Economics Letters 75 (2002) 123–127

economics letters

www.elsevier.com/locate/econbase

A state–space approach to calculating the Beveridge–Nelson decomposition

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Received 27 February 2001; accepted 1 October 2001

Abstract

A state–space approach provides a general unified framework for calculation of the Beveridge–Nelson decomposition for a wide variety of time series models, including all univariate and vector ARIMA models.

Keywords: Beveridge–Nelson decomposition; State–space models; Kalman filter

JEL classification: C22; C32

1. Introduction

The decomposition method introduced by Beveridge and Nelson (1981) provides a convenient way to estimate the permanent and transitory components of an integrated time series. Given a forecasting model for the first-differences of the series, the Beveridge–Nelson (BN) trend is the long-run forecast of the level of the series (minus any deterministic drift) and the BN cycle is the gap between the present level of the series and its long-run forecast.

In practice, calculation of the exact BN trend and cycle is often complicated by the presence of infinite sums in the long-run forecast. This paper points out, however, that exact calculation of the BN trend and cycle is relatively straightforward if the forecasting model can be cast into state–space form. Examples include all univariate and vector ARIMA models. Thus, the state–space approach provides a general unified framework for BN trend/cycle calculation for all of the cases discussed in the previous literature, including Cuddington and Winters (1987), Miller (1988), Newbold (1990), and Ariño and Newbold (1998).

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0165-1765/02/$ – see front matter
PII: S0165-1765(01)00581-X

2. Motivation

To motivate the state–space approach, first consider an integrated time series $\{Y_t\}_{-\infty}^{+\infty}$ that can be most accurately forecast using a stationary univariate AR(1) model for its first differences:

$$(\Delta Y_t - \mu) = \phi(\Delta Y_{t-1} - \mu) + \epsilon_t \tag{1}$$

where $\epsilon_t \sim i.i.d.N(0, \sigma^2)$, $|\phi| < 1$, and $\Delta Y_t \equiv Y_t - Y_{t-1}$. By considering the implied Wold form from the AR(1) model, it is straightforward to show that, under the assumption of normality, the minimum mean squared error (MSE) j-period-ahead forecast of the first difference is:

$$E_t[(\Delta Y_{t+j} - \mu)] = \phi^j(\Delta y_t - \mu) \tag{2}$$

where lower-case Δy_t denotes the realized value of the random variable ΔY_t. The BN trend, denoted τ_t, is defined as the minimum MSE forecast of the long-run level of the series (minus the deterministic drift) or, equivalently, the present level of the series plus the infinite sum of the minimum MSE j-period-ahead first difference forecasts:

$$\tau_t \equiv \lim_{J\to\infty} E_t[Y_{t+J} - J \cdot \mu] = y_t + \lim_{J\to\infty} \sum_{j=1}^{J} E_t[(\Delta Y_{t+j} - \mu)] \tag{3}$$

Thus, substituting (2) into (3), the BN trend of observation y_t for the AR(1) case is:

$$\tau_t = y_t + \frac{\phi}{1-\phi}(\Delta y_t - \mu) \tag{4}$$

That is, the trend is the present level plus the long-run impact of the transitory momentum in the series implied by the deviation of Δy_t from its steady-state level $\mu \equiv E[\Delta Y_t]$. Meanwhile, the BN cycle, denoted c_t, for the AR(1) case is:

$$c_t = -\frac{\phi}{1-\phi}(\Delta y_t - \mu) \tag{4}$$

Note that the cycle is defined in the conventional way as the deviations from the trend (i.e. $c_t \equiv y_t - \tau_t$).[1] Then, as discussed in Morley et al. (2001), the BN trend and cycle provide estimates of the permanent and transitory components of y_t.

3. A state–space approach

Given the BN trend/cycle decomposition for the AR(1) case, it is straightforward to generalize to any case where the first differences of $\{Y_t\}_{-\infty}^{+\infty}$ can be most accurately forecast by a model that can be cast into state–space form, including all univariate and vector ARMA models. In particular, suppose $(\Delta Y_t - \mu)$ is a linear combination of the elements of a $kx1$ state vector X_t:

[1] Beveridge and Nelson (1981) define the BN cycle as the trend minus the level.

$$\Delta Y_t - \mu = [h_1 \quad h_2 \quad \cdots \quad h_k] X_t \tag{5}$$

where h_i, $i = 1, 2, \ldots, k$, is the weight of the ith element of X_t in determining $(\Delta Y_t - \mu)$. Furthermore, suppose state vector X_t evolves according to the following first-order stochastic difference equation:

$$X_t = FX_{t-1} + v_t \tag{6}$$

where $v_t \sim N(0, \Omega)$ and the eigenvalues of F are less one in modulus. Then, it is straightforward to show that the minimum MSE j-period-ahead forecast of the first difference ΔY_{t+j} is:

$$E_t[\Delta Y_{t+j}] = [h_1 \quad h_2 \quad \cdots \quad h_k] F^j E_t[X_t] \tag{7}$$

Note that, since the state vector may contain unobserved elements (see, for example, the ARIMA(2,1,2) example considered below), the expectation $E_t[X_t]$ may have to be obtained prior to calculating (7). Fortunately, the Kalman filter, which can be employed to obtain exact maximum likelihood estimates for state–space models with unobserved elements, provides this expectation.[2] Thus, denoting the Kalman filtered calculation of the expected state vector $X_{t|t} \equiv E_t[X_t]$, the BN trend of observation y_t for the general case is:

$$\tau_t = y_t + [h_1 \quad h_2 \quad \cdots \quad h_k] F(I-F)^{-1} X_{t|t} \tag{8}$$

Meanwhile, the BN cycle of y_t for the general case is:

$$c_t = -[h_1 \quad h_2 \quad \cdots \quad h_k] F(I-F)^{-1} X_{t|t} \tag{9}$$

Again, the BN trend and cycle provide estimates of the permanent and transitory components of y_t.

