14.281 Contract Theory Notes

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Contents

1 Introduction

1.1 Situating Contract Theory

Think of (at least) three types of modelling environments

1. Competitive Markets: Large number of players $\rightarrow$ General Equilibrium Theory
2. Strategic Situations: Small number of players $\rightarrow$ Game Theory
3. Small numbers with design $\rightarrow$ Contract Theory & Mechanism Design

- Don't take the game as given
- Tools for understanding institutions

1.2 Types of Questions

1.2.1 Insurance

- 2 parties A & B
- A faces risk - say over income $Y_A = 0, 100, 200$ with probabilities $1/3, 1/3, 1/3$ and is risk-averse
- B is risk-neutral
- Gains from trade
- If A had all the bargaining power the risk-sharing contract is B pays A 100
- But we don't usually see full insurance in the real world
    1. Moral Hazard (A can influence the probabilities)
    2. Adverse Selection (There is a population of A's with different probabilities & only they know their type)

1.2.2 Borrowing & Lending

- 2 players
- A has a project, B has money
- Gains from trade
- Say return is $f(e, \theta)$ where e is effort and θ is the state of the world
- B only sees f not e or θ
- Residual claimancy doesn't work because of limited liability (could also have risk-aversion)
- No way to avoid the risk-return trade-off

1.2.3 Relationship Specific Investments

- A is an electricity generating plant (which is movable *pre hoc*)
- B is a coal mine (immovable)
- If A locates close to B (to save transportation costs) they make themselves vulnerable
- Say plant costs 100
- "Tomorrow" revenue is 180 if they get coal, 0 otherwise
- B's cost of supply is 20
- Zero interest rate
- NPV is 180-20-100=60
- Say the parties were naive and just went into period 2 cold
- Simple Nash Bargaining leads to a price of 100
- $\pi_A = (180 - 100) - 100 = -20$
- An illustration of the **Hold-Up Problem**
- Could write a long-term contract: bounded between 20 and 80 due to zero profit prices for A & B, maybe it would be 50
- But what is contract are incomplete – the optimal contract may be closer to no contract than a very fully specified one
- Maybe they should merge?

2 Mechanism Design

- Often, individual preferences need to be aggregated
- But if preferences are private information then individuals must be relied upon to reveal their preferences
- What constraints does this place on social decisions?
- Applications:
  - Voting procedures
  - Design of public institutions
  - Writing of contracts
  - Auctions
  - Market design
  - Matching

2.1 The Basic Problem

- Suppose there are I agents
- Agents make a collective decision x from a choice set X
- Each agent privately observes a preference parameter $\theta_i \in \Theta_i$
- Bernoulli utility function $u_i(x, \theta_i)$
- Ordinal preference relation over elements of $X \succcurlyeq_i (\theta_i)$
- Assume that agents have a common prior over the distribution of types
  - (i.e. the density $\phi(\cdot)$ of types on support $\Theta = \Theta_1 \times ... \times \Theta_I$ is common knowledge)

Remark 1. *The common prior assumption is sometimes referred to as the Harsanyi Doctrine. There is much debate about it, and it does rule out some interesting phenomena. However, it usefully rules out "betting pathologies" where participants can profitably bet against one another because of differences in beliefs.*

- Everything is common knowledge except each agent's own draw

Definition 1. *A Social Choice Function is a map $f : \Theta \to X$.*

Definition 2. *We say that f is Ex Post Efficient if there does not exist a profile $(\theta_1, ..., \theta_I)$ in which there exists any $x \in X$ such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$ for every i with at least one inequality strict.*

- ie. the SCF selects an alternative which is Pareto optimal given the utility functions of the agents
- There are multiple ways in which a social choice function ("SCF") might be implemented
  - Directly: ask each agent her type
  - Indirectly: agents could interaction through an institution or *mechanism* with particular rules attached
    - eg. an auction which allocates a single good to the person who announces the highest price and requires them to pay the price of the second-highest bidder (a second-price sealed bid auction).
- Need to consider both direct and indirect ways to implement SCFs

Definition 3. *A Mechanism $\Gamma = (S_1, ..., S_I, g(\cdot))$ is an $I+1$ tuple consisting of a strategy set S_i for each player i and a function $g : S_1 \times ... \times S_I \to X$.*

- We'll sometimes refer to g as the "outcome function"
- A mechanism plus a type space $(\Theta_1, ..., \Theta_I)$ plus a prior distribution plus payoff functions $u_1, ..., u_I$ constitute a game of incomplete information. Call this game $\mathcal{G}$

Remark 2. *This is a normal form representation. At the end of the course we will consider using an extensive form when we study subgame perfect implementation.*

- In a first-price sealed-bid auction $S_i = \mathbb{R}_+$ and given bids $b_1, ..., b_I$ the outcome function $g(b_1, ..., b_I) = \left(\{y_i(b_1, ..., b_I)\}_{i=1}^{I}, \{t_i(b_1, ..., b_I)\}_{i=1}^{I}\right)$ such that $y_i(b_1, ..., b_I) = 1$ iff $i = \min\{j : b_j = \max\{b_1, ..., b_I\}\}$ and $t_i(b_1, ..., b_I) = -b_i y_i(b_1, ..., b_I)$

Definition 4. *A strategy for player i is a function $s_i : \Theta_i \to S_i$.*

Definition 5. *The mechanism Γ is said to Implement a SCF f if there exists equilibrium strategies $(s_1^*(\theta_1), ..., s_I^*(\theta_I))$ of the game $\mathcal{G}$ such that $g(s_1^*(\theta_1), ..., s_I^*(\theta_I)) = f(\theta_1, ..., \theta_I)$ for all $(\theta_1, ..., \theta_I) \in \Theta_1 \times ... \times \Theta_I$.*

- Loosely speaking: there's an equilibrium of $\mathcal{G}$ which yields the same outcomes as the SCF f *for all possible profiles of types.*
- We want it to be true no matter what the actual types (ie. draws) are

Remark 3. *We are requiring only* ***an*** *equilibrium, not a unique equilibrium.*

Remark 4. *We have not specified a solution concept for the game. The literature has focused on two solution concepts in particular: dominant strategy equilibrium and Bayes Nash equilibrium.*

- The set of all possible mechanisms is enormous!
- The *Revelation Principle* provides conditions under which there is no loss of generality in restricting attention to direct mechanisms in which agents truthfully reveal their types in equilibrium.

Definition 6. *A Direct Revelation Mechanism is a mechanism in which $S_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all $\theta \in (\Theta_1 \times ... \times \Theta_I)$.*

Definition 7. *The SCF f is Incentive Compatible if the direct revelation mechanism Γ has an equilibrium $(s_1^*(\theta_1), ..., s_I^*(\theta_I))$ in which $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all i.*

2.2 Dominant Strategy Implementation

- A strategy for a player is weakly dominant if it gives her at least as high a payoff as any other strategy for all strategies of all opponents.

Definition 8. *A mechanism Γ Implements the SCF f in dominant strategies if there exists a dominant strategy equilibrium of Γ, $s^*(\cdot) = (s_1^*(\cdot), ..., s_I^*(\cdot))$ such that $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$.*

- A strong notion, but a robust one
  - eg. don't need to worry about higher order beliefs
  - Doesn't matter if agents miscalculate the conditional distribution of types
  - Works for any prior distribution $\phi(\cdot)$ so the mechanism designer doesn't need to know this distribution

Definition 9. *The SCF f is Truthfully Implementable in Dominant Strategies if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, ..., I$ is a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, ..., \Theta_I, f(\cdot))$, ie*

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i\left(f\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) \text{ for all } \hat{\theta}_i \in \Theta_i \text{ and } \theta_{-i} \in \Theta_{-i}. \tag{1}$$

Remark 5. *This is sometimes referred to as being "dominant strategy incentive compatible" or "strategy-proof".*

Remark 6. *The fact that we can restrict attention* ***without loss of generality*** *to whether $f(\cdot)$ in incentive compatible is known as the Revelation Principle (for dominant strategies).*

- This is very helpful because instead of searching over a very large space we only have to check each of the inequalities in (1).
  - Although we will see that this can be complicated (eg. when there are an uncountably infinite number of them).

Theorem 1. *(Revelation Principle for Dominant Strategies) Suppose there exists a mechanism Γ that implements the SCF f in dominant strategies. Then f is incentive compatible.*

Proof. The fact that Γ implements f in dominant strategies implies that there exists $s^*(\cdot) = (s_1^*(\cdot), ..., s_I^*(\cdot))$ such that $g(s^*(\theta)) = f(\theta)$ for all θ and that, for all i and $\theta_i \in \Theta_i$, we have

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(\hat{s}_i(\theta_i), s_{-i}), \theta_i) \text{ for all } \hat{s}_i \in S_i, s_{-i} \in S_{-i}.$$

In particular, this means that for all i and $\theta_i \in \Theta_i$

$$u_i\left(g\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right), \theta_i\right) \geq u_i\left(g\left(s_i^*\left(\hat{\theta}_i\right), s_{-i}^*(\theta_{-i})\right), \theta_i\right),$$

for all $\hat{\theta}_i \in \Theta_i, \theta_{-i} \in \Theta_{-i}$. Since $g(s^*(\theta)) = f(\theta)$ for all θ, the above inequality implies that for all i and $\theta_i \in \Theta_i$

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i\left(f\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) \text{ for all } \hat{\theta}_i \in \Theta_i, \theta_{-i} \in \Theta_{-i},$$

which is precisely incentive compatibility. ■

- Intuition: suppose there is an indirect mechanism which implements f in dominant strategies and where agent i plays strategy $s_i^*(\theta_i)$ when she is type θ_i. Now suppose we asked each agent her type and played $s_i^*(\theta_i)$ on her behalf. Since it was a dominant strategy it must be that she will truthfully announce her type.

2.2.1 The Gibbard-Satterthwaite Theorem

Notation 1. *Let $\mathcal{P}$ be the set of all rational preference relations $\succcurlyeq$ on X where there is no indifference*

Notation 2. *Agent i's set of possible ordinal preference relations on X are denoted $\mathcal{R}_i = \{\succcurlyeq_i : \succcurlyeq_i = \succcurlyeq_i(\theta_i) \text{ for some } \theta_i \in \Theta_i\}$*

Notation 3. *Let $f(\Theta) = (x \in X : f(\theta) = x \text{ for some } \theta \in \Theta)$ be the image of $f(\cdot)$.*

Definition 10. *The SCF f is Dictatorial if there exists an agent i such that for all $\theta \in \Theta$ we have:*

$$f(\theta) \in \{x \in X : u_i(x_i, \theta_i) \geq u_i(y, \theta_i), \forall y \in X\}.$$

- Loosely: there is some agent who always gets her most preferred alternative under f.

