neither attacks, then from Lemma 2, we must have $\frac{x_i}{R_i} \leq x$ for $i = 1, 2$. Rearranging and adding the conditions for $i = 1, 2$ we get $x \leq x$, which implies that the condition must hold with equality for both players, and hence

$$\begin{aligned} \frac{x_1}{R_1} &= \frac{x_2}{R_2} = x \\ \Rightarrow \quad \frac{x_i}{x} &= R_i \quad i = 1, 2. \end{aligned} \tag{3}$$

With these arms investments, the war payoff $\frac{x_i}{x}[1 - x]$ is equal to the peace payoff $R_i - x_i$, so neither player goes to war. But for this to be equilibrium, it must be further true that neither player prefers to vary her arms investment and go to war, which implies that her war payoff is maximized at the current investment given the opponent's investment. Thus we must have

$$\begin{aligned} \frac{\partial}{\partial x_i}\left[\frac{x_i}{x}(1 - x)\right] &= 0 \\ \Rightarrow \quad \frac{x_j}{x^2} &= 1 \\ \Rightarrow \quad x_j = x^2 &= \frac{x_j^2}{R_j^2} \end{aligned}$$

which from (3) implies that $R_j = \frac{x_j}{R_j}$ for both players $j = 1, 2$. But this can only be true if $R_1 = R_2 = \frac{1}{2}$, and not otherwise.

Further when $R_1 \neq R_2$, it follows from Lemma 2 that exactly one player attacks.

(iii) Follows directly from the proof of (ii) above.

(iv) Follows from substituting Lemma 3 and the specification of the payoffs.

■

Proof of Lemma 4.

Let g be given and set $x_j = 0$. If i attacks with arms x_i, then his expected payoff is

$$\Pi_i(x_i, g) \;=\; \frac{x_i}{x_i + g}[1 - (x_i + g)].$$

This is maximised at $\tilde{x}_i(g) = \sqrt{g} - g$, and is increasing in x_i for $x_i < \sqrt{g} - g$. Thus i will choose to arm up to $x_i(g, w_i)$ where

$$x_i(g, w_i) = \begin{cases} \sqrt{g} - g & \text{if} \quad w_i \geq \sqrt{g} - g \\ w_i & \text{if} \quad w_i < \sqrt{g} - g \end{cases}$$

His corresponding payoff can be obtained from the previous two expressions:

$$\Pi_i(g, w_i) = \begin{cases} (1 - \sqrt{g})^2 & \text{if} \quad w_i \geq \sqrt{g} - g \\ \frac{w_i}{w_i + g}[1 - (w_i + g)] & \text{if} \quad w_i < \sqrt{g} - g \end{cases}$$

Define $g(w_i)$ as the value of g that solves $w_i = \Pi_i(g, w_i)$. Since $\Pi_i(g, w_i)$ is decreasing in g, it follows that g deters i if and only if $g \geq g(w_i)$.

Now let $g = g(w_i)$.

First suppose $w_i \geq \sqrt{g} - g$, so $w_i = \Pi_i(g, w_i) = (1 - \sqrt{g})^2$. Then $(1 - \sqrt{g})^2 \geq \sqrt{g} - g \quad \Rightarrow g \leq \frac{1}{4} \Rightarrow w_i \geq \frac{1}{4}$. Conversely note that $\sqrt{g} - g \leq \frac{1}{4} \;\; \forall g \in [0, 1]$, so if $w_i \geq \frac{1}{4}$ then $\Pi_i(g, w_i) = (1 - \sqrt{g})^2$. Hence it follows that if $w_i \geq \frac{1}{4}$ then $g(w_i) = (1 - \sqrt{w_i})^2$.