4. Two examples

To illustrate the general usefulness of this approach, two examples are provided. The first example is a bivariate vector error correction model (VECM) as used for aggregate income and consumption in Cochrane (1994). The second example is a univariate ARIMA(2,1,2) model as used for real GDP in Morley et al. (2001).

4.1. A bivariate VECM

Cochrane (1994) employs a special case of the approach proposed here to calculate the BN trend and cycle of observed aggregate income (y_t) given a VECM forecasting model of aggregate income $\{Y_t\}_{-\infty}^{+\infty}$ and consumption $\{Y_t^c\}_{-\infty}^{+\infty}$. A slightly simplified form of his model has the following state–space representation:[3]

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[2]See Harvey (1990) for the full details of the Kalman filter and state–space models. Technically, the Kalman filter calculates the minimum MSE linear projection of the state vector on the observable data. This is equal to the expected value under a Normality assumption.

[3]Cochrane (1994) considers second-order dynamics for the first differences of output and consumption.

$$\begin{bmatrix} \Delta Y_t - \mu \\ \Delta Y_t^c - \mu \\ Y_t - Y_t^c - \alpha \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{11} - \gamma_{21} & \gamma_{12} - \gamma_{22} & 1 + \gamma_{13} - \gamma_{23} \end{bmatrix} \begin{bmatrix} \Delta Y_{t-1} - \mu \\ \Delta Y_{t-1}^c - \mu \\ Y_{t-1} - Y_{t-1}^c - \alpha \end{bmatrix} + \begin{bmatrix} v_{yt} \\ v_{ct} \\ v_{yt} - v_{ct} \end{bmatrix} \tag{10}$$

or, more compactly,

$$X_t = FX_{t-1} + v_t \tag{10'}$$

where $v_t \sim N(0,\ \Omega)$ and the eigenvalues of F are less one in modulus, which corresponds to cointegration of aggregate income and consumption with cointegrating vector $[1\ -1]$. Then, noting that $(\Delta Y_t - \mu) = [1\ \ 0\ \ 0]\, X_t$, the BN trend of observed y_t is:

$$\tau_t = y_t + [1\ \ 0\ \ 0]\, F(I - F)^{-1}\, x_t \tag{11}$$

where the $X_{t|t}$ term in (8) is set to the realized value of the state vector x_t since, in this example, its elements are all observable at time t. The BN cycle of y_t is:

$$c_t = -[1\ \ 0\ \ 0]\, F(I - F)^{-1}\, x_t \tag{12}$$

4.2. A univariate ARIMA(2,1,2)

Morley et al. (2001) employ the state–space approach proposed here to calculate the exact BN trend and cycle of observed real GDP (y_t) given a reduced form ARMA(2,2) forecasting model of the first differences of $\{Y_t\}_{-\infty}^{+\infty}$. It is important to note that the calculation of the exact BN trend and cycle is nontrivial in this case due to the presence of unobservable moving average terms in the forecasting equation. Also, it should be noted that there are multiple possible state–space representations for the model. However, the companion form representation for an ARMA(2,2) is convenient since it has $(\Delta Y_t - \mu)$ as the first element of the state vector X_t:

$$\begin{bmatrix} \Delta Y_t - \mu \\ \Delta Y_{t-1} - \mu \\ e_t \\ e_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \theta_1 & \theta_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta Y_{t-1} - \mu \\ \Delta Y_{t-2} - \mu \\ e_{t-1} \\ e_{t-2} \end{bmatrix} + \begin{bmatrix} e_t \\ 0 \\ e_t \\ 0 \end{bmatrix} \tag{13}$$

or, more compactly:

$$X_t = FX_{t-1} + v_t \tag{13'}$$

where $v_t \sim N(0,\ \Omega)$ and the eigenvalues of F are less one in modulus (equivalently, the roots of $(1 - \phi_1 z - \phi_2 z^2) = 0$ lie outside the unit circle). Then, the BN trend of observed y_t is:

$$\tau_t = y_t + [1\quad 0\quad 0\quad 0]\, F(I - F)^{-1}\, X_{t|t} \tag{14}$$

Meanwhile, the BN cycle is:

$$c_t = -[1\quad 0\quad 0\quad 0]\, F(I - F)^{-1}\, X_{t|t} \tag{15}$$

Again, the problem of unobserved moving average terms in the forecasting equation is solved by using the Kalman filter calculation of $X_{t|t} = E_t[X_t]$.

5. Conclusion

A state–space approach provides a straightforward and general unified framework for BN trend/cycle calculation for a wide variety of models, including all univariate and vector ARIMA models.

Acknowledgements

I have received helpful comments from Charles Nelson and Eric Zivot.

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