Theorem 2. *(Gibbard-Satterthwaite) Suppose: (i) X is finite and contains at least three elements, (ii) $\mathcal{R}_i = \mathcal{P}$ for all i, and (iii) $f(\Theta) = X$. Then the SCF f is dominant strategy implementable if and only if f is dictatorial.*

Remark 7. *Key assumptions are that individual preferences have unlimited domain and that the SCF takes all values in X.*

- The idea of a proof is the following: identify the pivotal voter and then show that she is a dictator
  - See Benoit (Econ Lett, 2000) proof
  - Very similar to Geanakoplos (Cowles, 1995) proof of Arrow's Impossibility Theorem
  - See Reny paper on the relationship
- This is a somewhat depressing conclusion: for a wide class of problems dominant strategy implementation is not possible unless the SCF is dictatorial
- It's a theorem, so there are only two things to do:
  - Weaken the notion of equilibrium (eg. focus on Bayes Nash equilibrium)
  - Consider more restricted environments
- We begin by focusing on the latter

2.2.2 Quasi-Linear Preferences

- An alternative from the social choice set is now a vector $x = (k, t_1, ..., t_I)$, where $k \in K$ (with K finite) is a choice of "project".
- $t_i \in \mathbb{R}$ is a monetary transfer to agent i
- Agent i's preferences are represented by the utility function

  $$u_i(x, \theta) = v_i(k, \theta_i) + (\bar{m}_i + t_i),$$

  where $\bar{m}_i$ is her endowment of money.
- Assume no outside parties
- Set of alternatives is:

$$X = \left\{(k, t_1, ..., t_I) : k \in K, t_i \in \mathbb{R} \text{ for all } i \text{ and } \sum_i t_i \leq 0\right\}.$$

- Now consider the following mechanism: agent i receives a transfer which depends on how her announcement of type affects the other agent's payoffs through the choice of project. Specifically, agent i's transfer is exactly the externality that she imposes on the other agents.

- A SCF is ex post efficient in this environment if and only if:

$$\sum_{i=1}^{I} v_i\left(k\left(\theta\right), \theta_i\right) \geq \sum_{i=1}^{I} v_i\left(k, \theta_i\right) \text{ for all } k \in K, \theta \in \Theta, k\left(\theta\right).$$

Proposition 1. *Let $k^*(\cdot)$ be a function which is ex post efficient. The SCF $f = (k^*(\cdot), t_1, ..., t_I)$ is truthfully implementable in dominant strategies if, for all $i = 1, ..., I$*

$$t_i\left(\theta\right) = \left(\sum_{j \neq i} v_j\left(k^*\left(\theta\right), \theta_j\right)\right) + h_i\left(\theta_{-i}\right), \tag{2}$$

where h_i is an arbitrary function.

- This is known as a Groves-Clarke mechanism

Remark 8. *Technically this is actually a Groves mechanism after Groves (1973). Clarke (1971) discovered a special case of it where the transfer made by an agent is equal to the externality imposed on other agent's if she is pivotal, and zero otherwise.*

- Groves-Clarke type mechanisms are implementable in a quasi-linear environment

- Are these the only such mechanisms which are?

- Green and Laffont (1979) provide conditions under which this question is answered in the affirmative

- Let $\mathcal{V}$ be the set of all functions $v : K \to \mathbb{R}$

Theorem 3. *(Green and Laffont, 1979) Suppose that for each agent $i = 1, ..., I$ we have $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathcal{V}$. Then a SCF $f = (k^*(\cdot), t_1(\cdot), ..., t_I(\cdot))$ in which $k^*(\cdot)$ satisfies*

$$\sum_{i=1}^{I} v_i\left(k\left(\theta\right), \theta_i\right) \geq \sum_{i=1}^{I} v_i\left(k, \theta_i\right),$$

for all $k \in K$ (efficient project choice) is truthfully implementable in dominant strategies only if $t_i(\cdot)$ satisfies (2) for all $i = 1, ..., I$.

- ie. if every possible valuation function from K to $\mathbb{R}$ arises for some type then a SCF which is truthfully implementable must be done so through a mechanism in the Groves class

- So far we have focused on only one aspect of ex post efficient efficiency–that the efficient project be chosen

- Another requirement is that none of the numeraire be wasted

- The condition is sometimes referred to as "budget balance" and requires

$$\sum_i t_i(\theta) = 0 \text{ for all } \theta \in \Theta.$$

- Can we satisfy both requirements?
- Green and Laffont (1979) provide conditions under which this question is answered in the negative

Theorem 4. *(Green and Laffont, 1979) Suppose that for each agent $i = 1, ..., I$ we have $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathcal{V}$. Then there does not exists a SCF $f = (k^*(\cdot), t_1(\cdot), ..., t_I(\cdot))$ in which $k^*(\cdot)$ satisfies*

$$\sum_{i=1}^{I} v_i(k(\theta), \theta_i) \geq \sum_{i=1}^{I} v_i(k, \theta_i),$$

for all $k \in K$ (efficient project choice) and

$$\sum_i t_i(\theta) = 0 \textit{ for all } \theta \in \Theta,$$

(budget balance).

- Either have to waste some of the numeraire or give up on efficient project choice
- Can get around this if there is one agent whose preferences are known
  - Maybe one agent doesn't care about project choice
  - eg. the seller in an auction
  - Maybe the project only affects a subset of the population...
- Need to set the transfer for the "no private information" type to $t_{BB}(\theta) = -\sum_{i \neq 0} t_i(\theta)$ for all θ.
- This agent is sometime referred to as the "budget breaker"
- We will return to this theme later in the course (stay tuned for Legros-Matthews)

2.3 Bayesian Implementation

- Now move from dominant strategy equilibrium as the solution concept to Bayes-Nash equilibrium
- A strategy profile implements an SCF f in Bayes-Nash equilibrium if for all i and all $\theta_i \in \Theta_i$ we have

$$E_{\theta_{-i}}\left[u_i\left(g\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \geq$$
$$E_{\theta_{-i}}\left[u_i\left(g\left(\hat{s}_i(\theta_i), s_{-i}^*(\theta_{-i})\right), \theta_i\right)|\theta_i\right],$$

  for all $\hat{s}_i \in S_i$.

- Again, we are able to make use of the revelation principle
- Same logic as in dominant strategy case
  - If an agent is optimizing by choosing $s_i^*(\theta_i)$ in some mechanism Γ then if we introduce an intermediary who will play that strategy for her then telling the truth is optimal conditional on other agents doing so. So truth telling is a (Bayes-Nash) equilibrium of the direct revelation game (ie. the one with the intermediary).

Remark 9. *Bayesian implementation is a weaker notion than dominant strategy implementation. Every dominant strategy equilibrium is a Bayes-Nash equilibrium but the converse is false. So any SCF which is implementable in dominant strategies can be implemented in Bayes-Nash equilibrium, but not the converse.*

Remark 10. *Bayesian implementation requires that truth telling give the highest payoff* ***averaging*** *over all possible types of other agents. Dominant strategy implementation requires that truth telling be best for* ***every*** *possible type of other agent.*

- Can this relaxation help us overcome the negative results of dominant strategy implementation
- Again consider a quasi-linear environment
- Under the conditions of Green-Laffont we couldn't implement a SCF truthfully and have efficient project choice and budget balance
- Can we do better in Bayes-Nash?
- A direct revelation mechanism known as the "expected externality mechanism" due to d'Aspremont and Gérard-Varet (1979) and Arrow (1979) answers this in the affirmative
- Under this mechanism the transfers are given by:

$$t_i(\theta) = E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} v_j\left(k^*\left(\theta_i, \tilde{\theta}_{-i}\right), \tilde{\theta}_j\right)\right] + h_i(\theta_{-i}).$$

- The first term is the expected benefit of other agents when agent i announces her type to be θ_i and the other agents are telling the truth

2.4 Participation Constraints

- So far we have worried a lot about incentive compatibility
- But we have been assuming that agents have to participate in the mechanism
- What happens if participation is voluntary?

2.4.1 Public Project Example

- Decision to do a project or not $K = \{0, 1\}$
- Two agents with $\Theta_i = \{L, H\}$ being the (real-valued) valuations of the project
- Assume that $H > 2L > 0$
- Cost of project is $c \in (2L, H)$
- An ex post efficient SCF has $k^*(\theta_1, \theta_2) = 1$ if either $\theta_1 = H$ or $\theta_2 = H$ and $k^*(\theta_1, \theta_2) = 0$ if (and only if) $\theta_1 = \theta_2 = L$
- With no participation constraint we can implement this SCF in dominant strategies using a Groves scheme
- By voluntary participation we mean that an agent can withdraw at any time (and if so, does not get any of the benefits of the project)
- With voluntary participation agent 1 must have $t_1(L, H) \geq -L$
  - Can't have to pay more than L when she values the project at L because won't participate voluntarily
- Suppose both agents announce H. For truth telling to be a dominant strategy we need:

$$\begin{aligned} Hk^*(H,H) + t_1(H,H) &\geq Hk^*(L,H) + t_1(L,H) \\ H + t_1(H,H) &\geq H + t_1(L,H) \\ t_1(H,H) &\geq t_1(L,H) \end{aligned}$$

- But we know that $t_1(L, H) \geq -L$, so $t_1(H, H) \geq -L$
- Symmetrically, $t_2(H, H) \geq -L$
- So $t_1(L, H) + t_2(H, H) \geq -2L$
- But since $c > 2L$ we can't satisfy $t_1(L, H) + t_2(H, H) \geq -c$
- Budget breaker doesn't help either, because $t_{BB}(\theta_1, \theta_2) \geq 0$ for all (θ_1, θ_2) and hence $t_0(H, H) \geq 0$ and we can't satisfy

$$t_0(H,H) + t_1(H,H) + t_2(H,H) \leq -c.$$

2.4.2 Types of Participation Constraints

- Distinguish between three different types of participation constraint depending on timing (of when agents can opt out of the mechanism)
- Ex ante: before the agents learn their types, ie:

$$U_i(f) \geq E_{\theta_i}\left[\bar{u}_i(\theta_i)\right]. \tag{3}$$

- Interim: after agents know their own types but before the take actions (under the mechanism), ie:

$$U_i(\theta|f) = E_{\theta_{-i}}\left[u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right)|\theta_i\right] \geq \bar{u}_i\left(\theta_i\right) \quad \text{for all } \theta_i. \tag{4}$$

- Ex post: after types have been announced and an outcome has been chosen (it's a direct revelation mechanism)

$$u_i\left(f\left(\theta_i, \theta_{-i}\right), \theta_i\right) \geq \bar{u}_i\left(\theta_i\right) \quad \text{for all } \left(\theta_i, \theta_{-i}\right) \tag{5}$$

- A question of when agents can agree to be bound by the mechanism
- Constraints are most severe when agents can withdraw ex post and least severe when they can withdraw ex ante. This can be seen from the fact that (5)⇒(4)⇒(3) but the converse doesn't hold

Theorem 5. *(Myerson-Satterthwaite) Suppose there is a risk-neutral seller and risk-neutral buyer of an indivisible good and suppose their respective valuations are drawn from* $[\underline{\theta}_1, \bar{\theta}_1] \in \mathbb{R}$ *and* $[\underline{\theta}_2, \bar{\theta}_2] \in \mathbb{R}$ *according to strictly positive densities with* $(\underline{\theta}_1, \bar{\theta}_1) \cap (\underline{\theta}_2, \bar{\theta}_2) \neq \varnothing$. *Then there does not exist a Bayesian incentive compatible SCF which is ex post efficient and gives every type non-negative expected gains from participation.*

- Whenever gains from trade are possible but not certain there is no ex post efficient SCF which is incentive compatible and satisfies interim participation constraints

Remark 11. *This applies to all voluntary trading institutions, including all bargaining processes.*

2.5 Optimal Bayesian Mechanisms

2.5.1 Welfare in Economies with Incomplete Information

- We have been concerned thus far with which SCFs are implementable
- We turn to evaluation of different implementable SCFs
- Want to be able to evaluate different "decision rules" or mechanisms
- Need to extend the notion of Pareto optimality where agents' preference are not known with certainty
- Pareto: "A decision rule is efficient if and only if no other feasible decision rule can be found that makes some individual better-off without making any worse-off"
- Need a notion of: (i) a feasible SCF, (ii) know what better-off means in this context, and (iii) specify who's doing the finding
- Feasibility: Bayesian incentive compatible plus individually rational
- Call this set the "incentive feasible set" F^* (Myerson, 1991)
- Better-off: depends on the timing

