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Managing Cost Overrun Risk in Project Funding Allocation

CHUNG-LI TSENG* chungli@umr.edu
Department of Engineering Management & Systems Engineering, University of Missouri-Rolla, Rolla, MO 65409, USA

KYLE Y. LIN kylin@nps.edu
Operations Research Department, Naval Postgraduate School, Monterey, CA 93943, USA

SATHEESH K. SUNDARARAJAN
Jacobs PFI Professional Services, Reading, RG1 6QF, UK

Abstract. This paper discusses decision making of project funding allocation under uncertain project costs. Because project costs are uncertain and funding allocations may not necessarily match the costs required, each project is inherently subject to a cost overrun risk (COR). In this paper, a model is proposed in which project cost is treated as a factor with a probability density function. The decision maker then allocates the total funding to the projects while minimizing a weighted sum of mean and variance of the COR of the project portfolio. Some properties of project COR are derived and interpreted. Optimal funding allocation, in relationship to factors such as various project sizes and riskiness, project interdependency, and the decision maker's risk preference, is analyzed. The proposed funding allocation model can be integrated with project selection decision-making and provides a basis for more effective project control.

Keywords: resource allocation, project management, decision making under uncertainty, cost overrun risk, portfolio optimization

Introduction

Project funding allocation is of paramount importance to all management decisions. It typically refers to an investment decision concerning what portion of total budget to allocate in each project, and how to meet funding requirements throughout the development of the projects, while maximizing the total reward.

Recognizing project risk is an important issue in making project funding allocation decisions. Obel and Vander Weide (1979) considered uncertain cash flows in the presence of a nonlinear utility function and nonlinear resource constraints. Zinn, Lesso, and Motazed (1977) proposed a probabilistic approach to compute the expected value, variance, and semivariance of the NPV of an investment. Giaccotto (1984) studied risk analysis in capital budgeting with serially correlated cash flows. Chiu and Park (1998)

*Corresponding author.

solves a capital budgeting model with cash flows represented by fuzzy numbers, which capture the risks and uncertainties involved. Gerchak (1999) models the success rate of a R&D project to be stochastically increasing in its funding level. Because this paper focuses on parallel development of R&D projects, detailed project characteristics such as project scale, cash requirements, interdependency, development risks are not considered.

Decisions made on funding levels of projects are critical to project success. Because project costs are uncertain and funding allocations may not necessarily match the costs required, each project is inherently subject to a cost overrun risk (COR). The Association for Project Management of the United Kingdom defines the cost overrun of a project as "the amount by which a contractor exceeds or expects to exceed the estimated costs, and/or the final limitations (the ceiling) of a contract." A research study representing 3,500 projects drawn from all over the world in several different industries shows that cost overruns are typically between 40 and 200 percent of the initial project cost (Laufer, 1997). The actual causes of cost overrun for a project may be internal and external to the project and typically are connected with project performance parameters, time overruns, and inflation, etc. Though several attempts have been made to identify and incorporate various risks into project-funding allocation, COR has not been addressed specifically.

It is not uncommon that a project portfolio is comprised of diverse projects of varying complexity and investment requirements, which complicate the management of COR. In management literature, rules of thumb have been developed for managing COR. For example, Taylor (1997) reports that the most expensive projects are the ones that are most subject to inflation and needs utmost funding attention in the portfolio. This is because projects with higher costs are prone to the impact of inflation, which, in other words, have higher exposure to COR. Similarly, traditional cost control technique Pareto's law states "20% of the elements effect 80% of the outcome" (Gould, 1997). Therefore, a manager should identify a small percentage of critical cost components, which together account for the largest percent of cost variances. These rules, though general and useful, neither quantify COR nor reveal quantitatively how to allocate funding among projects, especially in the presence of various investment requirements and potential risks of projects and their interdependencies. In reality, the funding level of a project needs constant attention of managers due to dynamic nature of project parameters for each individual project. Therefore, decisions made on funding levels of projects are of great importance for managing COR and should be made based on individual project requirements. This paper will specifically address this important issue.

To minimize the COR of a project portfolio, the decision maker (DM) is expected to minimize a weighted sum of the mean and variance of the portfolio COR, similar to the approach of the mean-variance optimization in portfolio investment problems (e.g., Markowitz, 1952). Balancing the mean and variance of the portfolio COR depends on the DM's risk preference. In this paper, it will be shown how the DM's risk preference, as well as project interdependencies, affects portfolio COR and budget allocation. In a later section, the feasibility for integrating project-funding allocation and COR minimization will be demonstrated. Project selection decisions are normally made in the concept phase of a project's life cycle, where the ability to influence project parameters has the highest likelihood. This integration during the concept phase can add value to ability to influence the project and provide a basis for more effective project control.

This paper is organized as follows. In Section 1, the COR of projects is defined, based on which the budget allocation problem is formulated. Some properties of COR of projects are discussed. Section 2 presents the optimality conditions of the budget allocation problem. Two cases are discussed: all projects are independent and some projects are dependent. Section 3 reports numerical results. This paper concludes in Section 4.

1. Project funding allocation and COR analysis

In this paper, we consider the decision making of project funding allocation subject to uncertain project costs. Because project costs are uncertain and project funding allocation may not necessarily match the costs required, each project is exposed to a risk of cost overrun. In this paper, we assume that the DM allocates funding with the COR of the project portfolio minimized. Detailed mathematical model is presented in the next section.

1.1. The mathematical model

In the development the following standard notation will be used.

i: index for projects $i = 1, \ldots, I$, and I is the number of projects.
B: total budget ($).
w_i: a decision variable representing amount of funding ($) awarded to project i.
X_i: a random variable representing the actual cost of project i.
$f_i\colon \Omega_i \to R^+$ is continuously differentiate representing the probability density function associated with X_i, defined over a set $\Omega_i \subset \Re^+$.

Additional symbols will be introduced when necessary.

It is assumed that the actual cost of each project, say project i, is a random variable X_i with a known probability density function (p.d.f.) f_i. Unlike in the deterministic capital budgeting formulation where each project proposes a fixed amount for the budget, it is assumed that each project presents a p.d.f. of the forecast project cost. This assumption may seem more unusual than it actually is. It has become popular to perform cost estimation using Monte Carlo simulation in project management (e.g., Smith, 1994). Using Monte Carlo simulation to forecast the total cost of a project, the p.d.f. of each cost component must be estimated. The total cost of the project is simply the sum of all cost components. To make project selection decisions, companies or government agencies normally estimate the fixed budget cost for a project based on some percentile of the p.d.f. of the project cost, obtained from the Monte Carlo simulation. Therefore, the assumption of requiring a p.d.f. for each project is merely a utilization of existing information, rather than demanding new data. Since the total project cost is a lump sum of many cost components, its p.d.f. tends to be bell-shaped and smooth according to the central limit theorem under some additional assumption such as statistical independence of cost components. In this paper, the p.d.f. f_i for each project is assumed to be continuously differentiable in $\Re^+$. The cumulative distribution function (c.d.f.) for project j is also defined by

$$F_i(w_i) \equiv \int_0^{w_i} f_i(s)\, ds \tag{1}$$

and $\bar{F}_i(w_i) \equiv 1 - F_i(w_i)$ is the complement of $F_i(w_i)$.

Because actual cost is random, each project is exposed to a COR. Suppose project i is funded with w_i dollars, its COR is defhed to be

$$\text{COR}_i(w_i) \equiv (X_i - w_i)^+. \tag{2}$$

which is a function of w_i and is also a random variable. In (2), the notation $x^+ \equiv \max(x, 0)$ is used.

In this paper, the focus is to allocate the total budget to projects with the COR of the project portfolio minimized. The COR of the project portfolio is quantified as a weighted sum of the mean and variance of the portfolio COR, similar to the approach of the mean-variance optimization in portfolio investment problems. The formulation of the project funding allocation problem is as follows and is denoted by (P_0):

(P_0)

$$\min_{\mathbf{w}} \quad g(\mathbf{w}) \equiv \alpha E\left[\sum_{i=1}^{I} \text{COR}_i(w_i)\right] + \beta \text{Var}\left[\sum_{i=1}^{I} \text{COR}_i(w_i)\right] \tag{3a}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} w_i \le B \tag{3b}$$

$$a_i \le w_i \le b_i, \quad i = 1, \ldots, I \tag{3c}$$

where a_i and b_i are the predetermined lower and upper bounds of funding level for project j, respectively. Setting appropriate values for a_i and b_i is important in practice. Intuitively, $\text{COR}_i(w_i)$ decreases (stochastically) as w_i increases. That is, the more funding a project is given, the lower its COR is. Therefore, an underestimated a_i implies feasibility of an underfunded solution that may eventually cause the project to be underperformed or unfulfilled. An overly large $b_i > \max\Omega_i$ implies a feasible solution that overfunds the project without any real effect in reducing its COR. Therefore, the interval $[a_i, b_i]$ is assumed to be finite and is a subset of Ω_i, i.e.,

$$[a_i, b_i] \subseteq \Omega_i, \quad i = 1, \ldots, I \tag{4}$$

It is clear that if $B \geq \sum_{i=1}^{I} b_i$, each project will be funded at its upper bound b_i, which achieves the minimal COR. Without loss of generality, it is assumed that competition for funding exists among projects, and in the rest of this paper, only the equality case of (3b) will be discussed. That is, (3b) will be replaced by

$$\sum_{i=1}^{I} w_i = B \tag{5}$$

Implicitly,

$$\sum_{i=1}^{I} a_i \leq B \leq \sum_{i=1}^{I} b_i \tag{6}$$

is also assumed to guarantee feasibility.