Next let $w_i < \sqrt{g} - g$ so $w_i = \frac{w_i}{w_i + g}[1 - (w_i + g)] \quad \Rightarrow \quad g = \frac{1}{2} - w_i$. Then $w_i < \sqrt{\frac{1}{2} - w_i} - (\frac{1}{2} - w_i) \;\Rightarrow\; w_i < \frac{1}{4}$. Conversely, if $w_i < \frac{1}{4}$ then by the previous paragraph we cannot have $(1 - \sqrt{g})^2 \geq \sqrt{g} - g$, hence indeed $w_i < \sqrt{g} - g$.

from the above it follows that:

$$g(w_i) = \begin{cases} (1 - \sqrt{w_i})^2 & \text{if} \quad w_i \geq \frac{1}{4} \\ \frac{1}{2} - w_i & \text{if} \quad w_i < \frac{1}{4} \end{cases}$$

Finally, suppose $w_j \geq w_i$. If $w_i \geq \frac{1}{4}$ then $g(w_i) = (1 - \sqrt{w_i})^2 > (1 - \sqrt{w_j})^2 = g(w_j)$. If $w_i < \frac{1}{4}$ and $g(w_i) = \frac{1}{2} - w_i > \frac{1}{4}$ then $w_j = 1 - w_i - g(w_i) = \frac{1}{2}$ and $g(w_j) = (1 - \frac{1}{\sqrt{2}} < \frac{1}{4} < g(w_i)$. Hence if $w_j \geq w_i$, then $g(w_i)$ is sufficient to deter j. ■

Proof of Observation 1.

This follows from solving $\min_{g,w_1,w_2} \hat{g}(\mathbf{w})$ subject to $g + w_1 + w_2 = 1$; $w_1,\ w_2 > 0$. ■

Proof of Lemma 5. Let $g \geq \hat{g}(w)$, then by Lemma 4, $x_1 = 0,\ x_2 = 0$ are optimal responses to each other. Hence $x = 0,\ a = 0$ is an equilibrium of $\Gamma_2|\mathbf{g}$. ■

Proof of Lemma 6. Let $g \geq \hat{g}(w)$, and suppose in the subgame $\Gamma_2|\mathbf{g}$ there is an equilibrium with $x \neq 0,\ a = 0$. then by Lemma 4 and Assumption 2, we must have $x \gg 0$, for suppose $x_1 = 0$, then 2 can weakly improve his payoff by setting $x_2 = 0$.

But if $x \gg 0$ then it must be true that in $\Gamma_3|(g, x)$ both players $i = 1, 2$ are exactly indifferent between $a_i = 0$ and $a_i = 1$, i.e., the condition for preferring peace

$$\frac{x_i}{x+g}[1 - x - g] \ \leq \ w_i - x_i \tag{4}$$

holds with equality. For if i strictly prefers $a_i = 0$, which implies that the condition (4) holds with a strict inequality, then there is $\epsilon > 0$ such that the inequality continues to hold if j reduces x_j by ϵ. But this increases j's payoff, hence x_j could not be the optimal response to x_i given $\mathbf{g}$.

But then adding together (4) for $i = 1, 2$ and rearranging, we get

$$\begin{aligned}
& \frac{x}{x+g}[1 - x - g] \ = \ w - x \\
\Rightarrow \ & \frac{x}{x+g} - x \ = \ w - x \\
\Rightarrow \ & \frac{x}{x+g} \ = \ w \ \equiv \ 1 - g \\
\Rightarrow \ & x \ = \ x + g - gx - g^2 \\
\Rightarrow \ & 1 - g - x \ = \ 0
\end{aligned} \tag{5}$$

But then each player i has zero consumption, and can increase his payoff by reducing x_i, hence this cannot be an equilibrium. ■