  - Before agents learn their types: ex ante efficiency
  - After agents learn their types: interim efficiency

- Putting (i) and (ii) together we refer to "ex ante incentive efficiency" and "interim incentive efficiency" (Holmström and Myerson, 1983)
- These are different from our previous definition of ex post efficiency
- Here that would require evaluation of SCFs after all information has been revealed
- The two definitions are equivalent if and only if $F = \{f : \Theta \to X\}$
- Who's doing the finding? Outside planner or the informed individuals within the economy
- Basic notion: the economist is an outside observer
  - Can't predict what decision or allocation will prevail without having all the private information
- With incomplete information the informed individuals might be able to agree (unanimously) to change a decision rule which a planner could not identify as an improvement

2.5.2 Durable Mechanisms

- Holmström-Myerson (Ecta, 1983)
- Suppose a mechanism M is interim incentive efficient
- A social planner can't propose another incentive-compatible decision rule which ever type is sure to prefer to M
- But it could be that there exists another mechanism M' such that:

$$u_i (M'|\theta_i) > u_i (M|\theta_i) \text{ for all } i.$$

- So if the types were $\theta_1, ..., \theta_I$ then all agents would prefer M' to M
- Are we done?
- Not even nearly
- Suppose agent 2 announces that she prefers M' to M, then agent 1 might want to say that she prefers M to M'
  - Agent 2 has revealed some new information to agent 1
- If agents unanimously agreed to change from M to M' then it would be common knowledge that all individuals prefer M' to M
- Recall Aumann (1976): If agents have a common prior and their posteriors are common knowledge then those posteriors must be equal
- Recall also the "no-trade theorems" (see Milgrom and Stokey, JET, 1982)

  - Common prior and risk-neutrality
  - No trade based solely because of differences in beliefs.
  - Suppose we're at a Pareto optimum and then some new information is received. Can their be trade with RE?
  - No, because there is no valid insurance motive since the initial allocation was Pareto optimal–so willingness to accept the best implies that the other agent doesn't like her claim/position

- Denote agent i's prior as π_i

Definition 11. *An event R is Common Knowledge if and only if $R = R_1 \times ... \times R_I$ with $R_i \subseteq \Theta_i$ for all i and*

$$\pi_i \left(\hat{\theta}_{-i} | \theta_i \right) = 0,$$

for all $\theta_i \in R, \hat{\theta}_{-i} \notin R$ and for i.

- ie, the information state R of the economy is common knowledge iff all individuals assign zero probability to events outside R

Definition 12. *We say that M' Interim Dominates M within R if and only if $R \neq \varnothing$ and*

$$u_i \left(M' | \theta_i \right) \geq u_i \left(M | \theta_i \right),$$

for all $\theta \in R$, for all i, with at least one inequality strict.

- If M is incentive efficient and each agent knows her own type then it can't be common knowledge that the agents unanimously prefer another mechanism M'

Theorem 6. *(Holmström-Myerson) An incentive compatible mechanism M is interim incentive efficient if and only if there does not exist any event R which is common-knowledge such that M is interim dominated within R by another incentive-compatible mechanism.*

- Doesn't mean they couldn't unanimously agree to move to another incentive efficient mechanism M'
  - But if unanimous agreement is reached then every agent must know more than her own type
  - ie, there must have been communication
- Now want to ask the following question: if a mechanism is determined by the agents themselves, after their types are privately observed, what are the properties of the rules which will emerge?
- We will be interested in *durable mechanisms*
  - ie. mechanisms which the agents will never unanimously agree to change

An example

- Suppose there are two agents: 1 and 2
- Each agent can be type a or b
- So there are four possible combinations of types
- Assume that each are ex ante equally likely
- Decision from the set $\{A, B, C\}$
- Payoffs (vNM)

	u_{1a}	u_{1b}	u_{2a}	u_{2b}
A	2	0	2	2
B	1	4	1	1
C	0	9	0	-8

- Note: sticking with the assumption that payoffs depend only on own type
- Note that agent 2, when of either type, prefers A to B and B to C
- So does agent $1a$
- Agent $1b$ prefers C to B to A
- The following incentive compatible mechanism maximizes the sum of utilities (among IC mechanisms)

$$\begin{aligned} M(1a, 2a) &= A \\ M(1b, 2a) &= C \\ M(1a, 2b) &= B \\ M(1b, 2b) &= B \end{aligned}$$

- This mechanism selects C is the types are $1b$ and $2a$
- It selects B if the types are $1b$ and $2b$
- Note that $2a$ can ensure either A or C by reporting truthfully, or ensure B by lying
- Since agent 2 has a 50-50 prior over agent 1 being type a or b she gets the same expected utility from reporting truthfully and lying
  - So we presume that she reports truthfully
- M is both ex ante and interim incentive efficient
  - So no planner could come up with a better mechanism
- Now suppose agent 1 is type $1a$
- Knowing this, she knows that both she and agent 2 prefer A to what the mechanism will give rise to

- And if she proposed that they choose A then agent 2 would be happy to accept that

- So, M is incentive efficient but there's an improvement to be made
  - The deterministic mechanism $M'(\cdot,\cdot) = A$
- Now suppose that agent 1 insists of using M rather than changing to choice A
  - Agent 2 would know that agent 1 was $1b$
  - Now M isn't incentive compatible because both $2a$ and $2b$ would announce $2b$ and ensure B (rather than announce $2a$ and get C)
- Conclusion: if agents already know their types then M could not be implemented even though it is incentive compatible and incentive efficient
  - It's not durable

Existence

- Do durable mechanisms exist?

Definition 13. *We say that an incentive compatible mechanism M is "Uniformly Incentive Compatible" if and only if*

$$u_i\left(M\left(\theta\right),\theta\right) \geq u_i\left(M\left(\theta_{-i},\hat{\theta}_i\right),\theta\right),$$

for all i, for all $\theta \in \Theta$ and for all $\hat{\theta}_i \in \Theta_i$.

- ie. no individual would ever want to lie under the mechanism, even if she knew the other agents' types, assuming that they were going to report truthfully
- This is now usually called "ex post incentive compatibility"

Theorem 7. *Suppose a mechanism M if uniformly incentive compatible and interim incentive efficient. Then M is durable.*

- The main (and encouraging) result is the following

Theorem 8. *There exists a nonempty set of decision rules that are both durable and incentive efficient*

- Are there decision rules that are durable but not incentive efficient?
- Sure
- Suppose the same type structure as above, but now two possible decisions A and B
- Preferences

$$\begin{aligned} u_1(A,\theta) &= u_2(A,\theta) = 2 \text{ for all } \theta \\ u_1(B,\theta) &= u_2(B,\theta) = 3 \text{ if } \theta = (1a,2a) \text{ or } (1b,2b) \\ u_1(B,\theta) &= u_2(B,\theta) = 0 \text{ if } \theta = (1a,2b) \text{ or } (1b,2a) \end{aligned}$$

- Consider the deterministic mechanism M which always selects A
- M is not interim incentive efficient but it is durable
  - Would be better to do B when the types match
  - But with any alternative mechanism there is an equilibrium in which there are reports which are independent of θ
  - Individuals cannot be foreced to communicate effectively in a non-cooperative game with incomplete information

2.5.3 Robust Mechanism Design

- Bergemann and Morris (Ecta, 2005)
- A key assumption in all that we have done so far is that the mechanism designer knows the prior distribution π
- Harsanyi's important idea: an agent's type should include beliefs about the strategic environment, beliefs about other players beliefs, ...
  - A sufficiently rich type space can then describe *any* environment
  - This is sometimes called the *implicit approach* to modelling higher order beliefs (see Heifetz and Samet, JME 1999 for further details)
- With a sufficiently rich type space it is a tautology that there is common knowledge of each agent's set of types and beliefs about other agents' types
- This notion is formalized in the universal type space of Mertens and Zamir (1985) (see also Brandenburdger and Dekel, 1993)
- If we assume a smaller type space and still maintain the assumption of common knowledge then the model may not be internally consistent
- What happens to Bayesian implementation without a common prior?
- Bergemann-Morris refer to this as *interim implementation*
- We have focused thus far on *payoff type spaces*
- But there many be many types of an agent who share the same payoff type
  - eg. they have different higher order beliefs
  - These are (much) smaller than the universal type space
- What we have done up until now is work with a very small type space (the payoff type space) and then assume that all agents (including the planner) have a common knowledge prior over that type space
- The largest type space we could work with is the union of all possible type space that could have arisen from the payoff environment
  - This is equivalent to the universal type space

- The paper also considers environments where there are both private values and common values

Definition 14. *An environment is said to be Separable if there exists* $\tilde{u}_i : X_0 \times ... \times X_I \times \Theta \to \mathbb{R}$ *such that*

$$\tilde{u}_i\left(\left(x_0, x_1, ..., x_I\right), \theta\right) = \tilde{u}_i\left(x_0, x_i, \theta\right)$$

for all $i, x \in X$ *and* $\theta \in \Theta$*; and there exists a function* $f_0 : \Theta \to X_0$ *and, for each agent* i*, a nonempty valued correspondence* $F_i : \Theta \to 2^{X_i}/\varnothing$ *such that*

$$F\left(\theta\right) = f_0\left(\theta\right) \times F_1\left(\theta\right) \times ... \times F_I\left(\theta\right).$$

- The bite comes from the implication that the set of permissible private components for any agent does not depend on the choice of the private component for the other agents
- Quasi-linear environments with no restrictions on transfers (eg. don't require budget balance) are special cases of separable environments
- So are environments where utility depends only on the common component and payoff type profile θ

Remark 12. *Any SCF is separable. It is only social choice* ***correspondences*** *which may not be separable*

- BM show that there can be social choice correspondences which are interim implementable on all payoff type spaces but not interim implementable on all type spaces
- They also show that in separable environments all of the following statements are equivalent for a social choice correspondence F
  - F is interim implementable on all type spaces
  - F is interim implementable on all common prior type spaces
  - F is interim implementable on all payoff type spaces
  - F is interim implementable on all common prior payoff type spaces
  - F is ex post implementable

3 Adverse Selection (Hidden Information)

3.1 Static Screening

3.1.1 Introduction

- A good reference for further reading is Fudenberg & Tirole chapter 7
- Different to "normal" Adverse Selection because 1 on 1, not a market setting
- 2 players: Principal and the Agent

- Payoff: Agent $G(u(q,\theta) - T)$, Principal $H(v(q,\theta) + T)$ where $G(\cdot), H(\cdot)$ are concave functions and q is some verifiable outcome (eg. output), T is a transfer, θ is the Agent's private information
- Don't use the concave transforms for now
- Say Principal is a monopolistic seller and the Agent is a consumer
- Let $v(q,\theta) = -cq$
- Principal's payoff is $T - cq$ where T is total payment (pq)
- $u(q,\theta) = \theta V(q)$
- Agent's payoff is $\theta V(q) - T$ where $V(\cdot)$ is strictly concave
- θ is type (higher $\theta \rightarrow$ more benefit from consumption)
- $\theta = \theta_1, ..., \theta_n$ with probabilities $p_1, ..., p_n$
- Principal only knows the distribution of types
- Note: relationship to non-linear pricing literature
- Assume that the Principal has all the bargaining power
- Start by looking at the first-best outcome (ie. under symmetric information)