Unlike the mean-variance portfolio optimization where the DM prefers the mean but not the variance of return, in (P_0) both of the mean and variance of the COR are not desirable. Therefore, in the objective function (3a), both α and β are nonnegative and are predetermined constants reflecting risk preference of the DM or the funding agency. Three types of risk preference can be categorized:

- *The DM is risk neutral.* This corresponds to the case where $\alpha > 0$ and $\beta = 0$, since the DM values the COR purely based on its expected value.
- The DM is *risk averse.* This corresponds to the case where $\alpha > 0$ and $\beta > 0$, implying that the DM likes neither the mean nor the variance of the COR.
- The DM is a *variance minimizer.* This corresponds to the case where $\alpha = 0$ and $\beta > 0$.

In this paper, $\alpha + \beta = 1$ is further imposed as a way to easily identify the relative values of these two parameters. As aforementioned, since the DM dislikes both mean and variance of the COR, the objective function of (P_0), (3a), does not include the case that the DM is risk prone.

1.2. *Optimal project funding allocation*

The project funding allocation problem in (3a) to (3c) is now restated with (3b) replaced by (5), and denote it by (P).

(P)

$$\min_{\mathbf{w}} \quad g(\mathbf{w}) \tag{7a}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} w_i = B \tag{7b}$$

$$a_i \leq w_i \leq b_i, \quad i = 1, \ldots, I \tag{7c}$$

Let $\mathbf{w}^* = (w_1^*, w_2^*, \ldots, w_I^*)$ be an optimal solution of (P). Then $\mathbf{w}^*$ satisfies the following necessary condition of optimality with a $\lambda^* \in \Re$ for all $i = 1, \ldots, I$:

$$\partial g(\mathbf{w}^*)/\partial w_i = \lambda^*, \quad \text{for } a_i < w_i^* < b_i \tag{8a}$$

$$\partial g(\mathbf{w}^*)/\partial w_i \geq \lambda^*, \quad \text{for } w_i^* = a_i \tag{8b}$$

$$\partial g(\mathbf{w}^*)/\partial w_i \leq \lambda^*, \quad \text{for } w_i^* = b_i. \tag{8c}$$

$\partial g(\mathbf{w}^*)/\partial w_i$ will be referred to as the *marginal* COR reduction for the portfolio by funding project j an additional dollar, or simply the marginal COR.[1] Optimality conditions (8a) to (8c) state that at optimality all projects are funded at a marginal COR that either equals the same fixed value lambda, or equals the marginal COR corresponding to the upper or lower bound of a project's funding level, whichever is closer to lambda. Here, lambda is used to denote the marginal COR, and this property shall be referred to as the "equal-lambda" property in the sequel. If g is strictly convex, the optimality conditions above are also sufficient. Although the convexity of g is not a necessary condition for (P) to be solvable, it does provide desirable properties such as uniqueness of the optimal solution. Moreover, with the convexity of g (and convexities for both $E[\sum_{i=1}^{I} \text{COR}_i(w_i)]$ and $\text{Var}[\sum_{i=1}^{I} \text{COR}_i(w_i)]$), a one-to-one correspondence exists between any *efficient* trade-off (Pareto optima) and a risk preference (α, β) in (P). That is, the DM can obtain *all* efficient trade-offs between the mean and variance of the project COR, i.e., the entire efficient frontier, by merely varying the weights (α, β) in (P). In a later section of this paper, it will be shown that, although g is not necessarily convex everywhere, it is indeed convex over the subset of domain that represents most "real" situations.

1.3. Properties of the COR for a single project

Before attacking the optimization of (P), we examine the properties of the COR for a single project in this section. The expected value of the COR for project i is

$$m_i(w_i) \equiv E[(X_i - w_i)^+] = \int_{w_i}^{\infty} (s - w_i) f_i(s)\, ds \tag{9a}$$

$$= (s - w_i)(-\bar{F}_i(s))|_{w_i}^{\infty} - \int_{w_i}^{\infty} (-\bar{F}_i(s))\, ds \tag{9b}$$

$$= \int_{w_i}^{\infty} \bar{F}_i(s)\, ds, \tag{9c}$$

where the preceding follows from integration by part and the fact that $(-\bar{F}_i(s))' = f_i(s)$. Therefore,

$$m_i'(w_i) = -\bar{F}_i(w_i) \leq 0 \tag{10a}$$

$$m_i''(w_i) = f_i(w_i) \geq 0 \tag{10b}$$

That is, $m_i(w_i)$ is strictly decreasing and convex over Ω_i. The variance of the COR for project i is

$$v_i(w_i) \equiv \mathrm{Var}[(X_i - w_i)^+] = \int_{w_i}^{\infty} (s - w_i)^2 f_i(s)\, ds - m_i(w_i)^2 \tag{11}$$

An illustration of typical function shapes for $m_i(\cdot)$ and $v_i(\cdot)$ is given in figure 1. To determine the structural properties of $v_i(w_i)$, the *completion rate function* of project i is introduced and is defined as

$$r_i(w_i) \equiv \frac{f_i(w_i)}{\bar{F}_i(w_i)}. \tag{12}$$

Given that one has already spent w_i on project i, $r_i(w_i)$ measures the likelihood that the project will be completed if one spends an additional amount of Δw_i dollars. In reliability theory, the completion rate is referred to as the failure rate function of the lifetime of a component. If the completion rate function of a project $r_i(w_i)$ is increasing in w_i, it means that the probability of completing the project with one additional dollar increases as w_i increases. It is believed that all real projects should have an increasing completion rate function. To facilitate our analysis, the following assumption is made.

Assumption 1. For each project i, the completion rate function $r_i(w_i)$ is increasing in w_i when w_i is sufficiently large. That is, there exists a $h_i \geq 0$ such that $r_i(w_i)$ is increasing in w_i for all $w_i > h_i$.

Lemma 1. If $r_i(w_i)$ is increasing in w_i for $\forall w_i > h_i \in \Omega_i$, then

$$m_i(w_i) f_i(w_i) \leq F_i(w_i) \bar{F}_i(w_i) \tag{13}$$

for all $w_i \geq \max(h_i, F_i^{-1}(1/2))$.

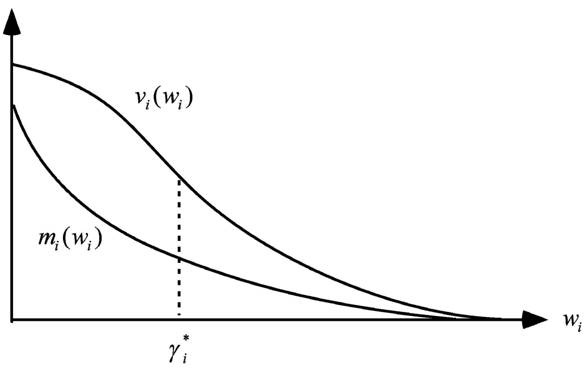


Figure 1. Typical function shapes for $m_i(w_i)$ and $v_i(w_i)$ (not to scale).

Proof. By definition,

$$\frac{m_i(w_i)}{\bar{F}_i(w_i)} = \int_{w_i}^{\infty} (s - w_i) \frac{f_i(s)}{\bar{F}_i(w_i)} \, ds = E[X_i - w_i \mid X_i > w_i] \tag{14}$$

Equation (14) is decreasing in w_i for all $w_i > h_i$, because $r_i(w_i)$ is increasing in w_i for all $w_i > h_i$. This implies that its first derivative is negative for all $w_i > h_i$. That is,

$$\frac{d(m_i(w_i)/\bar{F}_i(w_i))}{dw_i} = \frac{m_i(w_i) f_i(w_i)}{\bar{F}_i^2(w_i)} - 1 \le 0$$

This implies

$$m_i(w_i) f_i(w_i) \le \bar{F}_i^2(w_i)$$

Furthermore, when $w_i \ge F_i^{-1}(1/2)$, $\bar{F}_i(w_i) \le 1/2 \le F_i(w_i)$. We then have

$$m_i(w_i) f_i(w_i) \le \bar{F}_i(w_i)\bar{F}_i(w_i) \le F_i(w_i)\bar{F}_i(w_i)$$

for all $w_i \ge \max(h_i, F_i^{-1}(1/2))$. □

Proposition 2. The following statements are true:

(i) $v_i(\cdot)$ is decreasing, i.e., $v_i'(w_i) \le 0, \forall w_i \in \Omega_i$.