Proof of Lemma 7.
Let $g \geq \hat{g}(w)$, and suppose in the subgame $\Gamma_2|\mathbf{g}$ there is a war equilibrium $s|\mathbf{g}$. By Lemma 1, exactly one player attacks in stage 3, *wlog* let this be player 1. Let (x_1, x_2) be the arms profile chosen in stage 2. Since $s|\mathbf{g}$ is an equilibrium of Γ_2, x_1 and x_2 must be optimal responses to each other given g and $\mathbf{a} \neq 0$. Hence, by Lemma 3, we have

$$x_1 = \min\{\sqrt{x_2 + g} - (x_2 + g),\ w_1\}$$
$$x_2 = \max\{0, \min\{\sqrt{x_1} - (x_1 + g),\ w_2\}\} \tag{6}$$

Note that $(\sqrt{x_1} - x_1) \leq \frac{1}{4}$ for all values of x_1, so if $g \geq \frac{1}{4}$ then 2's optimal response to any value of x_1 is $x_2 = 0$. But then by Lemma 4, 1 cannot improve his payoff by arming and attacking unilaterally, hence by Assumption 2 there does not exist a war equilibrium.

So let $g < \frac{1}{4} \implies (w_1, w_2) \gg (\frac{1}{4}, \frac{1}{4})$. Then by (6) we have $x_1 = x_2 + g = \frac{1}{4}$, and $\Pi_1 = \Pi_2 = \frac{1}{4}$.

Now consider the deviation $s_1'|\mathbf{g}$ by 1, with $x_1' = 0,\ a_1' = 0$. Suppose in the subgame Γ_3 following the deviation, 2 does not attack. Then 1 gets the payoff $\Pi_1' = w_1 > \Pi_1$. Alternatively suppose 2 does attack. Then 1 gets the payoff

$$\Pi_1' = \frac{g}{g + x_2}[1 - (g + x_2)] = \frac{g}{\frac{1}{4}}[1 - \frac{1}{4}] = 3g$$

$\Pi_1' > \frac{1}{4}$ if $3g > \frac{1}{4} \implies g > \frac{1}{12}$, which is always true by Observation 1.

Hence given g and $s_2|\mathbf{g}$, the deviation $s_1'|\mathbf{g}$ dominates $s_1|\mathbf{g}$ for 1, and s cannot be an equilibrium. ■.

Proof of Lemma 8.

Contrary to hypothesis, suppose there is a peace equilibrium with outcome $z|\mathbf{g} = (x, a)$ where $g < \hat{g}(\mathbf{R} - \mathbf{g})$. Let $\mathbf{w} = (\mathbf{R} - \mathbf{g})$, and suppose wlog that $w_1 \leq w_2$. Since g is nfd, at least player 1 is undeterred.

Suppose both players are undeterred, then we must have $x' \gg 0$, for if $x_i = 0$ then $j \neq i$ will arm and attack.

Suppose player 2 is deterred, then too $x' \gg 0$, for if $x_1' = 0$ then 2 will not arm, but if $x_2' = 0$ then 1 will arm and attack, hence with $x_2' = 0$ we cannot have $\mathbf{a} = 0$.

Thus both x_1' and x_2' must be strictly positive. But then by the argument in the proof of Lemma 6, (4) must hold with equality, and (5) must be true, hence $z|\mathbf{g}$ cannot be an equilibrium of $\Gamma_2|\mathbf{g}$. ■

Proof of Proposition 2.

The proposition follows directly from lemmas 5, 6, 7 and 8 above. ■

Proof of Proposition 3.

Part (i) follows directly from Proposition 2. Further, it is easy to see that $g(\sigma) > \hat{g}(w)$ cannot be chosen in equilibrium. So it remains to prove part (ii) for $g(\sigma) < \hat{g}(\mathbf{w})$.

If $g(\sigma) = 0$ then the result follows from Proposition 1.

So suppose $0 < g(\sigma) < \hat{g}(\mathbf{w})$. By Lemma 8, if σ is an equilibrium then it is a war equilibrium. By Lemma 1 exactly one player attacks in this equilibrium; let i be the attacker, and j the player that is attacked (the "defender"). Recall that this implies $x_j < w_j$ (otherwise i has no incentive to attack).