First Best Case I: Ex ante no-one knows θ, ex post θ is verifiable

- Principal solves

$$\max_{(q_i,T_i)}{}^{n}_{i=1} p_i(T_i - cq_i)$$
$$s.t.{}^{n}_{i=1} p_i(\theta_i V(q_i) - T_i) \geq \overline{U} \tag{PC}$$

First Best Case II: Ex ante both know θ

- Normalize $\overline{U}$ to 0
- Principal solves

$$\max_{(q_i,T_i)} \{T_i - cq_i\}$$
$$s.t. \theta_i V(q_i) - T \geq 0 \tag{PC}$$

- The PC will bind, so $T_i = \theta_i V(q_i)$
- So they just solve $\max_{q_i} \{\theta_i V(q_i) - cq_i\}$

- FOC $\theta_i V'(q_i) = c$
- This is just perfect price discrimination – efficient but the consumer does badly
- Case I folds into II by offering a contingent contract

Second-Best

- Agent knows θ_i but the Principal doesn't
- First ask if we can achieve/sustain the first best outcome
- ie. will they naturally reveal their type
- say the type is θ_2
- if they reveal themselves their payoff is $\theta_2 V(q_2^*) - T_2^* = 0$
- if they pretend to be θ_1 their payoff is $\theta_2 V(q_2^*) - T_1^* = \theta_2 V(q_1^*) - \theta_1 V(q_1^*) = (\theta_2 - \theta_1)V(q_1^*) > 0$ since $\theta_2 > \theta_1$
- can't get the first-best

Second-best with n types

- First to really look at this was Mirrlees in his 1971 optimal income tax paper – normative
- Positive work by Akerlof, Spence, Stiglitz
- Revelation Principle very useful: can look at / restrict attention to contracts where people reveal their true type *in equilibrium*
- Without the revelation principle we would have the following problem for the principal

$$\max_{T(q)} \{{}_{i=1}^{n} p_i(T(q_i) - cq_i)\}$$
$$\text{subject to}$$
$$\theta_i V(q_i) - T(q_i)) \geq 0, \forall i \quad \text{(PC)}$$
$$q_i = \arg\max_q \{\theta_i V(q) - T(q))\}, \forall i \quad \text{(IC)}$$

- But the revelation principle means that there is no loss of generality in restricting attention to optimal equilibrium choices by the buyers
- We can thus write the Principal's Problem as

$$\max_{(q_i, T_i)} \{{}_{i=1}^{n} p_i(T_i - cq_i)\}$$
$$\text{subject to}$$
$$\theta_i V(q_i) - T_i) \geq 0, \forall i \quad \text{(PC)}$$
$$\theta_i V(q_i) - T_i \geq \theta_i V(q_j) - T_j, \forall i, j \quad \text{(IC)}$$

- Incentive compatibility means the Agent truthfully reveals herself
- This helps a lot because searching over a schedule $T(q)$ is hard
- Before proceeding with the n types case return to a two type situation

Second-best with* 2 *types

- Too many constraints to be tractable (there are $n(n-1)$ constraints of who could pretend to be whom)
- 2 types with $\theta_H > \theta_L$
- Problem is the following:

$$\begin{aligned} &\max \{p_H(T_H - cq_H) + p_L(T_L - cq_L)\} \\ &s.t.(i)\ \theta_H V(q_H) - T_H \geq \theta_H V(q_L) - T_L \qquad \text{(IC)} \\ &(ii)\ \theta_L V(q_L) - T_L \geq 0 \qquad \text{(PC)} \end{aligned}$$

- We have eliminated two constraints: the IC constraint for the low type and the PC constraint for the high type
- Why was this ok?
- The low type constraint must be the only binding PC (high types can "hide behind" low types)
- And the low type won't pretend to be the high type
- PC must bind otherwise we could raise T_L and the Principal will always be happy to do that
- IC must always bind otherwise the Principal could raise T_H (without equality the high type's PC would not bind) – also good for the Principal
- So $\theta_H V(q_H) - T_H = \theta_H V(q_L) - T_L$ and $\theta_L V(q_L) - T_L = 0$
- Now substitute to get an unconstrained problem:

$$\max_{q_L, q_H} \{p_H(\theta_H V(q_H) - \theta_H V(q_L) + \theta_L V(q_L) - cq_H) + p_L(\theta_L V(q_L) - cq_L)\}$$

- The FOCs are

$$p_H \theta_H V'(q_H) - p_H c = 0$$

  and

$$p_L \theta_L V'(q_L) - p_L c + p_H \theta_L V'(q_L) - p_H \theta_H V'(q_L) = 0$$

- The first of these simplifies to $\theta_H V'(q_H) = c$ (so the high type chooses the socially efficient amount)

- The second of these simplifies to the following:

$$\begin{aligned}\theta_L V'(q_L) &= \frac{c}{1 - \frac{1-p_L}{p_L}\frac{\theta_H - \theta_L}{\theta_L}} \\ &> c\end{aligned}$$

  (so the low type chooses too little)

- $q_H = q_H^*$ and $q_L < q_L^*$
- No incentive reason for distorting q_H because the low type isn't pretending to be the high type
- But you do want to discourage the high type from pretending to be the low type – and hence you distort q_L
- We can check the IC constraint is satisfied for the low type

$$\theta_H V(q_H) - T_H = \theta_H V(q_L) - T_L \text{ (high type's IC is binding)}$$

  now recall that (recalling that $\theta_H > \theta_L, q_H > q_L$), so we have

$$\theta_L V(q_L) - T_L \geq \theta_L V(q_H) - T_H$$

- So the low type's IC is satisfied
- High type earns rents – PC does not bind
- Lots of applications: optimal taxation, banking, credit rationing, implicit labor contracts, insurance, regulation (see Bolton-Dewatripont for exposition)

3.1.2 Optimal Income Tax

- Mirrlees (Restud, 1971)
- Production function $q = \mu e$ (for each individual), where q is output, μ is ability and e is effort
- Individual knows μ and e but society does not
- Distribution of μs in the population, μ_L and μ_H in proportions π and $1-\pi$ respectively
- Utility function $U(q - T - \psi(e))$ where T is tax (subsidy if negative) and $\psi(e)$ is cost of effort (presumably increasing and convex)
- The government's budget constraint is $\pi T_L + (1-\pi)T_H \geq 0$
- Veil of Ignorance – rules are set up before the individuals know their type

- So the first-best problem is:

$$\max_{e_L,e_H,T_L,T_H} \{\pi U(\mu_L e_L - T_L - \psi(e_L)) + (1-\pi)U(\mu_H e_H - T_H - \psi(e_H))\}$$

$$\text{subject to}$$

$$\pi T_L + (1-\pi)T_H \geq 0$$

- But the budget constraint obviously binds and hence $\pi T_L + (1-\pi)T_H = 0$
- Then we have $T_H = -\pi T_L/(1-\pi)$
- The maximization problem can be rewritten as

$$\max_{e_L,e_H,T_L} \{\pi U(\mu_L e_L - T_L - \psi(e_L)) + (1-\pi)U(\mu_H e_H + (\pi T_L/1-\pi) - \psi(e_H))\}$$

- The FOCs are

(i) $-U'(\mu_L e_L - T_L - \psi(e_L)) = U'(\mu_H e_H + (\pi T_L/1-\pi) - \psi(e_H))$

(ii) $\mu_L = \psi'(e_L)$

(iii) $\mu_H = \psi'(e_H)$

- Choose e_L, e_H efficiently in the first-best
- Everyone has same marginal cost of effort so the higher marginal product types work harder
- (i) just says the marginal utilities are equated
- Hence $\mu_L e_L - T_L - \psi(e_L) = \mu_H e_H + T_H - \psi(e_H)$
- The net payoffs are identical so you are indifferent between which type you are
- Consistent with Veil of Ignorance setup
- There is no DWL because of the lump sum aspect of the transfer

Second-Best

- Could we sustain the first-best?
- No because the high type will pretend to be the low type, $\mu_H e = q_L$ so $q_L - T_L - \psi(q_L/\mu_H) > q_L - T_L - \psi(e_L)$ since $q_L/\mu_H < e_L$
- Basically the high type can afford to slack because they are more productive - hence no self sustaining first-best

- The Second-Best problem is

$$\max_{e_L,e_H,T_L,T_H} \{\pi U(\mu_L e_L - T_L - \psi(e_L)) + (1-\pi)U(\mu_H e_H - T_H - \psi(e_H))\}$$
$$s.t.(i)\mu_H e_H - T_H - \psi(e_H) \geq \mu_L e_L - T_L - \psi(\mu_L e_L/\mu_H)$$
$$(ii)\pi T_L + (1-\pi)T_H \geq 0$$

- Solving yields $e_H = e_H^*$
- and $\mu_L = \psi'(e_L) + \beta(1-\pi)(\mu_L - \mu_L/\mu_H \psi'(\mu_L e_L/\mu_H))$
- where $\beta = \frac{U_L' - U_H'}{U_L'}$ (marginal utilities evaluated at their consumptions levels)
- but $U_L < U_H$ so $U_L' > U_H'$ (by concavity) and hence $0 < \beta < 1$
- Since $\psi(\cdot)$ is convex we have $\psi'\left(\frac{\mu_L e_L}{\mu_H}\right) < \psi'(e_L)$
- $\mu_L > \psi'(e_L) + \beta(1-\pi)(\mu_L - \mu_L/\mu_H \psi'(e_L))$
- and hence:
$$\psi'(e_L) < \frac{\mu_L - \beta(1-\pi)\mu_L}{1 - \beta(1-\pi)\mu_L/\mu_H} < \mu_L$$
- (the low type works too little)
- To stop the high type from misrepresenting themselves we have to lower the low type's required effort and therefore subsidy
- High type is better off → lose the egalitarianism we had before for incentive reasons
- Can offer a menu (q_L, T_L), (q_H, T_H) and people self select
- If you have a continuum of types there would be a tax schedule $T(q)$
- Marginal tax rate of the high type is zero (because they work efficiently) so $T'(q) = 0$ at the very top and $T'(q) > 0$ elsewhere with a continuum of types

3.1.3 Regulation

- Baron & Myerson (Ecta, 1982)
- The regulator/government is ignorant but the firm knows its type
- Firm's characteristic is $\beta \in \{\underline{\beta}, \overline{\beta}\}$ with probabilities ν_1 and $1 - \nu_1$
- Cost is $c = \beta - e$
- Cost is verifiable
- Cost of effort is $\psi(e) = e^2/2$
- Let $\Delta\beta = \overline{\beta} - \underline{\beta}$ and assume $\Delta\beta < 1$

- Government wants a good produced with the lowest possible subsidy - wants to minimize expected payments to the firm
- The First-Best is simply

$$\min_{e}\left\{\beta - e + e^2/2\right\}$$

- The FOC is $e^* = 1$ and the firm gets paid $\beta - 1/2$
- Can we sustain the FB?
- No because $p_L = \beta_L - 1/2$ and $p_H = \beta_H - 1/2$