(ii) There exists a $\gamma_i \in \Omega_i$ such that $v_i(w_i)$ is convex for all $w_i > \gamma_i$.

Proof. To prove (i), from (11) with some algebra we have

$$\begin{aligned} v_i'(w_i) &= -2m_i(w_i) - 2m_i(w_i)m_i'(w_i) \\ &= -2m_i(w_i)(1 + m_i'(w_i)) \\ &= -2m_i(w_i)F_i(w_i) \le 0. \end{aligned} \tag{15}$$

To prove (ii), from (15) we have

$$\begin{aligned} v_i''(w_i) &= -2m_i(w_i) f_i(w_i) - 2m_i'(w_i)F_i(w_i) \\ &= -2(m_i(w_i) f_i(w_i) - F_i(w_i)\bar{F}_i(w_i)) \end{aligned} \tag{16}$$

From Lemma 1, it is clear that there exists a $\gamma_i \in \Omega_i$ such $v_i''(w_i) \ge 0$, for all $w_i > \gamma_i$. □

Let the point γ_i^* be the minimal possible value of γ_i defined in Proposition 2. It is evident that γ_i^* satisfies the following equation

$$m_i(\gamma_i^*) f_i(\gamma_i^*) = F_i(\gamma_i^*)\bar{F}_i(\gamma_i^*) \tag{17}$$

That is, γ_i^* is an inflection point of $v_i(\cdot)$ such that to the right of γ_i^*, $v_i(\cdot)$ is convex (e.g. see figure 1). The values of γ_i^* for some popular distributions have been obtained

Table 1
The values of γ_i^* for some distributions.

Distribution of X_i	$F_i(\gamma_i^*)(\%)$	Remark
Normal $N(\mu, \sigma^2)$	35.37	$\gamma_i^* = \mu - 0.367\sigma$
Exponential	50	
Uniform	33.33	
Beta(3, 3)	34.37	Symmetric
Beta(4, 2)	29.86	Skewed to right ($\gamma_i^* = 0.5771$)
Beta(2,4)	38.57	Skewed to left ($\gamma_i^* = 0.2588$)

using numerical analysis and are summarized in Table 1, along with the values of $F_i(\gamma_i^*)$. Note that it can be shown that normal, uniform, and beta distributions all have an strictly increasing completion rate function, and therefore by Lemma 1 their $F_i(\gamma_i^*)$ are bounded above by 50%. For an exponential distribution, its completion rate is a constant and its $F_i(\gamma_i^*)$ is exactly at 50%.

The results in Table 1 and the fact that γ_i^* can be bounded by the median of X_i, i.e. $F_i^{-1}(1/2)$, under the assumption of increasing completion rate suggest that γ_i^* can serve a satisfactory lower bound of a_i. This is justified because it seems unlikely for a funding agency to fund a project that has a only 50% chance of being completed within budget. In later development of this paper, the lower bound of the funding level a_i will be assigned to γ_i^*, i.e.,

$$a_i = \gamma_i^*, \quad i = 1, \ldots, I \tag{18}$$

With (18), both $m_i(\cdot)$ and $v_i(\cdot)$ are now strictly convex in the feasible region.

The COR for a single project at least reveals the following two mathematical properties:

1. *Monotonically decreasing*: Regardless of the risk preference of the DM, every additional funding dollar to a project helps reduce the mean and variance of the COR of the project.

2. *Convexity*: Regardless of the risk preference of the DM, as the funding level for a project increases, the marginal COR reduction decreases. That is, it becomes harder and harder to reduce the mean and variance of the COR of a project by merely increasing the funding level.

The first property of monotonicity is substantiated by the time-cost trade-off relation of a project, which describes that every additional funding dollar may reduce the duration required for project completion (e.g., Hillier and Lieberman, 2000). This is because that additional up-front funding level provides an opportunity for deploying more resources and better project processes and management, which may lead to reduction of the project duration. In other words, the project funding level can be increased to prevent not only cost but time overrun of a project. One example is that a fast-track project, whose acceleration is achieved by overlapping project phases, demands a much higher funding level for resources, processes, and management than otherwise with sequential phases. This situation is further strengthened in case of fixed-time projects (e.g., launching a new product, in which time-to-market is crucial), for which preventing cost overrun is extremely important and an increase in funding level is highly required.

To interpret the convexity of the COR, note that project performance parameters are normally inflienced by both internal and external environments. The time-cost-performance triangle requires a balance throughout the life cycle of the project, which unfortunately is not the case for most projects (Chapter 16, Kerzner, 1998). Employing trade-off decisions for additional funding to reduce the overrun can handle issues within the control of the firm. Depending upon the magnitude of the issue, the timeliness of its identification, and the potential impact on the project results, it is possible that beyond some level, any additional funding dollar has little or no effect for reducing overruns. This may occur when alternatives for reducing overruns such as additional resources and processes within the reach of the firm have been completely exploited. Moreover, restrictions in resource allocation and process changes exist. These restrictions are likely to be augmented as more resource requiring reallocation. This situation is prevalent in R&D projects whose cost overruns cannot be controlled by only increasing the funding levels. Also the effects of external environment such as changes of regulatory and market requirements can cause overruns, which cannot be reduced by just increasing the funding level beyond the limits that are controlled by the firm.

2. Solving the project funding allocation problem

2.1. All projects are independent

If the costs of all projects (X_i) are independent, their CORs are also independent. The objective function of the project funding allocation problem (P) becomes separable, i.e., $g(\mathbf{w}) = \sum_i g_i(w_i)$, where

$$\begin{aligned} g_i(w_i) &\equiv \alpha E[\mathrm{COR}_i(w_i)] + \beta \mathrm{Var}[\mathrm{COR}_i(w_i)] \\ &= \alpha m_i(w_i) + \beta v_i(w_i) \end{aligned} \tag{1}$$

The project funding allocation problem (P) is reduced to a standard resource allocation problem (e.g., Ibaraki and Katoh, 1988):

(P_1)

$$\min_{\mathbf{w}} \quad \sum_{i=1}^{I} g_i(w_i) \tag{2a}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} w_i = B \tag{2b}$$

$$a_i \le w_i \le b_i, \quad i = 1, \ldots, I \tag{2c}$$

From (10a) and (15),

$$\begin{aligned} g_i'(w_i) &= \alpha m_i'(w_i) + \beta v_i'(w_i) \\ &= -\alpha \bar{F}_i(w_i) - 2\beta m_i(w_i) F_i(w_i) \\ &= F_i(w_i)(\alpha - 2\beta m_i(w_i)) - \alpha \end{aligned} \tag{3}$$

If $\{\tilde{w}_i\}$ is optimal to (P_1), there exists a $\tilde{\lambda}$ such that

$$F_i(\tilde{w}_i)(\alpha - 2\beta m_i(\tilde{w}_i)) = \tilde{\lambda}, \quad \text{for } a_i < \tilde{w}_i < b_i \tag{4a}$$

$$F_i(\tilde{w}_i)(\alpha - 2\beta m_i(\tilde{w}_i)) \geq \tilde{\lambda}, \quad \text{for } \tilde{w}_i = b_i \tag{4b}$$

$$F_i(\tilde{w}_i)(\alpha - 2\beta m_i(\tilde{w}_i)) \leq \tilde{\lambda}, \quad \text{for } \tilde{w}_i = b_i. \tag{4c}$$

Note that the last term $-\alpha$ in (3) has been absorbed in $\tilde{\lambda}$ since it is also a constant. Because (P_1) is a convex problem, a dual method can be developed to solve (P_1) by iterating the dual variable λ. Given a λ, the w_i that satisfies (4a) to (4c) is solved for each i. The iteration terminates when (2b) is satisfied.

Proposition 3. When X_i are independent, two special cases for optimal project funding allocation are:

(i) If the DM is risk neutral, at optimality all of the projects will be funded such that the probability for each project to be completed within budget is a constant, or close to the constant to the greatest extent subject to the project's funding limits.

(ii) If the DM is a variance minimizer, at optimality all of the projects will be funded such that the probability for each project to be completed within budget, multiplied by the expected cost overrun, equals to a constant, or close to the constant to the greatest extent subject to the project's funding limits.

Proof.

