A Note on Constraining AR(2) Parameters in Estimation

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The following is a note on constraining second-order autoregressive (AR(2)) parameters in maximum likelihood estimation. The method presented here constrains AR(2) parameters to imply stationary dynamics. This is essential to estimation since parameters, if not constrained, often stray into regions that lead to nonsensical results (e.g., complex likelihood values), ultimately resulting in the optimization program crashing. Meanwhile, it is important to allow for complex roots for an AR(2), even though it might be much easier to constrain the roots if we force them to be real.

The main trick to constraining the AR(2) parameters to imply stationary and potentially complex dynamics is to note that the roots will be complex conjugates if they are complex. That is,

$$l_1 = a + bi, \tag{1}$$

$$l_2 = a - bi, \tag{2}$$

where, for simplicity, we consider the eigenvalues of the companion matrix (i.e., the inverse roots of the AR(2) process). Then, recall that the eigenvalues of the companion matrix F are solved as follows: [1]

$$\left| F - l I_2 \right| = 0 \tag{3}$$

where the companion matrix F is given by

$$F \equiv \begin{bmatrix} f_1 & f_2 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

[1] See Chapter 1 of Hamilton (1994) for a discussion of the companion form representation of an AR process.

From equation (3), we can solve for the AR(2) parameters in terms of the eigenvalues as follows:

$$f_1 = l_1 + l_2 , \tag{5}$$

$$f_2 = -l_1 l_2 . \tag{6}$$

If we substitute equations (1) and (2) into equations (5) and (6), we get

$$f_1 = 2a , \tag{7}$$

$$f_2 = -(a^2 + b^2) , \tag{8}$$

which makes use of the fact that $i^2 \equiv -1$.

Note that equations (7) and (8) apply to the case that the roots are real. To see why, consider $b^2 < 0$. This will imply the roots in equations (1) and (2) are real. So, we can use equations (7) and (8) for our constraints without making any assumptions about whether the roots are real or complex.

Then, to constrain a, consider the trigonometric argument that implies that, if the eigenvalues are inside a unit circle, a must be less than one in absolute value:

$$a = a_{uc} /(1 + |a_{uc}|) , \tag{9}$$

where a_{uc} is unconstrained, except that it must be real. In terms of b, it turns out that it makes more sense to constrain b^2, but to allow it to be negative, which, again, corresponds to the case in which the roots of the AR process are real. Given that f_1 roams between –2 and 2 according to equations (7) and (9), the following constraint on b^2 keeps f_2 in the appropriate region of the “triangle” diagram on p. 17 of Hamilton (1994):

$$b^2 = (1-|a|)b_{uc}^2 + |a| - a^2 , \tag{10}$$

where b_{uc}^2 is unconstrained, except that it must be real. Again, it can be negative.

Reference

Hamilton, James D., 1994, *Time Series Analysis* (Princeton University Press).

Erratum to "A Note on Contraining AR(2) Parameters in Estimation"[1]

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In the original note, I stated that b_{uc}^2 in equation (10) is unconstrained, except that it must be real. This is incorrect. It must lie between -1 and 1. All of the posted code on my website that makes use of these constraints for AR(2) parameters include this constraint on b_{uc}^2. Apologies for any confusion.

[1] I thank Yunjong Eo for finding the error in the original note.