Second-Best

- Two cost levels $\underline{c}$ and $\overline{c}$
- Two price levels $\underline{p}$ and $\overline{p}$ (payments)
- Government solves

$$\begin{gathered}\min\left\{\nu_1 \underline{p} + (1-\nu_1)\overline{p}\right\} \\ s.t.(i)\ \underline{p} - \underline{c} - e^2/2 \geq \overline{p} - \overline{c} - (\overline{e} - \Delta\beta)^2/2 \\ (ii)\ \overline{p} - \overline{c} - \overline{e}^2/2 \geq 0\end{gathered}$$

- noting that $\underline{e} = \overline{e} - \Delta\beta$ (from cost equation and low type pretending to be high type)
- Define $\underline{s} = \underline{p} - \underline{c} = \underline{p} - \underline{\beta} + \underline{e}$ and $\overline{s} = \overline{p} - \overline{c} = \overline{p} - \overline{\beta} + \overline{e}$ (these are the "subsidies")
- The government's problem is now

$$\begin{gathered}\min\left\{\nu_1\left(\underline{s} + \underline{\beta} - \underline{e}\right) + (1-\nu_1)\overline{s} + \overline{\beta} - \overline{e}\right\} \\ s.t.(i)\ \underline{s} - e^2/2 \geq \overline{s} - (\overline{e} - \Delta\beta)^2/2 \\ (ii)\ \overline{s} - \overline{e}^2/2 \geq 0\end{gathered}$$

- Since the constraints must hold with equality we can substitute and write this as an unconstrained problem

$$\min_{\underline{e},\overline{e}}\left\{\nu_1\left(\frac{\overline{e}^2}{2} + \underline{e}^2/2 - \frac{(\overline{e} - \Delta\beta)^2}{2}\right) + (1-\nu_1)\left(\frac{\overline{e}^2}{2} - \overline{e}\right)\right\}$$

- The FOCs are

(1) $\underline{e} = 1$

(2) $\nu_1\overline{e} - \nu_1(\overline{e} - \Delta\beta) + (1-\nu_1)\overline{e} - (1-\nu_1) = 0$

- (2) implies that:

$$\overline{e} = \frac{1-\nu_1-\nu_1\Delta\beta}{1-\nu_1} = 1 - \frac{\nu_1\Delta\beta}{1-\nu_1}$$

- The low cost ("efficient") type chooses $\underline{e} = 1$
- The high cost ("bad") types chooses $\overline{e} = 1 - \frac{\nu_1\Delta\beta}{1-\nu_1}$
- Offer a menu of contracts: fixed price or a cost-sharing arrangement
- The low cost firm takes the fixed price contract, becomes the residual claimant and then chooses the efficient amount of effort
- See also Laffont & Tirole (JPE, 1986) – costs observable

3.1.4 The General Case – n types and a continnum of types

- Problem of all the incentive compatibility constraints
- It turns out that we can replace the IC constraints with downward adjacent types
- The constraints are then just:

  (i) $\theta_i V(q_i) - T_i \geq \theta_i V(q_{i-1}) - T_{i-1} \quad \forall i = 2, ..., n$

  (ii) $q_i \geq q_{i-1} \quad \forall i = 2, ..., n$

  (iii) $\theta V(q_1) - T_1 \geq 0$

- (ii) is a monotonicity condition
- It is mathematically convenient to work with a continuum of types – and we will
- Let $F(\theta)$ be a cdf and $f(\theta)$ the associated density function on the support $\left[\underline{\theta}, \overline{\theta}\right]$
- The menu being offered is $T(\theta), q(\theta)$
- The problem is

$$\max_{T(\cdot),q(\cdot)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} [T(\theta) - cq(\theta)] f(\theta) d\theta \right\}$$

$$s.t.(i)\ \theta V(q(\theta) - T(\theta) \geq \theta V\left(q\left(\widehat{\theta}\right)\right) - T\left(\widehat{\theta}\right) \forall \theta, \widehat{\theta} \quad \text{(IC)}$$

$$(ii)\ \theta V(q(\theta) - T(\theta) \geq 0, \forall \theta \quad \text{(PC)}$$

- We will be able to replace all the IC constraints with a Local Adjacency condition and a Monotonoicity condition

Definition 15. *An allocation $T(\theta), q(\theta)$ is implementable if and only if it satisfies IC $\forall \theta, \widehat{\theta}$*

Proposition 2. *An allocation* $T(\theta), q(\theta)$ *is implementable if and only if* $\theta V'(q(\theta)) \frac{dq(\theta)}{d\theta} - T'(\theta) = 0$ *(the local adjacency condition) and* $\frac{dq(\theta)}{d\theta} \geq 0$ *(the monotonicity condition).*

Proof. $\Rightarrow$ direction:

Let $\widehat{\theta} = \arg\max_{\theta} \left\{\theta V\left(q\left(\widehat{\theta}\right)\right) - T\left(\widehat{\theta}\right)\right\}$. Now $\frac{d}{d\widehat{\theta}} = \theta V'\left(q\left(\widehat{\theta}\right)\right) - \frac{dq(\widehat{\theta})}{d\widehat{\theta}} - T'\left(\widehat{\theta}\right)$
so $\theta V'(q(\theta)) - \frac{dq(\theta)}{d\theta} - T'(\theta) = 0, \forall\theta$
Now, by revealed preference:

$$\theta V(q(\theta)) - T(\theta) \geq \theta V(q(\theta')) - T(\theta')$$

and

$$\theta' V(q(\theta')) - T(\theta') \geq \theta' V(q(\theta)) - T(\theta)$$

combining these yields:

$$\theta\left[V(q(\theta)) - V(q(\theta'))\right] \geq T(\theta) - T'(\theta) \geq \theta'\left[V(q(\theta)) - V(q(\theta'))\right]$$

the far RHS can be expressed as $(\theta - \theta')(V(q(\theta)) - V(q(\theta'))) \geq 0$
hence if $\theta > \theta'$ then $q(\theta) \geq q(\theta')$ ■

- This really just stems from the **Single-Crossing Property** (or **Spence-Mirrlees Condition**), namely $\frac{\partial U}{\partial q}$ is increasing in θ
- Note that this is satisfied with the separable functional form we have been using–but need not be satisfied in general
- Higher types are "even more prepared" to buy some increment than a lower type

Proof. $\Leftarrow$ direction

Let $W\left(\theta, \widehat{\theta}\right) = \theta V\left(q\left(\widehat{\theta}\right)\right) - T\left(\widehat{\theta}\right)$. Fix θ and suppose the contrary. This implies that $\exists\widehat{\theta}$ such that $W\left(\theta, \widehat{\theta}\right) > W(\theta, \theta)$.

Case 1: $\widehat{\theta} > \theta$

$$W\left(\theta, \widehat{\theta}\right) - W(\theta, \theta) = \int_{\theta}^{\widehat{\theta}} \frac{\partial W}{\partial \tau}(\theta, \tau)\, d\tau = \int_{\theta}^{\widehat{\theta}} \theta V'(q(\tau)) \frac{dq}{d\tau} - T'(\tau)\, d\tau$$

But $\tau > \theta$ implies that:

$$\begin{aligned} & \int_{\theta}^{\widehat{\theta}} \theta V'(q(\tau)) \frac{dq}{d\tau} - T'(\tau)\, d\tau \\ \leq \quad & \int_{\theta}^{\widehat{\theta}} \left(\tau V'(q(\tau)) \frac{dq}{d\tau} - T'(\tau)\right) d\tau = 0 \end{aligned}$$

because the integrand is zero. Contradiction. Case 2 is analogous. ■

- This proves that the IC constraints are satisfied globally, not just the SOCs (the common error)

- Now we write the problem as:

$$\max_{T(\cdot),q(\cdot)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \left[T\left(\theta\right) - cq\left(\theta\right)\right] f(\theta) d\theta \right\}$$

$$s.t.(i)\ \theta V'(q\left(\theta\right)) \frac{dq\left(\theta\right)}{d\theta} - T'\left(\theta\right) \geq 0 \quad \forall \theta \qquad \text{(Local Adjacency)}$$

$$(ii)\ \frac{dq\left(\theta\right)}{d\theta} \geq 0 \quad \forall \theta \qquad \text{(Monotonicity)}$$

$$(iii) \underline{\theta} V\left(q\left(\underline{\theta}\right)\right) - T\left(\underline{\theta}\right) = 0 \qquad \text{(PC-L)}$$

- Let $W\left(\theta\right) \equiv W\left(\theta, \theta\right) = \theta V\left(q\left(\theta\right)\right) - T\left(\theta\right) = \max_{\hat{\theta}} \left\{ \theta V(q(\hat{\theta})) - T\left(\hat{\theta}\right) \right\}$
- Recall that in the 2 type case we used the PC for the lowest type and the IC for the other type
- We could have kept on going for higher and higher types
- Now, from the FOCs:

$$\frac{dW\left(\theta\right)}{d\theta} = \theta V'\left(q\left(\theta\right)\right) \frac{dq}{d\theta} - \frac{dT}{d\theta} + V\left(q\left(\theta\right)\right) = V\left(q\left(\theta\right)\right)$$

  (by adding $V(q(\theta))$ to both sides)

$$W\left(\theta\right) - W\left(\underline{\theta}\right) = \int_{\underline{\theta}}^{\theta} \frac{dW\left(\tau\right)}{d\tau} d\tau = \int_{\underline{\theta}}^{\theta} V\left(q\left(\tau\right)\right) d\tau$$

(change of measure trick)

- But $W\left(\underline{\theta}\right) = 0$ (PC of low type binding at the optimum)
- Now $T\left(\theta\right) = -\int_{\underline{\theta}}^{\theta} V\left(q\left(\tau\right)\right) d\tau + \theta V\left(q\left(\theta\right)\right)$ (by substitution)
- So the problem is now just

$$\max_{q(\cdot)} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta V\left(q\left(\theta\right)\right) - \int_{\underline{\theta}}^{\theta} V\left(q\left(\tau\right)\right) d\tau - cq\left(\theta\right) \right] f\left(\theta\right) d\theta \right\}$$

$$s.t. \frac{dq\left(\theta\right)}{d\theta} \geq 0 \quad \forall \theta$$

- Proceed by ignoring the constraint for the moment and tackle the double integral using integration by parts
- Recall that

$$\int_{\underline{\theta}}^{\overline{\theta}} uv' = uv \Big|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} u'v$$

- So let $v' = f(\theta)$ and $u = \int V(q(\tau))\,d\tau$, and we then have

$$\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{\theta}}^{\theta} V(q(\tau))\,d\tau\right] f(\theta)\,d\theta &= \int_{\underline{\theta}}^{\theta} V(q(\tau))\,d\tau F(\theta)\Big|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))\,F(\theta)\,d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} V(q(\tau))\,d\tau - \int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))\,F(\theta)\,d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} V(q(\theta))\,[1-F(\theta)]\,d\theta
\end{aligned}$$

- So we can write the problem as:

$$\max_{q(\cdot)}\left\{\int_{\underline{\theta}}^{\overline{\theta}} ((\theta V(q(\theta) - cq(\theta))\,f(\theta) - V(q(\theta)\,[1-F(\theta)])\,d\theta\right\}$$

- Now we can just do pointwise maximization (maximize under the integral for all values of θ)

$$\theta V'(q(\theta)) = V'(q(\theta))\left(\frac{1-F(\theta)}{f(\theta)}\right) + c, \quad \forall\theta \tag{6}$$

- From 6 we can say the following:
  (1)
  $$\theta = \overline{\theta} \rightarrow \overline{\theta} V\left(q\left(\overline{\theta}\right)\right) = c$$
  (2)
  $$\theta < \overline{\theta} \rightarrow \overline{\theta} V\left(q\left(\overline{\theta}\right)\right) > c$$
  ($q(\theta)$ is too low)