1. Let $\beta = 0$, (4a) is reduced to

$$F_i(\tilde{w}_i) = \tilde{\lambda}/\alpha \in [0, 1] \tag{5}$$

   That is, each project is funded such that the probability to be completed within budget is a constant $\tilde{\lambda}/\alpha$. For a project i such that the constant $\tilde{\lambda}/\alpha \notin [F_i(a_i), F_i(b_i)]$, the project is funded at either a_i or b_i, whichever is closer to $F_i^{-1}(\tilde{\lambda}/\alpha)$.

2. With $\alpha = 0$, (4a) is reduced to $F_i(\tilde{w}_i)m_i(\tilde{w}_i) = -\tilde{\lambda}/2\beta$. Statement (ii) follows a similar argument in (i). □

From Proposition 3, it is worth noting that a risk neutral DM would treat all projects equal regardless of project sizes. This is because the equal-lambda condition only involves the c.d.f. F_i, which is between 0 and 1 for all projects. However, for a risk-averse DM the expected COR, m_i, plays a role in the equal-lambda condition. In this situation, funding is leaned toward projects with a greater m_i. That is, a bigger project receives greater funding than a smaller one.

2.2. Some projects are dependent

When some project costs are dependent, the project funding allocation problem (P) becomes more complicated and is equivalent to the following problem (P_2):

(P_2)

$$\begin{aligned}\min_{\mathbf{w}} \quad & \alpha \sum_{i=1}^{I} m_i(w_i) + \beta \sum_{i=1}^{I} v_i(w_i) + 2\beta \sum \sum_{i_1 < i_2} \operatorname{Cov}\big(\big(X_{i_1} - w_{i_1}\big)^+, \big(X_{i_2} - w_{i_2}\big)^+\big) \\ \text{s.t.} \quad & \sum_{i=1}^{I} w_i = B \\ & a_i \le w \le b_i, \quad i = 1, \ldots, I\end{aligned}$$

Because of the covariance terms, the objective function is no longer separable. Consider two projects j and k such that X_j and X_k are dependent. The covariance of the COR between projects j and k is defined as:

$$\begin{aligned}C_{jk}(w_j, w_k) &\equiv \operatorname{Cov}((X_j - w_j)^+, (X_k - w_k)^+) \\ &= E[(X_j - w_j)^+(X_k - w_k)^+] - m_j(w_j)m_k(w_k)\end{aligned} \tag{6}$$

Proposition 4. The following equalities are true:

(i) $\partial C_{jk}(w_j, w_k)/\partial w_j = -\bar{F}_j(w_j)(E[(X_k - w_k)^+ \mid X_j > w_j] - E[(X_k - w_k)^+])$

(ii) $\partial^2 C_{jk}(w_j, w_k))/\partial w_j \partial w_k = \Pr\{X_j > w_j, X_k > w_k\} - \Pr\{X_j > w_j\}\Pr\{X_k > w_k\}$

(iii) $\partial^2 C_{jk}(w_j, w_k)/\partial w_j^2 = f_j(w_j)\{E[(X_k - w_k)^+ \mid X_j = w_j] - E[(X_k - w_k)^+]\}$

Proof.

1. By definition,

$$\begin{aligned}
&E[(X_j - w_j - \Delta w)^+(X_k - w_k)^+] \\
&\quad = \int_{w_j+\Delta w}^{\infty} \int_{w_k}^{\infty} (s - w_j - \Delta w)(t - w_k) f_{jk}(s,t)\, dt\, ds \\
&\quad = \int_{w_j+\Delta w}^{\infty} \int_{w_k}^{\infty} (s - w_j)(t - w_k) f_{jk}(s,t)\, dt\, ds \\
&\qquad - \Delta w \int_{w_j+\Delta w}^{\infty} \int_{w_k}^{\infty} (t - w_k) f_{jk}(s,t)\, dt\, ds
\end{aligned} \tag{7}$$

   where $f_{jk}(\cdot,\cdot)$ denotes the joint p.d.f. of X_j and X_k. Therefore,

$$\begin{aligned}
&\frac{\partial E[(X_j - w_j)^+(X_k - w_k)^+]}{\partial w_j} \\
&= \lim_{\Delta w \to 0} \frac{E[(X_j - w_j - \Delta w)^+(X_k - w_k)^+] - E[(X_j - w_j)^+(X_k - w_k)^+]}{\Delta w} \\
&= -\int_{w_j}^{\infty} \int_{w_k}^{\infty} (t - w_k) f_{jk}(s,t)\, dt\, ds \\
&= -\bar{F}_j(w_j) E[(X_k - w_k)^+ \mid X_j > w_j]
\end{aligned} \tag{8}$$

   From (6),

$$\begin{aligned}
\frac{\partial C_{jk}(w_j, w_k)}{\partial w_j} &= -\bar{F}_j(w_j) E[(X_k - w_k)^+ \mid X_j > w_j] + \bar{F}_j(w_j) E[(X_k - w_k)^+] \\
&= -\bar{F}_j(w_j)(E[(X_k - w_k)^+ \mid X_j > w_j] - E[(X_k - w_k)^+]),
\end{aligned} \tag{9}$$

   and (i) is proved.

2. To prove (ii), from (10a) and (9) we have

$$\begin{aligned}
\frac{\partial^2 \mathrm{Cov}((X_j - w_j)^+, (X_k - w_k))^+}{\partial w_j \partial w_k} &= \Pr\{X_j > w_j, X_k > w_k\} \\
&\quad - \Pr\{X_j > w_j\} \Pr\{X_k > w_k\}
\end{aligned} \tag{10}$$

3. To derive $\partial^2 C_{jk}(w_j, w_k)/\partial w_j^2$, note that

$$E[(X_k - w_k)^+ \mid X_j > w_j] = \int_{w_k}^{\infty} \int_{w_j}^{\infty} \frac{f_{jk}(s,t)(t - w_k)}{\bar{F}_j(w_j)}\, ds\, dt$$

$$E[(X_k - w_k)^+ \mid X_j > w_j + \Delta w_j] = \int_{w_k}^{\infty} \int_{w_j+\Delta w}^{\infty} \frac{f_{jk}(s,t)(t - w_k)}{\bar{F}_j(w_j + \Delta w)}\, ds\, dt$$

Therefore,

$$
\begin{aligned}
&E[(X_k - w_k)^+ \mid X_j > w_j + \Delta w] - E[(X_k - w_k)^+ \mid X_j > w_j] \\
&\quad = \int_{w_k}^{\infty} \int_{w_k+\Delta w}^{\infty} (t - w_j) f_{jk}(s,t) \frac{\bar{F}_j(w_j) - \bar{F}_j(w_j + \Delta w)}{\bar{F}_j(w_j)\bar{F}_j(w_j + \Delta w)} \, ds \, dt \\
&\quad\quad - \int_{w_k}^{\infty} \int_{w_j}^{w_j+\Delta w} \frac{f_{jk}(s,t)(t - w_k)}{\bar{F}_j(w_j)} \, ds \, dt \qquad (11)
\end{aligned}
$$

Dividing (11) by Δw and taking $\Delta w \to 0$ yields

$$
\begin{aligned}
&\frac{\partial E[(X_k - w_k)^+ \mid X_j > w_j]}{\partial w_j} \\
&\quad = \int_{w_k}^{\infty} \int_{w_j}^{\infty} (t - w_j) f_{jk}(s,t) \frac{f_j(w_j)}{\bar{F}_j^2(w_j)} \, ds \, dt - \int_{w_k}^{\infty} \frac{f_{jk}(w_j,t)(t - w_k)}{\bar{F}_j(w_j)} \, dt \\
&\quad = \frac{f_j(w_j)}{\bar{F}_j(w_j)} \{E[(X_k - w_k)^+ \mid X_j > w_j] - [(X_k - w_k)^+ \mid X_j = w_j]\}
\end{aligned}
$$

We have

$$
\begin{aligned}
&\partial^2 C_{jk}(w_j, w_k)/\partial w_j^2 \\
&\quad = f_j(w_j)\{E[(X_k - w_k)^+ \mid X_j = w_j] - E[(X_k - w_k)^+]\}, \qquad (12)
\end{aligned}
$$

and (iii) is proved.

□

The properties of $C_{jk}(w_j, w_k)$ derived thus far in this section are general, applicable to any joint p.d.f. of X_j and X_k. If X_j and X_k are bivariate normal, their joint p.d.f. can be uniquely identified by a correlation coefficient, along with the means and variances of X_j and X_k. In reality, it seems unlikely to specify a joint p.d.f. for each pair of correlated projects. The following assumption is imposed.

Assumption 2. For any pair of correlated projects j and k, X_j and X_k follow a bivariate normal distribution with a correlation coefficient ρ_{jk}.

With Assumption 2, the properties of $C_{jk}(w_j, w_k)$ in Proposition 4 can be further quantifed, as shown in the following corollary:

Corollary 5. With Assumption 2, if $\rho_{jk} > 0$ (<0), then the following-statements are true for all feasible w_j and w_k:

(i) $\partial C_{jk}(w_j, w_k)/\partial w_j < 0$ (>0).