Since σ is an equilibrium, it must be true that (dropping the argument σ for convenience) x_i and $g + x_j$ are best responses to each other in the subgame $\Gamma_2|\mathbf{g}$, as described in Lemma 3. First suppose that $g_i = 0$ so $g = g_j$. Then $g + x_j$ must equal the resources that j would have invested in private arms in the pure conflict subgame, and the result follows.

Hence we need to consider $\mathbf{g}$ such that $g_i > 0$.

In the subgame following $\mathbf{g}$, we must have g weakly less than the defender j's best response to the attacker's arms x_i. Otherwise suppose $g > \sqrt{x_i} - x_i$, then the defender invests $x_j = 0$ in private arms. But then the attacker could reduce g_i by a small ϵ without affecting the defender's choice of arms. It can easily be seen that this strictly increases the attacker's payoff, so g_i was not optimal.

From the previous step it follows that $x_j + g = \sqrt{x_i} - x_i$. The attacker i's best response to this is $x_i = \min\{\sqrt{x_j + g} - (x_j + g),\ w_i\}$. If $w_i \geq \sqrt{x_j + g} - (x_j + g)$ then i invests this amount, and it follows that $x_i = x_j + g = \frac{1}{4}$. These are exactly the arms that the two contestants use in pure contest, and lead to the pure contest payoffs Π^{contest}.

Finally suppose $w_i < \sqrt{x_i + g} - (x_i + g)$, so $x_i = w_i$. Note that $\sqrt{a} - a \leq \frac{1}{4}$ for $a \in [0, 1]$. As in the previous step, j responds optimally to this choice. From Proposition 1 we have $\Pi_i = \sqrt{x_i}(1 - \sqrt{x_i})$, which is increasing in x_i for $x_i < \frac{1}{4}$. But then let i reduce his contribution to $g_i' = g_i - \epsilon$ and increase his arms to $x_i' = x_i + \epsilon$. Given that j responds optimally, i's payoff must increase. So the initial choice of g_i was suboptimal and this cannot be an equilibrium.

■

Proof of Proposition 4

Corresponding to Lemma 4 consider the two cases of minimal full deterrence:
(i) If $w_1 < \frac{1}{4}$ then $\hat{g}(w_1) = \frac{1}{2} - w_1$ and $w_2 = \frac{1}{2}$.
(ii)If $\min\{w_1, w_2\} = w_1 \geq \frac{1}{4}$. Then $\hat{g}(w_1) = (1 - \sqrt{w_1})^2$. Hence

$$w_2 = 1 - [w_1 + (1 - \sqrt{w_1})^2] = 2(\sqrt{w_1} - w_1)$$

It can be checked that $w_2 \geq w_1$ provided $w_1 \leq \frac{4}{9}$. When $w_1 = \frac{4}{9}$, we have $w_1 = w_2$, and $\hat{g}(w_1) = \frac{1}{9}$. We know that $\hat{g}(w)$ depends only on $\min\{w_1, w_2\}$ and $g > \hat{g}(w)$ is also full deterrent. Hence all values of w_2 between w_1 and the value derived above are consistent with full deterrence.
If $w_2 \geq w_1 > \frac{1}{4}$, then from (ii) above $w_1 + g(w_1) + w_2 \geq 2w_1 + (1 - \sqrt{w_1})^2 = 3w_1 - 2\sqrt{w_1} + 1$. But the last term is ≤ 1 only if $w_1 \leq \frac{4}{9}$. Hence full deterrence is not feasible with $\min\{w_1, w_2\} > \frac{4}{9}$. ■

Proof of Proposition 5.

We focus on the case $R_1 \leq R_2$. The proof for the complementary case is identical.