- Since efficiency requires $\theta V'(q(\theta)) = c$
- Now differentiate (6) and solve for $\frac{dq}{d\theta} \geq 0$
- This implies that $\frac{f(\theta)}{1-F(\theta)}$ is increasing in θ (*this is a sufficient condition in general, but is a necessary and sufficient condition in this buyer-seller problem*)
- This property is known as the **Monotone Hazard Rate Property**
- It is satisfied for all log-concave distributions
- We've been considering the circumstance where θ announces their type, θ^a and gets a quantity $q(\theta^a)$ and pays a tariff of $T(\theta^a)$
- This can be reinterpreted as: given $\widehat{T}(q)$, pick q
- For each q there can only be one $T(q)$ by incentive compatibility
- $\widehat{T}(q) = T(\theta^{-1}(q))$

- The optimization problem becomes

$$\max_q \left\{\theta V(q) - \widehat{T}(q)\right\}$$

- The FOC is $\theta V'(q) = \widehat{T}'(q) \equiv p(q)$

$$p(q) = \frac{p(q(\theta))}{\theta}\left(\frac{1-F(\theta)}{f(\theta)}\right) + c$$

$$\frac{p-c}{p} = \frac{1-F(\theta)}{\theta f(\theta)}$$

- Recall that we ignored the constraint $\frac{dq}{d\theta} \geq 0$
- The FOC implies

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) V'\left(q\left(\theta\right)\right) = c$$

- Differentiating this wrt θ yields

$$\frac{dq}{d\theta} = -\frac{g'\left(\theta\right) v'\left(q\left(\theta\right)\right)}{v''\left(q\left(\theta\right)\right) g\left(\theta\right)},$$

  where $g\left(\theta\right) = \theta - \frac{1-F(\theta)}{f(\theta)}$
- Since the following holds

$$\theta V'(q(\theta)) = V'(q(\theta))\left(\frac{1-F(\theta)}{f(\theta)}\right) + c$$

  we have

$$V'(q(\theta)) = \frac{c}{\theta - \left[\left(1-F(\theta)\right)/f(\theta)\right]}$$

- We require that $V'(q(\theta))$ be falling in θ and hence require that $\theta - \frac{1-F(\theta)}{f(\theta)}$ be increasing in θ
- That is, that the hazard rate be increasing
- Now turn attention to $T(q)$
- $\widehat{T'}(q) > c$ except for at the very top where $\widehat{T'} = c$
- Therefore it can't be convex
- Note that

$$1 - \frac{c}{p} = \frac{1-F\left(\theta\right)}{\theta f\left(\theta\right)}$$

$$\frac{\theta f(\theta)}{1-F(\theta)} \uparrow \theta \Leftrightarrow \frac{dp}{dq} < 0$$

- And note that $\frac{dp}{dq} = \widehat{T''}(q)$
- So the IHRC $\Rightarrow \frac{dp}{dq} < 0$
- If the IHRC does not hold the Monotonicity Constraint binds and we need to applying "Ironing" (See Bolton & Dewatripont)
- Use Pontryagin's Principle to find the optimal cutoff points
- Require $\lambda(\theta_1) = \lambda(\theta_2) = 0$, where λ is the Lagrange multiplier
- Still get optimality and the top and sub-optimality elsewhere

3.1.5 Random Schemes

- Key paper is Maskin & Riley (RAND, 1984)
- A deterministic scheme is always optimal if the seller's program is convex
- But if the ICs are such that the set of incentive feasible allocations is not convex then random schemes may be superior

dtbpF3.3529in2.0678in0ptFigure

- Both types are risk-averse
- So S loses money on the low type, but may be able to charge enough more to the high type to avoid the randomness if the high type is more risk-averse
- If they are sufficiently more risk-averse (ie. the types are far enough apart), then the random scheme dominates
- Say: announce $\theta = \theta^a$ and get a draw from a distribution, so get $(\widetilde{q}, \widetilde{T})$
- If the high type is less risk-averse than the low type then the deterministic contract dominates
  - The only incentive constraints that matter are the downward ones
  - So if the high type is less risk-averse then S loses money on that type from introducing randomness
  - And doesn't gain anything on the low type, because her IR constraint is already binding and so can't extract more rents from her

3.1.6 Extensions and Applications

- Jullien (2000) and Rochet & Stole (2002) consider more general PCs (egs. type dependent or random)
- Classic credit rationing application: Stiglitz & Weiss (1981)

Multi-Dimensional Types

- So far we have assumed that a single parameter θ captures all relevant information
- Laffont-Maskin-Rochet (1987) were the first to look at this
- They show that "bunching" is more likely to occur in a two-type case than a one-type case (ie. Monotone Hazard Rate condition violated)
- Armstrong (Ecta, 1996) provides a complete characterization
  - Shows that some agents are always excluded from the market at the optimum (unlike the one-dimensional case where there is no exclusion)
  - In one dimension if the seller increases the tariff uniformly by ε then profits go up by ε on all types whose IR was slack enough (so that they still participate), but lose on all the others
  - With multi-dimensional types the probability that an agent had a surplus lower than ε is a higher order term in ε – so the loss is lower from the increase even if there is exclusion
- Rochet-Chone (1997) shows that
  - Upward incentive constraints can be binding at the optimum
  - Stochastic constracts can be optimal
  - There is no generalization of the MHRC which can rule out bunching
- Armstrong (1997) shows that with a large number of independently valued dimensions the the optimal contract can be approximated by a two-part tariff

Aside: Multi-Dimensional Optimal Income Taxation

- Mirrlees (JPubE, 1976) considered the problem of multi-dimensional optimal income taxation
- Strictly harder than the above problems because he doesn't assume quasi-linear utility functions only
- He shows how, when $m < n$ (i.e. the number of characteristics is smaller than the number of commodities), the problem can be reduced to a single eliptic equation which can be solved by well-known method
- When $m \geq n$ (i.e. the number of characteristics is at least at large than the number of commodities) the above approach does not lead to a single second-order partial differential equation, but a system of m second-order partial differential equations for the m functions M_j
- Numerical evidence has shown recently that a lot of the conclusions from the one-dimensional case go away in multiple dimensions (eg. the no distortion at the top result)
- But the system of second-order PDEs seem very hard to solve

3.2 Dynamic Screening

- Going to focus on the situation where there are repeated interactions between an informed and uninformed party
- We will assume that the informed party's type is fixed / doesn't change over time
  - There is a class of models where the agent gets a new draw from the distribution of types each period (see BD §9.2 for details)
- The main new issue which arises is that there is (gradual) elimination of the information asymmetry over time
- Renegotiation a major theme
  - Parties may be subject to a contract, but can't prevent Pareto improving (and therefore mutual) changes to the contract

3.2.1 Durable good monopoly

- There is a risk-neutral seller ("S") and a risk-neutral buyer ("B")
- Normalize S's cost to zero
- B's valuation is $\bar{b}$ or $\underline{b}$ with probabilities $\mu, 1-\mu$ and assume that $\bar{b} > \underline{b} > 0$
  - This is common knowledge
- B knows their valuation, S does not
- Trade-off is $\underline{b}$ vs. $\mu_1 \bar{b}$ and assume that $\mu_1 \bar{b} > \underline{b}$ (i.e. low types are not very likely)
- 2 periods
- Assume that the good is a durable good and there is a discount factor of δ which is common to B and S

Commitment

- Assume that S can commit not to make any further offers
- Under this assumption it can be shown that the Revelation Principle applies
- Contract: if B announces $\bar{b}$ then with probability $\overline{x_1}$ B gets the good today and with probability $\overline{x_2}$ they get the good tomorrow. B pays $\overline{p}$ for this
- Similarly for $\underline{b} \to \underline{x_1}, \underline{x_2}, \underline{p}$
- S solves:

$$\max_{\overline{x_1}, \overline{x_2}, \underline{x_1}, \underline{x_2}} \left\{ \mu_1 \overline{p} + (1-\mu_1)\underline{p} \right\}$$

$$s.t.(i) \quad \bar{b}\left(\overline{x_1}(1+\delta) + (1-\overline{x_1})\overline{x_2}\delta\right) - \overline{p} \geq \bar{b}\left(\underline{x_1}(1+\delta) + (1-\underline{x_1})\underline{x_2}\delta\right) - \underline{p}$$

$$(ii) \quad \underline{b}\left[\underline{x_1}(1+\delta) + (1-\underline{x_1})\underline{x_2}\delta\right] - \underline{p} \geq 0$$

- In fact, both constraints will hold with equality
- Let

$$\overline{X_1} = \overline{x_1}(1+\delta) + (1-\overline{x_1})\overline{x_2}\delta$$
$$\underline{X_1} = \underline{x_1}(1+\delta) + (1-\underline{x_1})\underline{x_2}\delta$$

- $\underline{p} = \underline{b}\underline{X_1}$ and $\overline{p} = \overline{bX_1} - \overline{b}\underline{X_1} + \underline{b}\underline{X_1}$ since the high type's IC constraint and low type's IR constraint both bind at the optimum
- So:

$$\max \left\{\mu_1 \left[\overline{bX_1} - \overline{b}\underline{X_1} + \underline{b}\underline{X_1}\right] + (1-\mu_1)\underline{b}\underline{X_1}\right\}$$
$$s.t.(i) \quad 0 \leq \overline{X_1} \leq 1+\delta$$
$$(ii) \quad 0 \leq \underline{X_1} \leq 1+\delta$$

- The constraints are just physical constraints
- Notice that the coefficient on $\underline{X_1}$ is $\underline{b} - \mu_1\overline{b} < 0$
- Similarly for $\overline{X_1} : \mu_1\overline{b} > 0$
- Conclusion: since $\mu_1\overline{b} > \underline{b}$ by assumption it is optimal to set $\overline{X_1} = 1+\delta, \underline{X_1} = 0$ and $\underline{p} = 0$, $\overline{p} = \overline{b} + \delta\overline{b}$ (ie. what it's worth to the high type)
- Just a repetition of the one period model (S faces a stationary problem because of commitment)

No Commitment

- Now consider the case where S cannot commit
- Suppose S can't commit and date 1 not to make further offers in period 2
- Study the Perfect Bayesian Equilibria ("PBE") of the game
- Basically, S has the following choices
  - (1) Sell to both types at date 1
  - (2) Sell to both types at date 2
  - (3) Never sell to the low type
- Under (1) $\underline{p} = \underline{b} + \delta\underline{b}$, $\Pi_1 = \underline{b} + \delta\underline{b}$
- Under (2) $p_2 = \underline{b}$, $p_1 = \overline{b} + \delta\underline{b}$ since $\overline{b} + \delta\overline{b} - p_1 = \delta(\overline{b} - \underline{b})$, by incentive compatibility
- Notice that under (2) $\Pi_2 = \mu_1\left(\overline{b} + \delta\underline{b}\right) + (1-\mu_1)\delta\underline{b} = \mu_1\overline{b} + \delta\underline{b}$
- Hence $\Pi_2 > \Pi_1$ since $\mu_1\overline{b} > \underline{b}$
- Now consider strategy (3) – only sell to the high type in both periods