(ii) $\partial^2 C_{jk}(w_j, w_k)/\partial w_j w_k > 0$ (<0).

(iii) $\partial^2 C_{jk}(w_j, w_k)/\partial w_j^2 > 0$ (<0). when w_j is sufficiently large, for any given w_k.

Proof. Because the normal distribution is symmetric, it is sufficient to prove the cases when $\rho_{jk} > 0$. First note that when $\rho_{jk} > 0$.

$$\Pr\{X_j > w_j, X_k > w_k\} - \Pr\{X_j > w_j\}\Pr\{X_k > w_k\} > 0 \tag{13}$$

which proves (ii).

To see (i), note that the preceding equation implies

$$\Pr\{X_k > w_k | X_j > w_j\} > \Pr\{X_k > w_k\}, \forall w_k \tag{14}$$

or equivalently, $X_k | X_j > w_j$ is stochastically larger than X_k. Since the function $(x-a)^+$ increases in x, it then follows that

$$E[(X_k - w_k)^+ | X_j > w_j] > E[(X_k - w_k)^+] \tag{15}$$

which proves (i).

To prove (iii), observe that the random variable $(X_k - w_k)^+ | X_j = w_j$ increases stochastically in w_j: therefore $E[(X_k - w_k)^+ | X_j = w_j]$ increases in w_j. In addition, it is easy to see that $E[(X_k - w_k)^+ | X_j = w_j] \to \infty$ (or $-\infty$) as $w_j \to \infty$ (or $-\infty$). Hence for any w_k, there exists a $\hat{w}_j$ such that

$$E[(X_k - w_k)^+ | X_j = w_j] \begin{cases} <0, & \text{if } w_j < \hat{w}_j \\ =0, & \text{if } w_j = \hat{w}_j \\ >0, & \text{if } w_j > \hat{w}_j \end{cases} \tag{16}$$

and (iii) follows. □

The results in Corollary 5 imply that $C_{jk}\,(w_j, w_k)$ may not be convex for every w_j, w_k, and ρ_{jk}. For example, it can be shown that if $\rho_{jk} < 0$, C_{jk} need not be convex. In fact, in our experience we have hardly found that C_{jk} convex.

2.3. *Convexity analysis*

Consider a special case that all project costs are independent except two projects j and k. Assume that both X_j and X_k are normal with means μ_j and μ_k, variances σ_j^2 and σ_j^2, respectively, and their correlation is ρ_{jk}. In this special case, the objective function of (P_2) can be rewritten as follows:

$$\alpha \sum_{\forall i} m_i(w_i) + \beta \sum_{\forall i \neq j,k} v_i(w_i) + \beta[v_j(w_j) + v_k(w_k) + 2C_{jk}(w_j, w_k)]. \tag{17}$$

To study convexity of (17), attention is paid to the last three terms in the brackets because all other terms are already convex. As aforementioned, since $2C_{jk}$ is nonconvex, it is hoped that the convexities of v_j and v_k can bring their sum to be convex. To facilitate the analysis, X_i is transformed (normalized) to a standard normal $Z_i, = (X_i, -\mu_i)/\sigma_i, i = j, k$. Similarly, w_i is normalized to $w_i^n \equiv (w_i - \mu_i)/\sigma_i, i = j, k$. Also define $v_i^n(w_i^n) = \text{Var}[(Z_i - w_i^n)^+], i = j, k$, and $C_{jk}^n(w_j^n, w_k^n) = \text{Cov}[(Z_j - w_j^n)^+ , (Z_k - w_k^n)^+]$, where the superscript n stands for normalization. We have

$$\ell(w_j, w_k) \equiv v_j(w_j) + v_k(w_k) + 2C_{jk}(w_j, w_k) \tag{18}$$

$$= \sigma_j^2 v_j^n\left(w_j^n\right) + \sigma_k^2 v_k^n\left(w_k^n\right) + 2\sigma_j\sigma_k C_{jk}^n\left(w_j^n, w_k^n\right)$$

$$= \sigma_k^2\left(\xi_{jk}^2 v_j^n\left(w_j^n\right) + v_k^n\left(w_k^n\right) + 2\xi_{jk} C_{jk}^n\left(w_j^n, w_k^n\right)\right) \tag{19}$$

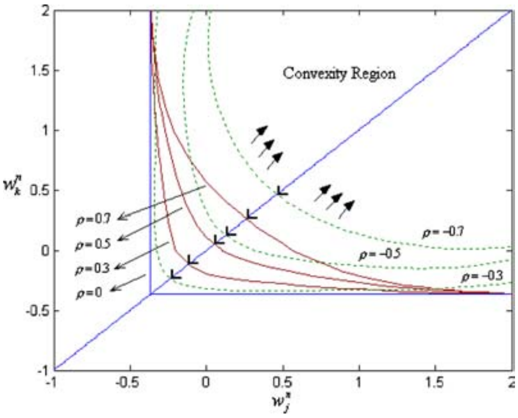
where $\xi_{jk} \equiv \sigma_j/\sigma_k$. Without loss of generality, assume that $\xi_{jk} \in (0, 1]$, i.e., $\sigma_k \geq \sigma_j$. If $\sigma_j \geq \sigma_k$, one can simply switch the roles of j and k.

Not expecting (19) to be convex everywhere, we explore its *convexity region*. Given a function $\ell : R^2 \to R$, defined in (18), and ℓ is continuously differentiable, we define the convexity region of ℓ as the maximal subset of domain in which (19) is convex. That is,

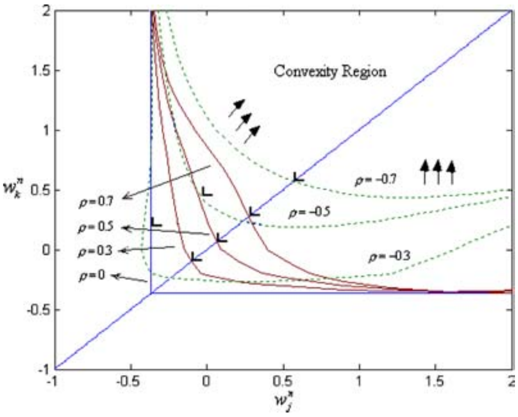
$$\text{Convexity region of } \ell(w_j, w_k)$$

$$= \{(w_j, w_t) \in R^2 \mid \text{both eigenvalues of } \nabla^2\ell(w_j, w_k) \geq 0\}, \tag{20}$$

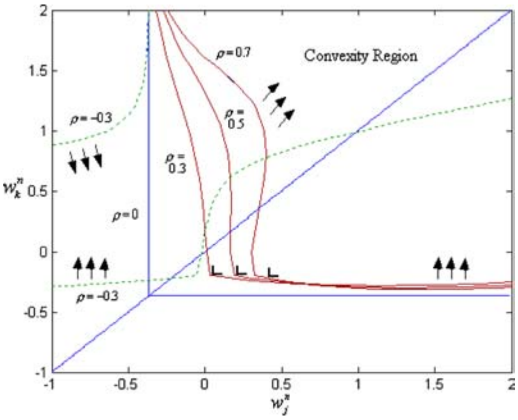
where $\nabla^2\ell$ is the *Hessian* of ℓ. Note that a convexity region may not necessarily be a convex set. From (19), it is clear that its convexity region depends on the correlation ρ_{jk} and the ratio of variances ξ_{jk}. To identify the convexity region, we plot the constant contour of the points of which the smallest eigenvalue of the Hessian of ℓ is zero. Figures 2(a) to (c) show some convexity regions with various values of ρ_{jk} and ξ_{jk}, in which a region enclosed by a constant contour is a convexity region. Note that a dotted constant contour corresponds to a negative ρ_{jk}. Figure 2(a) shows the convexity regions for $\xi_{jk} = 1$, i.e., projects j and k are of the same level of riskiness because $\sigma_j = \sigma_k$. Each convexity region is symmetric to the diagonal ($w_j^n = w_k^n$). When $\rho_{jk} = 0$, the contour is composed of two perpendicular lines crossing at $(w_1^n, w_2^n) = (0.367, 0.367)$, as obtained in Table 1. As ρ_{jk} is increased from 0, the perpendicular corner is gradually smoothed and moving northeast. On the other hand, when ρ_{jk} is decreased from 0, the whole convexity region moves towards northeast. In this paper, each convexity region is approximated by a quadrant subset: $\{(w_j^n, w_k^n) | w_j^n \geq a_j^n(\xi_{jk}, \rho_{jk}), w_k^n \geq a_k^n(\xi_{jk}, \rho_{jk})\}$. Implicitly, we set an upper bound for (w_j^n, w_k^n) at value 2, which corresponds to a probability 97.72% that a project can complete within budget if it is fully funded (at the upper bound of the funding level). To maximize the coverage of the quadrant, we move the vertex of the quadrant towards southwest as much as possible. While such a quadrant is not unique, in this paper we select the one whose vertex is closest to the 45° line, $w_j^n = w_k^n$, implying equal emphasis on each project. Table 2 summarizes the obtained values of (a_j^n, a_k^n) in terms of their c.d.f. value $(F^n(a_j^n),\ F^n(a_k^n))$ obtained by numerical methods, where F^n



(a)



(b)



(c)

Figure 2. Convexity regions for various values of ρ_{jk}, when $\xi_{jk} = 1$, (b) Convexity regions for various values of ρ_{jk}, when $\xi_{jk} = 0.1$, (c) Convexity regions for various values of ρ_{jk}, when $\xi_{jk} = 0.01$.