First suppose that $R_1 \leq \frac{1}{4}$. Then the post-contribution allocation must have $\min\{w_1, w_2\} \leq \frac{1}{4}$. Then full deterrence requires $g = \frac{1}{2} - \min\{w_1, w_2\}$, which gives $\max\{w_1, w_2\} = \frac{1}{2}$. Hence contributions by i where $w_i = \min\{w_1, w_2\}$ do not alter the contribution required from $j \neq i$. Thus the only contribution from i consistent with minimality is $g_i = 0$, and j must contribute $g_j = R_j - \frac{1}{2}$. It follows that when $R_1 \leq \frac{1}{4}$, the only contribution profile that is a candidate for equilibrium is $(g_1, g_2) = (0, R_2 - \frac{1}{2})$, which yields the consumption profile $(R_1, \frac{1}{2})$.

This leaves endowments with $(R_1, R_2) \geq (\frac{1}{4}, \frac{1}{4})$. By the argument in the preceding paragraph, a minimal fd allocation starting from this endowment must have $w_i \geq \frac{1}{4},\ i = 1, 2$. Hence w_1 and w_2 are related by the second line of equation (1). For each value of R_1 in the initial endowment, we want to identify the smallest and largest values of w_1 that is attainable in a mfd allocation.

Let $R_1 \in [\frac{1}{4}, \frac{1}{2}] \ \Leftrightarrow \ R_2 \in [\frac{1}{2}, \frac{3}{4}]$. By the previous argument, the largest contribution player 1 can make is $g_1 = R_1 - \frac{1}{4}$ which leaves him with $w_1 = \frac{1}{4}$. This calls for $g = \frac{1}{4} \geq g_1$, so player 2 must contribute the rest, resulting in the allocation $[\frac{1}{4}, \frac{1}{2}]$. Thus in this range the smallest allocation w_1 consistent with mfd is $\frac{1}{4}$.

To identify the largest allocation, consider two intervals. First let $R_1 \in [\frac{1}{4}, \frac{4}{9}] \ \Rightarrow \ R_2 \in [\frac{5}{9}, \frac{3}{4}]$. We know from Lemma 4 that if $g_1 = 0$ then player 1 can be deterred if player 2 contributes $(1 - \sqrt{R_1})^2$.

This leaves player 2 with consumption $w_2 = 2(\sqrt{R_1} - R_1) \geq w_1$. Since 1 does not contribute, $w_1 = R_1 = 1 - R_2$, the resultant consumption vector is $(R_1, 2[\sqrt{1-R_2} - (1-R_2)])$.

Next let $R_1 \in (\frac{4}{9}, \frac{1}{2}]$, if player 2 contributes sufficiently to deter player 1, this leaves him with $w_2 < w_1$. Hence to ensure full deterrence with no contribution from player 1, he must deter himself. This implies $g_2 = (1 - \sqrt{w_2})^2$. Since $g_2 + w_2 = R_2$, This leaves player 2 a consumption of $[\frac{1}{2}\{1 + \sqrt{(2R_2 - 1)}\}]^2$, which ranges from $w_2 = \frac{4}{9}$ when $R_2 = \frac{5}{9}$ to $w_2 = \frac{1}{4}$ when $R_2 = \frac{1}{2}$. Of course, since player 1 does not contribute to deterrence, his allocation remains at $w_1 = R_1$.

Finally, note that when player 1 with R_1 makes his minimum possible contribution, player 2 with $R_2 = 1 - R_1$ must make the maximum contribution. Putting these together we obtain the proposition. ■

Proof of Theorem 1

Without loss of generality we focus on $\mathbf{R} : R_1 \leq R_2$. We consider three cases.