- Under this strategy $p_1 = \bar{b} + \delta\bar{b}$, $p_2 = \bar{b}$
- Need to credibly commit to keep the price high in period 2
- The high type buys with probability ρ_1 in period 1 and $1 - \rho_1$ in period 2
  - No pure strategy equilibrium because if $p_2 = \underline{b}$ then the high type doesn't want to buy in period 1 and if $p_2 = \bar{b}$ then high type wants to buy in period 1 so that can't be a continuation equilibrium
- Use Bayes' Rule to obtain:

$$\begin{aligned} pr[\bar{b} \quad | \quad \text{declined first offer}] &= \frac{\mu_1(1-\rho_1)}{\mu_1(1-\rho_1) + (1-\mu_1)} \\ &= \frac{\mu_1(1-\rho_1)}{1-\mu_1\rho_1} = \sigma \end{aligned}$$

- Condition for the Seller to keep price high is:

$$\sigma \geq \underline{b}/\bar{b}$$

- Note: this is the Pareto efficient PBE
- If fact it will hold with equality (ρ_1 as high as possible), and can be written as:

$$\frac{\mu_1(1-\rho_1)}{1-\mu_1\rho_1} = \underline{b}/\bar{b}$$

- Early buyers are good, but can't have too many (in order to maintain credibility)
- Solving yields:

$$\rho_1^* = \frac{\mu_1\bar{b} - \underline{b}}{\mu_1\left(\bar{b} - \underline{b}\right)}$$

- Therefore the Seller's expected profit from strategy (3) is:

$$\begin{aligned} & \mu_1\rho_1\left(\bar{b} + \delta\bar{b}\right) + \mu_1(1-\rho_1)\delta\bar{b} \\ = \; & \mu_1\rho_1\bar{b} + \mu_1\delta\bar{b} \\ = \; & \mu_1\bar{b}\left[\frac{\mu_1\bar{b} - \underline{b}}{\mu_1\left(\bar{b} - \underline{b}\right)}\right] + \mu_1\delta\bar{b} \end{aligned}$$

- Expected profit from strategy (2) was $\mu_1\bar{b} + \delta\underline{b}$
- Strategy (3) is preferred to strategy (2) iff:

$$\begin{aligned} \mu_1 \quad &> \quad \frac{\bar{b}\underline{b}\,(1+\delta) - \delta\underline{b}}{\delta\bar{b}^2 - \delta\bar{b}\underline{b} + \bar{b}\underline{b}} \\ &\equiv \quad \overline{\mu_2} \end{aligned}$$

- Check that $\overline{\mu_2} > \overline{\mu_1} = \underline{b}/\bar{b}$ (and it is)
- Now consider a T period model (Hart & Tirole, 1988)

$$\exists 0 \leq \overline{\mu_1} \leq \overline{\mu_2} < ... < \overline{\mu_T} < 1 \text{ such that}$$

- $\mu_1 < \overline{\mu_1} \Rightarrow$ sell to low types at date 1
- $\overline{\mu_2} > \mu_1 > \overline{\mu_1} \Rightarrow$ sell to low types at date 2
- $\overline{\mu_3} > \mu_1 > \overline{\mu_2} \Rightarrow$ sell to low types at date 3
- $\overline{\mu_T} > \mu_1 > \overline{\mu_{T-1}} \Rightarrow$ sell to low types at date T
- $\mu_1 > \overline{\mu_1} \Rightarrow$ never sell to low types
- In addition it can be shown that $\overline{\mu_i}$ is independent of T
- Also: $\overline{\mu_i}$ is weakly decreasing in δ $\forall i$ - if people are more patient the seller will do more screening
- Also: $\overline{\mu_i}$ has a well defined limit as $\delta \to 1$
- $\overline{\mu_i} \to 1$ as $T \to \infty$
- COASE CONJECTURE (Coase 1972): When periods become very short it's like $\delta \to 1$
- *As period length goes to zero bargaining is over (essentially) immediately – so the price is the value that the low type puts on it* $\Rightarrow$ the seller looses all their monopoly power

3.2.2 Non-Durable Goods

- Every period S can sell 1 or 0 units of a good to B
- Can think of this as renting the good
- B ends up revealing her type in a separating equilibrium
- Commitment solution is essentially the same as the Durable Good case
- Non-Commitment solution is very different
- S offers

$$r_1; r_2(Y), r_2(N); r_3(YY), r_3(YN), r_3(NY), r_3(NN); ...$$

- Consider Perfect Bayesian Equilibria ("PBE")
- Here the problem is that S can't commit not to be tough in future periods (people effectively reveal their type) – a *Ratcheting Problem*
- 2 period model: is ratcheting a problem?

- Say they try to implement the durable good solution:

$$\begin{aligned} S1 &: \underline{b} + \delta \underline{b} \\ S3 &: \overline{b} + \delta \overline{b}, \underline{b} \\ S2 &: p_1 = \overline{b} + \delta \underline{b}, p_2 = \underline{b} \end{aligned}$$

- in the service model $\widehat{p}_2(N) = \underline{b}, \widehat{p}_2(Y) = \overline{b} \Rightarrow \widehat{p}_1 = \overline{b}(1-\delta) + \delta \underline{b}$ since $\underline{b} - \widehat{p}_1 + \delta(\overline{b} - \overline{b}) = \delta(\overline{b} - \underline{b})$
- So ratcheting isn't a problem with 2 periods
- But this breaks down with many periods
- *Screening fails because the price you have to charge in period 1 to induce the high types to buy is below the price at which the low type is prepared to buy*
- Take T large and suppose that $\mu_{i-1} < \mu_1 < \overline{\mu_i}$
- Consider date $i-1$:

$$\overline{b} - r_{i-1} \geq (\overline{b} - \underline{b}) \left(\delta + \delta^2 + ... \right) \simeq (\overline{b} - \underline{b}) \frac{\delta}{1-\delta}$$

- if T is large, and this $\geq \overline{b} - \underline{b}$ if $\delta > \frac{1}{2}$
- $\Rightarrow r_{i-1} < \underline{b}$
- Now the low type will buy at $i-1$
- *Screening blows-up*

Proposition 3. *Assume $\delta > \frac{1}{2}$. Then for any prior beliefs μ_1 $\exists k$ such that $\forall T$ and $t < T - l, r_t = \underline{b}$*

- Non Coasian dynamics: pools for a while and then separates
- In the Durable Goods case: Coasian dynamics - separates for a while and then pools
- Can get around it with a contract (long-term contract)
- Consider a service model but allow S to offer long-term contracts
- But don't prevent them from lowering the price (to avoid this just becoming a commitment technology)
- Can offer "better" contracts
- This returns us to the Durable Goods case - the ratcheting disappears (see Hart and Tirole)
- *A long-term contract is just like a durable good*
- *As soon as you go away from commitment in dynamic models the Revelation Principle fails – the information seeps out slowly here*

3.2.3 Soft Budget Constraint

- Kornai (re Socialist Economies)
- Dewatripont & Maskin
- Government faces a population of firms each needing one unit of capital
- Two types of firms: α good, quick types - project gets completed and yields $Rg > 1$ (money) and Eg (private benefit to firm / manager). There are also $1 - \alpha$ bad, slow types - no financial return, zero or negative private benefit, but can be refinanced at further cost $1 \rightarrow \Pi_b^*$ financial benefit and a private benefit of Eb $(1 < \Pi_b^* < 2)$
- Can the Government commit not to refinance?
- If yes then only the good types apply – and this is first-best
- If no then bad types also apply – and bad types are negative NPV so the outcome is sub-optimal
- Decentralization may aid commitment (free riding actually helps!)
- We will return to this idea when we study financial contracting
  - Dispersed creditors can act as a commitment no to renegotiate
- Transition in Eastern Europe (Poland and banking reform v. mass privatization)

3.2.4 Non Committment

- Go back to a regulation setting
- Firm has cost type $\theta \in \{\beta_L, \beta_H\}$ and let $\Delta\beta = \beta_H - \beta_L$
- τ indexes time periods, $\tau = 1, 2$
- Cost of production is $C_\tau = \beta - e_\tau$
- Cost of effort is $\psi(e_\tau)$
- Regulator pays subsidy t_τ
- Welfare is $W_\tau = S - (1 + \lambda)(C_\tau + t_\tau) + U_\tau$, where $U_\tau = t_\tau - \psi(e_\tau)$
- Firm gets $t - \psi(e)$
- FB: $\psi'(e^*) = 1$
- Suppose government's prior is $\Pr(\beta = \beta_L) = \nu_1$
- Let (t_L, C_L) be the contracts chosen by type β_L (so that effort is $e_L = \beta_L - C_L$) and similarly for type β_H
- In a one period problem the solution is just like before

$$\begin{aligned} \psi'(e_L^*) &= 1 \\ \psi'(e_H^*) &< 1. \end{aligned}$$

- Two periods without committment–can't commit to second period contracts
- Move away from the Revelation Principle
- First consider a continuum of types
- $\beta \in [\underline{\beta}, \bar{\beta}]$ with prior CDF $F_1(\cdot)$
- Posterior formed according to Bayes Rule and denoted $F_2(\cdot)$
- Going to consider the equilibria of the game between the regulator and the firm
- Solution concept: PBE
- The regulator's strategy is an incentive scheme in each period $C_1 \to t_1(C_1)$ and $C_2 \to t_2(C_2; t_1(\cdot), C_1)$
- The firm may quit the relationship and earn zero payoff
- Firm's strategy is a participation choice and an effort level
- An equilibrium will be said to be fully revealing if $\beta \to C_1(\beta) = \beta - e_1(\beta)$ is one to one

Theorem 9 ((Laffont-Tirole (1988))). *For any first period incentive scheme $t_1(\cdot)$ there does not exist a fully separating continuation equilibrium*

- If a firm reveals itself then it earns zero rent in the second period because the FB obtains
- Consider a firm of type β and think about a deviation from the revealing equilm strategy of producing at a cost as if it was type β and instead produces as if its cost was $\beta + d\beta$
- By the envelope theorem it makes a second-order loss of profit from this deviation
- But it gets a first-order rent in period 2 because in this equilm the regulator believes it to be type $\beta + d\beta$
- Striking result–highlights the importance of the committment assumption
- Moreover, there is no non-degenerate sub-interval of $[\underline{\beta}, \bar{\beta}]$ over which separation occurs
- If there is "small uncertainty" (i.e. $\left|\underline{\beta} - \bar{\beta}\right|$ is small) then the equilibrium must involve "much pooling" in the sense that one can find two types which are arbitrarily far apart who pool
- A natural kind of equilibrium to look for is a *partition equilibrium* (like in Crawford-Sobel cheap talk)
- In such an equilm $[\underline{\beta}, \bar{\beta}]$ is partitioned into a countable number of ordered intervals such that in period one all types in that interval choose the same cost level
- Full pooling is a degenerate partition equilm where there is only on such sub-interval

- When does such an equilm exist?