Table 2
Some values of $(F^n(a_j^n), F^n(a_k^n))$.

ρ_{jk}	$\xi_{jk} = 1$	$\xi_{jk} = 0.5$	$\xi_{jk} = 0.1$	$\xi_{jk} = 0.01$
−0.7	(67.8%, 67.8%)	(71.5%, 71.5%)	*	*
−0.5	(55.0%, 55.0%)	(48.3%, 67.1%)	*	*
−0.3	(40.3%, 40.3%)	(35.4%, 58.0%)	*	*
−0.1	(37.1%, 37.1%)	(35.4%, 39.2%)	*	*
0	(35.4%, 35.4%)	(35.4%, 35.4%)	(35.4%, 35.4%)	(35.4%, 35.4%)
0.1	(38.7%, 38.7%)	(39.4%, 39.4%)	(36.7%, 41.6%)	(49.7%, 35.4%)
0.3	(44.9%, 44.9%)	(45.7%, 45.7%)	(51.5%, 41.6%)	(56.9%, 35.4%)
0.5	(51.9%, 51.9%)	(52.5%, 52.5%)	(57.5%, 41.6%)	(66.0%, 35.4%)
0.7	(60.3%, 60.3%)	(60.9%, 60.9%)	(65.6%, 39.2%)	(76.9%, 35.4%)

*A useful quadrant does not exist.

$(\cdot)$ is the c.d.f. of a standard normal. In figures 2(a) to (c), each (a_j^n, a_k^n) is marked by a 'L' sign.

In figure 2(a) when $\xi_{jk} = 1$, a_j^n and a_k^n are chosen to be equal because of the symmetry of the convexity regions. Basically when ξ_{jk} is decreased, i.e., σ_j/σ_k is decreased, convexity regions corresponding to a positive ρ_{jk} move eastwards, and those corresponding to a negative ρ_{jk} move westwards (see figure 2(b)). When $\xi_{jk} < 0.1$, i.e., project k is more than 10 times as risky as project j, a useful quadrant cannot be identified to approximate a convexity region corresponding to negative ρ_{jk} (see figure 2(c)).

Summary of Convexity Regions

The results in Table 2 can be generalized as follows.

1. $\rho_{jk} > 0$: Convexity exists for a problem with a "reasonable" lower bound for funding level (roughly $F^n(a_i^n) > 60\%, i = j, k$) and a "common" correlation coefficient (roughly $\rho_{jk} \leq 0.5$).
2. $\rho_{jk} < 0$: Similar to the case with $\rho_{jk} > 0$, except that the level of riskiness between the two correlated projects should be close (roughly $\sigma_k < 10\sigma_j$).

We conclude that convexity exists in most practical situations for the project funding allocation problem. Even if convexity may not exist for some instances, it should be emphasized that the problem remains to be solvable. In theory, without convexity, (P_2) may have local optima.

The following proposition will show the effect of optimal project funding allocation due to correlation with convexity in place:

Proposition 6. Assume that all X_i are independent except j and k satisfying Assumption 2. Consider those cases that $a_i^n(\xi_{jk}, \rho_{jk})$ exists for $i = j, k$. Suppose $a_i = a_i^n(\xi_{jk}, \rho_{jk}) <$

b_i, for $i = j, k$. Let $\hat{\mathbf{w}}$ be the optimal solution of (P_2). If $\rho_{jk} > 0$ (or <0), then

$$\hat{w}_i > (<) \tilde{w}_i, \quad \text{if } i = j, k \tag{21a}$$

$$\hat{w}_i < (>) \tilde{w}_i, \quad \text{if } i = j, k, \tag{21b}$$

where $\tilde{\mathbf{w}}$ is the optimal funding levels when all projects are independent.

Proof. It suffices to show the case with $\rho_{jk} > 0$ only. We have

$$\partial g(\mathbf{w})/\partial w_j = \alpha m'_j(w_j) + \beta v'_j(w_j) + 2\beta \partial C_{jk}(w_j, w_k)/\partial w_j \tag{22a}$$

$$\partial g(\mathbf{w})/\partial w_k = \alpha m'_k(w_k) + \beta v'_k(w_k) + 2\beta \partial C_{jk}(w_j, w_k)/\partial w_k \tag{22b}$$

$$\partial g(\mathbf{w})/\partial w_i = \alpha m'_i(w_i) + \beta v'_i(w_i) \quad \forall i \neq j, k \tag{22c}$$

If $a_i = a_i^n(\xi_{jk}, \rho_{jk}) < b_i$, for $i = j, k$, then g is convex over the feasible region. From Corollary 5 (i), both $\partial C_{jk}(w_j, w_k)/\partial w_j$ and $\partial C_{jk}(w_j, w_k)/\partial w_k$ are negative. Obviously $\tilde{\mathbf{w}}$ is a feasible but not an optimal solution for (P_2). Since $\partial g(\mathbf{w})/\partial w_i$ is increasing in w_i for all i because of the convexity of g, projects j and k should increase their funding levels from $\tilde{w}_j$ and $\tilde{w}_k$ to make up the loss in lambda due to negative $\partial C_{jk}(w_j, w_k)/\partial w_j$ and $\partial C_{jk}(w_j, w_k)/\partial w_k$, respectively, while other projects should decrease their funding levels from $\tilde{w}$ (because of the budget constraint (5)) to reach the equal-lambda optimality condition. (This also includes the case in which some projects are funded at either an upper bound or a lower bound and cannot be further increased or decreased in funding level.) Since g is continuously differentiable, $\partial g(\mathbf{w})/\partial w_i$ is continuous, which ensures the existence of the optimal lambda. □

Summary of the properties when projects j and k are correlated

1. $\rho_{jk} > 0$: A positive correlation would help both projects *j* and *k* obtain more funding than when they are independent. Because doing so improves (i.e., decreases) the COR of the entire project portfolio.
2. $\rho_{jk} < 0$: A negative correlation would cause both projects *j* and *k* to have less funding than when they are independent.
3. The preceding two properties can be extended to a project portfolio with multiple pairs of projects that are mutually and exclusively correlated. Compared with the case that all projects are independent, those pairs that are positively correlated get more funding and those pairs that are negatively correlated get less. However, when some project, say j, is involved in two different correlated pairs, e.g., $\mathrm{Cov}(X_j, X_{k1}) \neq 0$ and $\mathrm{Cov}(X_j, X_{k2}) \neq 0$, the preceding properties may not apply.

While conventional project management techniques require managerial attention in funding and controlling project in isolation, our fhding shows that it may be benefical for an agency to identify projects that are correlated for funding allocation and management control. To further determine whether choosing projects with positive correlation or negative correlation is more benefical to an agency requires more information such as the nature of the correlation.

2.4. *Discrete distribution case*

In reality, cost uncertainty is often modeled in terms of scenarios, or more generally by discrete distributions. The findings obtained in this paper can be extended to this case. For example, suppose X_i is a random variable representing the actual cost of project i, which is associated with a probability mass function $p_i(x_i^j)$ of X_i by

$$p_i\left(x_i^j\right) = Pr\left(X_i = x_i^j\right), \tag{23}$$

where $\{x_i^1, x_i^2, \ldots, x_i^J\}$ are the possible scenarios of the realization of X_i and J is the number of total scenarios. Given a funding level w_i, the expected COR of the project is

$$\hat{m}_i(w_i) = E[(X_i - w_i)^+] = \sum_{j=1}^{J} p_i\left(x_i^j\right)\left(x_i^j - w_i\right)^+ \tag{24}$$

It has been shown in Section 2.3 that $m_i(w_i)$ is a smooth, strictly decreasing, and convex function when X_i, is continuous random variable. When X_i, is discrete, it can be shown that $\hat{m}_i$, is a piecewise-linear function, which is still strictly decreasing and convex. Similarly, the variance of COR is a piecewise-nonlinear function. Therefore, when the project cost is considered in discrete scenarios, the optimal funding allocation problem (P) becomes a nonlinear and nonsmooth optimization problem, which is generally very difficult to solve. The optimality conditions and properties derived in the previous sections at least can be used to approximate the optimal funding allocation.