Case 1: $R_1 \in [\frac{1}{4}, \frac{1}{2}) \Leftrightarrow R_2 \in (\frac{1}{2}, \frac{3}{4}]$.
Suppose there is an equilibrium z^* such that $\mathbf{a}^* \neq 0$, i.e., there is war. Then $\mathbf{g}^*$ is not full deterrent, and by Proposition 3 the players receive the pure contest payoffs $(\frac{1}{4}, \frac{1}{4})$.
Thus player 1 will never contribute more than $R_1 - \frac{1}{4}$ to public defence, hence in any equilibrium we must have $w_1 \geq \frac{1}{4} = \Pi_1^C$.
But then to ensure full-deterrence the largest contribution 2 may be required to make is $g_2 = \frac{1}{4} - g_1 \leq \frac{1}{4}$, which leaves him with $w_2 > \frac{1}{4} = \Pi_2^C$. Hence for any incentive compatible contribution from player 1, player 2 prefers to ensure full-deterrence rather than engage in contest. Thus if there is an equilibrium then it must be full-deterrence.
It is easy to verify that $\mathbf{g} = (R_1 - \frac{1}{4}, \frac{1}{2} - R_1)$ is an equilibrium. Hence there is at least one equilibrium, and any equilibrium is full-deterrence.

Case 2: $R_1 = R_2 = \frac{1}{2}$.
The arguments for Case 1 carry over for any contribution $0 < g_1 < \frac{1}{4}$, which lead to full-deterrence equilibria with payoffs $\Pi_i > \frac{1}{4},\ i = 1, 2$.
However, for $g_1 = 0$, player 2 has two optimal choices; he can contribute $g_2 = \frac{1}{4}$, which ensures full-deterrence and yields him a payoff of $\frac{1}{4}$, or he can set $g_2 \in [0, \frac{1}{4})$, leading to war (see Proposition 3) which also yields a payoff of $\frac{1}{4}$. An equivalent argument applies to player 1, hence, in particular, $\mathbf{g} = (0, 0)$ followed by a pure contest is an equilibrium.
It follows that any minimal full-deterrence equilibrium where each player contributes a strictly

positive amount to public defence strictly dominates the unique pure contest equilibrium. Further, if $g_1 = 0$, the deterrence equilibrium with $g_2 = \frac{1}{4}$ yields payoffs $(\frac{1}{2}, \frac{1}{4})$, which pareto dominates the contest outcome.

Case 3: $R_2 > \frac{3}{4} \Leftrightarrow R_1 < \frac{1}{4}$.
By Proposition 5, any full-deterrence equilibrium must have $g_1 = 0$ and $g_2 = R_2 - \frac{1}{2}$, yielding player 2 a payoff of $\frac{1}{2}$.
A pure contest equilibrium yields player 2 a payoff of $(1 - \sqrt{R_1})^2$.
Hence the nature of the equilibrium depends on player 2's choices, and he will choose to deter if and only if

$$\frac{1}{2} \geq (1 - \sqrt{R_1})^2 \quad \Leftrightarrow \quad R_1 \geq (\frac{3}{2} - \sqrt{2}).$$

■

Proof of Theorem 3

Let $R_1 \in [\frac{3}{2} - \sqrt{2}, \frac{1}{4}]$. We know from an earlier proposition that in this range player 2 unilaterally ensures full-deterrence in equilibrium, with $g_2 = \frac{1}{2} - R_1$. Hence the sum of consumptions in equilibrium is $c^* = \frac{1}{2} + R_1$.

Consider the corresponding pure contest outcome (see Section 2.4). Since player 1 is constrained, she invests her entire endowment in arms, and 2 responds optimally, which yields payoffs $\Pi_1 = \sqrt{R_1}(1 - \sqrt{R_1})$ and $\Pi_2 = (1 - \sqrt{R_1})^2$. Thus total consumption is $c^C = (1 - \sqrt{R_1})$.

Thus full-deterrence is efficient if and only if

$$\frac{1}{2} + R_1 \geq (1 - \sqrt{R_1}),$$

which reduces to $R_1 \geq 1 - \frac{\sqrt{3}}{2}$, which it can be verified is greater than $\frac{3}{2} - \sqrt{2}$. ■