2.5. *Integrating funding allocation and project selection*

The proposed project funding allocation problem can be integrated with the deterministic project selection problem as a multi-objective problem:

$$(O_1) \quad \max_{u_i} \sum_{i=1}^{I} R_i u_i$$

$$(O_2) \quad \min_{w} g(\mathbf{w})$$

$$\text{s.t.} \quad \sum_{i=1}^{I} w_i u_i \leq B \tag{25a}$$

$$a_i u_i \leq w_i \leq b_i u_i, \quad i = 1, \ldots, I \tag{25b}$$

$$u_i \in \{0, 1\}, \quad i = 1, \ldots, I, \tag{25c}$$

where R_i represents forecasted reward associated with project i and u_i is a zero-one variable indicating whether project i is selected or not. Both objectives (O_1) and (O_2) are indeed conflicting. Without minimizing the portfolio COR, (O_1) would select as many projects as possible, with each project funded at the minimal possible level a_i. On the other hand, without considering (O_1), (O_2) tends to select few or even a single project to ensure that each project is fully funded so as to achieve the minimal COR. Therefore, a clear trade-off exists between more projects and a lower portfolio COR. This problem can be viewed as a bi-level problem such that the upper level problem (O_1) performs project selection, and based on which the lower level problem (O_2) allocates funding with the portfolio COR minimized. How to obtain a project portfolio that balances the trade-off depends on the DM's return/risk preference.

3. Numerical results

The proposed method has been applied to a capital-budgeting decision-making of a government agency. Given budget limitations, five different projects out of 70 projects were selected to pursue. All five projects are different in cost requirements, completion time, risks involved, and managerial attention. From statistical results, the cost distributions for each project are obtained. The following are some brief descriptions of these five projects.

Project 1 is a typical building renovation project. Project cost data are collected from previous building renovation efforts. Due to the inherent complexity in renovation projects, the difficulty for estimating the actual up-front project cost has been observed. Historical data also show that the project cost is more likely to be high than low. The project cost is modeled as a right-skewed beta distribution, beta(4, 2) between \$80 K and \$120 K.

Project 2 is an equipment supply project for facility expansion. An equipment supply project normally has an uncertain and long lead time. The costs of this type of project include a large cushion amount to cope with uncertain project delivery time. The cushion is essentially used to absorb changes in material and labor costs over the lead period. Since the lead time may be reduced by improved coordination, the project cost does have a tendency to be greatly reduced and is modeled as a left-skewed beta distribution, beta(2, 4) between \$160 K and \$240 K.

Projects 3 to 5 are greenfield building projects. Since these projects are new, cost data are collected from similar projects both within and outside the agency. Upon identifying the data for project costs for similar projects, it was observed that the project cost distributions behave as a normal distribution, i.e., $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 3, \ldots, 5$, where μ_i and σ_i^2 are the mean and variance of the normal distribution. Projects 3 and 4 are of the same nature, but at different geographical locations. They are modeled to have the same mean, but Project 4 has a higher standard deviation. A correlation $\rho_{3,4}$ between X_3 and X_4 is also modeled. Project 5 involves a bigger area and is more costly than Projects 3 and 4.

Table 3
Basic information for the projects.

Project i	Distribution of X_i (\$K)	a_i (\$K)	$F_i(a_i)$	b_i (\$K)	$F_i(b_i)$
1	beta(4,2) over [80, 120]	103.08	29.86%	120	100.0%
2	beta(2,4) over [160, 240]	180.70	38.57%	240	100.0%
3	normal, $N(400, 60^2)$	377.44	35.35%	520	97.72%
4	normal, $N(400, 90^2)$	366.16	35.35%	580	97.72%
5	normal, $N(500, 100^2)$	462.40	35.35%	700	97.72%

For Projects 1 to 5, the lower bounds of funding level are assumed to be γ_i^*, $i = 1, \ldots, 5$, as in (18). The values of $F_i(\gamma_i^*)$, $i = 1, \ldots, 5$ for the two beta distributions assigned for Projects 1 and 2 and the normal distribution can be found in Table 1. For Projects 1 and 2, the upper bounds of funding level are simply assigned to be the maxima of the ranges for the beta distributions assigned, i.e. $F_i(b_i) = 100\%$, $i = 1, 2$. For Projects 3 to 5, their funding upper bounds b_i are assigned to be $\mu_i + 2\sigma_i$, $i = 3, \ldots, 5$. That means even if a project is fully funded with b_i, the DM still bears a 2.28% ($\bar{F}_i(b_i)$) chance of cost overrun. The distributions of the project costs and the bounds of funding-level are summarized in Table 3. The total budget B is \$1,700 K.

The case where all projects are independent is tested first, i.e., $\rho_{3,4} = 0$. The corresponding results are summarized in Table 4. When $\alpha = 1$ and $\beta = 0$, the DM is risk neutral. According to Proposition 3, all projects are funded such that they have the same probability of cost overrun, 65.25% in this case, regardless of the size of the project. Gradually increasing β shifts the emphasis of the COR from its mean to its variance. Since the scale of the variance of the COR is roughly the square of that of the mean value, the solution is very sensitive to the change of β. It can be seen that as the value of β is increased, funding is shifted from smaller projects, such as 1 and 2, to bigger projects, such as 4 and 5. Doing so, the variance of the portfolio COR is reduced but the mean is increased. When β is increased up to 0.03, both Projects 1 and

Table 4
Funding allocation (\$K) and the corresponding probability of completion within budget: all projects are independent.

α	β	w_1 $F_1(w_1)$	w_2 $F_2(w_2)$	w_3 $F_3(w_3)$	w_4 $F_4(w_4)$	w_5 $F_5(w_5)$	E[COR]	Var[COR]$^{\frac{1}{2}}$
1	0	110.37	191.51	423.55	435.32	539.25	63.41	66.52
		65.25%	65.25%	65.25%	65.25%	65.25%		
0.995	0.005	108.32	187.66	419.83	438.49	545.70	63.83	65.15
		54.47%	56.35%	62.95%	66.56%	67.61%		
0.970	0.030	103.08	180.70	413.25	445.55	557.42	67.30	62.80
		29.85%	38.57%	58.74%	69.36%	71.71%		
0	1	103.08	180.70	402.98	449.33	563.90	68.98	62.64
		29.85%	38.57%	51.98%	70.82%	73.86%		

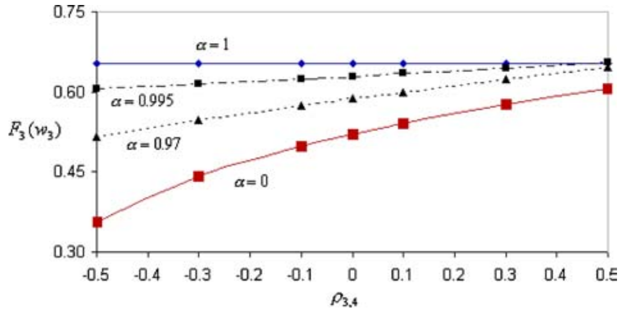
Table 5
Funding allocation ($K) and the corresponding probability of completion within budget: projects 3 and 4 are dependent ($\alpha = 0$, $\beta = 1$).

$\rho_{3,4}$	w_1 $F_1(w_1)$	w_2 $F_2(w_2)$	w_3 $F_3(w_3)$	w_4 $F_4(w_4)$	w_5 $F_5(w_5)$	E[COR]	Var[COR]$^{\frac{1}{2}}$
−0.50	103.08	180.70	377.88	436.55	601.79	73.39	56.46
	29.85%	38.57%	35.62%	65.77%	84.56%		
−0.30	103.08	180.70	391.22	442.97	582.02	72.80	59.03
	29.85%	38.57%	44.18%	68.35%	79.39%		
−0.10	103.08	180.70	399.58	447.41	569.22	69.87	61.41
	29.85%	38.57%	49.72%	70.08%	75.56%		
0	103.08	180.70	402.98	449.33	563.90	68.98	62.64
	29.85%	38.57%	51.98%	70.82%	73.86%		
0.10	103.08	180.70	406.07	451.15	559.00	68.33	63.85
	29.85%	38.57%	54.03%	71.51%	72.24%		
0.30	103.08	180.70	411.53	454.62	550.06	67.57	66.32
	29.85%	38.57%	57.62%	72.80%	69.17%		
0.50	103.08	180.70	416.17	458.22	541.82	67.37	68.89
	29.85%	38.57%	60.62%	74.11%	66.21%		

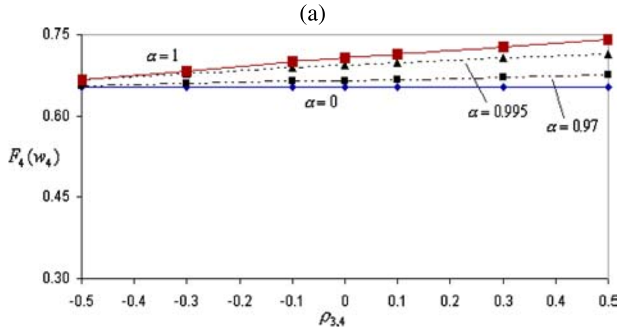
2 are already funded at their lower bounds (with only 29.85 and 38.57% chance for the projects to be completed within budget, respectively). Note that these two lower bounds for Projects 1 and 2 may seem too low. Indeed they are. These two lower bounds are assigned purely for illustrative purpose. The intention is to show the reader how a small project may not be as competitive as a big project in terms of securing funding from a project portfolio perspective. Therefore, setting an appropriate lower bound for a small project is equivalent to protecting the project. Failing to do so may result in incompletion and cost overruns.

Next, the case is tested with a nonzero correlation between Projects 3 and 4, i.e., $\rho_{3,4} \neq 0$. Because of space limit, in Table 5 only the results with $\alpha = 0$ and $\beta = 1$ are displayed. Consider the case with $\rho_{3,4} = 0$ as the baseline. With a positive correlation, the two smaller-sized Projects 3 and 4 can together "grab" some funding from the Project 5, and may lose some funding to Project 5 if the correlation is negative. In the last 2 columns of Table 5, it can be seen that the variance of the portfolio COR increases as $\rho_{3,4}$ increases, and vice versa. Also whenever the variance of the portfolio COR increases, its mean decreases. This is due to their conflicting nature in this optimization problem. In all cases summarized in Table 5 the funding of Projects 1 and 2 remain at their lower bounds. Figures 3(a) and (b) depict the funding level changes of Projects 3 and 4, respectively, at different risk preferences. It can be seen that as $\rho_{3,4}$ increases, the funding levels of both Projects 3 and 4 increases regardless of the risk preference. Apparently, Project 3 is more sensitive to the change of $\rho_{3,4}$ than Project 4 because it has a smaller variance and is less risky.

Next we observe the effect of project size and project riskiness to the funding-level. Consider a baseline case: $\mu_3 = \mu_4 = 400$ and $\sigma_3 = \sigma_4 = 60$, $\rho_{3,4} = 0.3$, and all other



(a)



(b)

Figure 3. (a) Funding level of Project 3 vs. the value of $\rho_{3,4}$, (b) Funding level of Project 4 vs. the value of $\rho_{3,4}$.

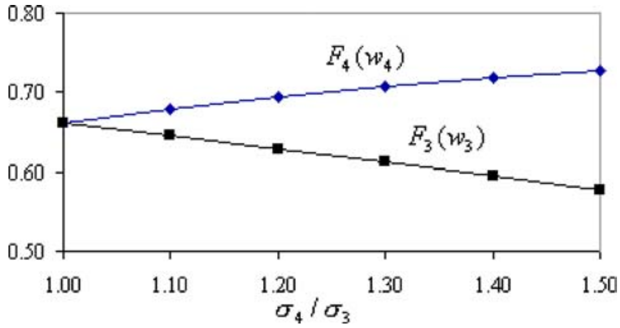


Figure 4. Funding levels of Projects 3 and 4 vs. the value of σ_4.

projects are the same as in Table 3. That is, both Projects 3 and 4 are equivalent in terms of project size and riskiness. With respect to the baseline, we first increase the riskiness of Project 4, i.e. σ_4, from 60 to 90 with other parameters fixed. The result is shown in figure 4(a). It can be seen that Project 4 gains additional funding as it becomes riskier, but Project 3 loses funding. In figure 4(a) it can be seen that the funding loss of Project 3 is always greater than the funding gain of Project 4, in terms of the probability of project

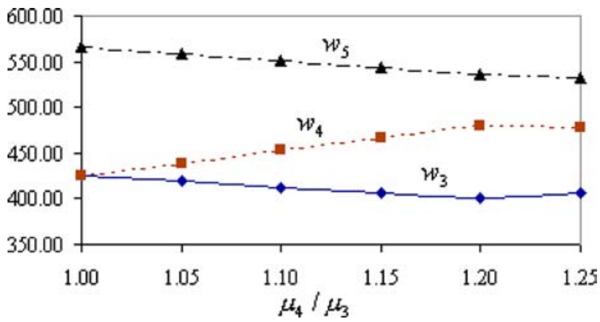


Figure 5. Funding levels of Projects 3 and 4 vs. the value of μ_4.

completion within budget. That is reasonable because as σ_4 increases, the entire project portfolio actually becomes riskier.

We next test the effect of project size. With respect to the same baseline above, now we increase the size of Project 4, i.e. μ_4, from 400 to 500, with other parameters fixed. In figure 5, as μ_4 increases, Project 4's funding level w_4 increases and Project 3's funding level w_3 decreases. Both the increasing and decreasing rates for Projects 4 and 3 are linear. However, when $\mu_4 = 500$, Project 4's size is the same as that of Project 5. Since Project 5 is riskier, it turns out that Project 4's funding level decreases. This result seems to suggest that when all projects have a different project size, project size dominates the allocation of the funding. But when two or more projects are of the same size, their riskiness will determine the funding level.

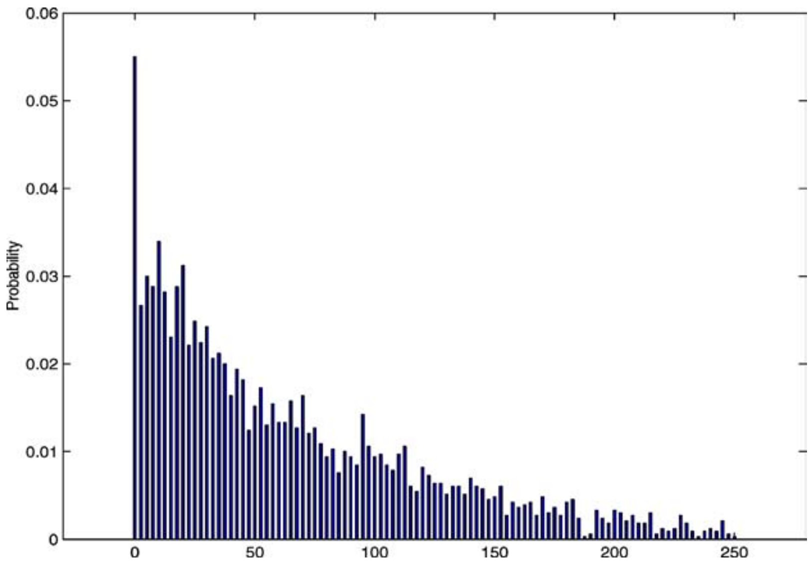


Figure 6. Frequency chart of the portfolio COR, with $\alpha = 0$, $\beta = 1$, $\rho_{3,4} = 0$.

Finally, it may seem that the mean value of the portfolio COR at the level of \$60 to \$70 K is not significant for a \$1,700 K total budget. Note that the standard deviation of the portfolio COR is also close to the mean value. The fact is that the shape of the distribution function of the portfolio COR normally has a long tail in the upside, as shown in figure 6, which displays the frequency chart of the portfolio COR, assuming $\alpha = 0$, $\beta = 1$, and $\rho_{3,4} = 0$. The following are the statistics corresponding to the case in figure 6: mean: \$68.98 K, standard deviation: \$62.64 K, median: \$51.13 K, skewness: 1.36, Kur-tosis: 5.18, and range maximum: \$520 K. The result shows the high riskiness of cost overrun.

Note that in Table 5 it can be seen that w_4 is always greater than w_3 because the costs in Project 4 have a greater variance, although their mean values are the same. In this problem, X_3 and X_4 are identical except that X_4 has a greater variance. If both X_3 and X_4 are identical, w_3 and w_4 should always have the same value.

4. Conclusions

Projects normally have to compete for funding, given a budget limitation. Once a project is funded, it then becomes the project manager's responsibility to complete the project within budget. Budgeting and cost control, which have been commonly treated as two disjointed functions, are actually closely related: if a project is underfunded, it is exposed to a higher COR. In this paper, the funding allocation problem is defined to be an optimization problem that minimizes portfolio COR subject to a fixed budget. It has been shown in this paper that with competition, the combination of a project's size, riskiness, its correlation with other projects, and DM's preference ultimately decides its funding level, from the perspective of portfolio optimization in the COR. Project selection decisions are made in the concept phase of a project's life cycle, where ability to influence project parameters has the highest likelihood. We have demonstrated the feasibility for integrating project selection and and project COR minimization. This integration during the concept phase can add value to ability to influence the project and provide a basis for more effective project control.

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Note

1. More precisely, it is the marginal reduction of some linear combination of the mean and variance of the portfolio COR.

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