Definition 16. *We say that ψ is α-convex if $\psi'' \geq \alpha$ everywhere*

- Necessary condition: suppose that ψ is α-convex and that the equilm is a partition equilm. If C^k and C^l are two equilm first-period cost levels then $\left|C^k - C^l\right| \geq \delta/\alpha$.
- i.e. the minimum distance between two equilm costs is equal to the discount factor divided by the curvature of the disutility of effort
- e.g. in the quadratic case $\psi(e) = \alpha e^2/2$
- Sufficient condition: suppose that ψ is α-convex and that the distribution is log-concave. Then if the regulator offers a finite set of cost transfer pairs $\left\{t^k, C^k\right\}$ such that $\left|C^k - C^l\right| \geq \delta/\alpha$ for all k, l then there exists a partition equilibrium of the continuation game
- Much more stringent assumptions than when sending a message is costless (as in Crawford-Sobel)
- Further results are available in the two-type case
- An issue of how many types of contracts are allowed–unlike the static or committment case there can be a benefit from offering more contracts than types
- Three possibilities in the two-contract case: (i) Efficient type's IC constraint only is binding, (ii) Inefficient type's IC constraint only is binding, (iii) Both IC constraints are binding
- Turns out that (iii) is a real possibility
- Ask the following: is the optimal static mechanism incentive compatible in period 1 in the non-committment case?
- In that allocation the efficient type's IC constraint binds and the inefficient type's IR constraint binds
- Not IC: the efficient type earns zero rent in period 2 by revealing herself, but a positive rent by not doing so
- To make it IC the regulator has to offer her a larger period 1 transfer, to compensate for the loss of period 2 surplus
- But if the discount factor is high enough then the inefficient type will choose this contract in period 1
- They are then believed to be the efficient type and have to produce a lot in period 2
- If we could force them to participate then this would be a deterrent
- But they can opt out and get zero in period 2
- This is called the "take the money and run" strategy
- To avoid this pooling can be better

Proposition 4. *There exists a $\bar{\delta} > 0$ such that for all $\delta \leq \bar{\delta}$ the separating equilibrium is preferred to the pooling equilibrium and for all $\delta > \bar{\delta}$ the pooling equilibrium is better than the separating equilibrium*

- A large discount factor is plausible–especially since there are just two periods here!
- By offering a single contract in period 1 the regulator is essentially committing not to update her beliefs and ensures the optimal static mechanism in period 2
- When period 2 matters a lot relative to period 1 then this is optimal

4 Persuason and Information Transmission

4.1 Cheap Talk

- Crawford-Sobel (Ecta, 1982)
- Main question: how much information can be transmitted when communication is costless, but interests are not necessarily aligned?
- cf. signalling models: the key ingredient there is that communication is *costly*
  - eg. Spence job market signalling: to get separation need education to more costly for certain types than for others
- Two parties: a decision maker who is uninformed, and an informed expert
- DM is to make a decision $d \in [0, 1]$
- State of nature is $\theta \in [0, 1]$
- DM has a uniform prior about θ
- DM's payoff is $U(d, \theta) = -(d - \theta)^2$
- The E(xpert) knows the value of θ and her payoff is $V(b, \theta, d) = -(d - (\theta + b))^2$
- $b \geq 0$ is a measure of the bias of the expert
- E may send a message $m \in [0, 1]$
- Timing:
  1. 1. E observes θ
     2. E sends m to DM
     3. DM chooses d
- Solution concept: PBE

Proposition 5. *For all b there exists a "babbling equilbrium" in which E sends a random message ("babbles") and hence no information is conveyed.*

- Intuition: in a babbling equilibrium DM believes there is no information content in the message. E then has no incentive to send an informative message, so is happy to babble
- Bigger question: are there informative equilibria?
- Preliminary question: are there equilibria in which information is truthfully conveyed?

Proposition 6. *There exists an equilibrium in which information is fully revealed if and only if* $b = 0$.

- Proof sketch: suppose $b > 0$ and E truthfully revealed θ. In this equilibrium she is believed, but her payoff could be increased in some states by deviating to a message $\theta + b$–a contradiction.
- Now we constuct an equilibrium in which *some* information is conveyed
- Let DM's posterior distribution about the value of θ given m be $G(\theta|m)$
- Given quadratic preferences

$$\begin{aligned} d^*(m) &\equiv \max_{d\in[0,1]} \left\{ \int U(d,\theta)\, G(\theta|m)\, d\theta \right\} \\ &= E[\theta|m]. \end{aligned}$$

- E knows this, of course, and could be faced with the following problem
- Suppose message m leads to action d and message m' leads to action $d' > d$
- Also, suppose that in state $\theta'' > \theta'$ E prefers d' to d but in state θ' prefers d to d'
- Noting that V satisfies single crossing, $d^2V/d\theta dd > 0$ and hence E prefers d' to d for all $\theta > \theta''$
- Therefore, by the Intermediate Value Theorem, there exists a state $\hat{\theta}$ such that $\theta' < \hat{\theta} < \theta''$ in which E is indifferent between d and d'
- This is the same as saying that the distance between E's bliss point and d in state $\hat{\theta}$ is the same as the distance between the bliss point and d'
- ie. $\hat{\theta} + b - d = d' - \left(\hat{\theta} + b\right)$
- So E sends message m for all $\theta < \hat{\theta}$ and message m' for all $\theta > \hat{\theta}$
- For this to be an equilbrium we need to find d, d' and $\hat{\theta}$ such that

$$\begin{aligned} \hat{\theta} + b - d &= d' - \left(\hat{\theta} + b\right), \text{ and} \\ d(m) &= E[\theta|m]. \end{aligned}$$

- Solving we have

$$\begin{aligned} d &= \frac{\hat{\theta}}{2}, \\ d' &= \frac{1+\hat{\theta}}{2}. \end{aligned}$$

- Substituting into $\hat{\theta} + b - d = d' - \left(\hat{\theta} + b\right)$ we have

$$\hat{\theta} = \frac{1}{2} - 2b.$$

- Clearly such an equilibrium exists
- Moreover the cutoff $\hat{\theta}$ is uniquely determined by b
- If bias is too big then non-existence (ie. $b > 1/4$)
- This is a particular equilm with just two partitions
- But when bias is "small" there exist equilibria with more than two partitions

Theorem 10 (Crawford-Sobel)**.** *There exists a partition equilibrium of every size (ie. number of partitions) from 1 (completely uninformative) to $N(b)$ (the most informative).*

- Many equilibria!
- One thing to focus on is the impossibility of perfectly informative communication
- Another is the following quite general message: when preferences are somewhat aligned cheap talk "can" improve both party's payoff
- Cheap talk is just that: not an announcement in a mechanism, not a costly signal, just an unverifiable utterance
- One might think it could never help much (eg. Yogi Berra: "a verbal contract isn't worth the paper it's written on"), but the CS theorem shows that it *could*
- A large literature explores concrete settings in which it *does*
- Basic idea: cheap talk (by construction) does not directly affect payoffs, but it can affect them indirectly

4.2 Improved Communication

4.2.1 Conversation

- Krishna-Morgan (JET, 2004)
- Mulitiple messages from E can be subsumed as one message
- What about two-sided communication

- DM knows "nothing" and hence has no new information to reveal, but can act as a randomization device
- Illustration: suppose $b = 1/12$
- If only E gets to talk then the most informative equilm reveals whether θ is above or below 1/3
- Timing:

  1. 1. E observes θ
     2. E and DM meet face to face
     3. E delivers a "report"
     4. DM chooses d

- Consider the following equim:
  - In the meeting E reveals whether $\theta > 1/6$ or not and sends some other message (this determines whether the meeting is a "success" or not
  - If E sends the message that $\theta \leq 1/6$ then the meeting is deemed a failure and DM chooses $d = 1/12$
  - If E sends $\theta > 1/6$ then the report is conditional on the success or failure of the meeting
  - If the meeting was a failure then $d = 7/12$ (the optimal action conditional on $\theta > 1/6$
  - But if the meeting was a "success" then the report further partitions the interval $[1/6, 1]$ into $[1/6, 5/12]$ and $[5/12, 1]$
  - In the first subinterval $d^* = 7/24$ and in the second $d^* = 17/24$
  - If $\theta = 1/6$ then E prefers $d = 1/12$ to $d = 7/12$
  - So we need "uncertainty" about the outcome of the meeting–otherwise E would not be willing to reveal whether the state was above or below 1/6
  - If $\theta < 1/6$ then E would say $\theta \in [1/6, 5/12]$ and induce $d = 7/24$ and if $\theta > 1/6$ then E would announce $\theta < 1/6$ and induce $d = 1/12$ rather than $d = 7/12$
  - It turns out that when $\theta = 1/6$ then with probability $p = 16/21$ E is indifferent between $d = 1/12$ and the lottery where she gets $d = 7/12$ with probability p and $d = 7/24$ with probability $1 - p$ gets $d = 7/12$
  - When $\theta < 1/6$ E prefers $d = 1/12$ to the lottery and when $\theta > 1/6$ E prefers the lottery
  - So can we get the meeting to be successful with probability $p = 16/21$?
  - KM show that we can, as follows
  - Suppose E sends message (low, A_i) or $(high, A_i)$ and DM sends a message A_j with $i, j \in \{1, ..., 21\}$
  - Low means $\theta \leq 1/6$ and high means $\theta > 1/6$
  - The A_i and A_j parts of the message serve as a coordination device about the success of the meeting

- E chooses A_i randomly (i.e. from a uniform distribution)
- DM does similarly for A_j
- The meeting is deemed a success if $0 \leq i - j < 16$ or of $j - 1 > 5$ and a failure otherwise
- With this structure the probability that the meeting is a success is exactly $p = 16/21$
- So more information is conveyed than in any CS equilm
- Striking thing: having the DM participate in the conversation helps even though she is completely uninformed
- Aumann and Hart (2003) show that even with unlimited communication full revelation is impossible (cf. Geanakoplos-Polemarcharkis)

4.2.2 Delegation

- Can we do better by delegating to E?
- Tradeoff: E has her own preferences and is thus biased, but she is also informed
- Suppose $b = 1/12$ then direct computation yields a payoff of $-1/36$ in the most informative partition equilm of the CS model, but under delegation the action which is chosen is $d = \theta + b$, by construction, and the payoff is $-b^2 = -1/144$, so delegation is optimal
- This conclusion is more general than this example (see Dessein, 2002)
- DM can do even better by cambining the amount of delegation/discretion
- Here the optimal thing to do is limit E's discretion to $d \in [0, 1 - b]$

4.2.3 Compensation

- An obvious ommision in what we did is to preclude the possibility of compensating E for her advice
- Can we do better with an optimal contract?
- Now add a transfer such that the payoffs are
  - DM's payoff is $U(d, \theta) = -(d - \theta)^2 - t$
  - E's payoff is $V(b, \theta, d) = -(d - (\theta + b))^2 + t$
- Again use mechanism design to find the optimal contract
- Can apply the revelation principle here and restrict attention to mechanisms/contracts whereby E announces d and θ truthfully in equilm
- Aside: this isn't cheap talk any more–talk affects payoffs directly here

- Suppose $t\left(\hat{\theta}\right) = 2b\left(1-\hat{\theta}\right)$ and $d\left(\hat{\theta}\right) = \hat{\theta}$, then the FB decision is acheived and there is full revelation
- But this is costly for DM
- eg. when $b = 1/12$ her payoff is $-1/12$, whereas it is $-1/36$ in the best CS equilm
- General result: Krishna-Morgan (2004): Full revelation is in general feasible, but never optimal

4.2.4 Multiple Senders and Multiple Dimensions

- Battaglini (2002): two sender cheap talk with a one dimensional state space
  - Also showed that with a multi-dimensional Euclidean state space a perfectly revealing PBE can be constructed
  - Moreover, there are no out of equilm messages and so these equilbria survive any refinements which place restrictions on out of equilm beliefs
  - Construction: each sender conveys information only along directions in which her interests coincide with DM (ie. directions which are orthogonal to the bias of E)
  - Since these generically span the whole state space DM can extract all the information and perfectly identify the true state
- Ambrus-Takahashi (2007) consider restricted state spaces
  - eg. some policies may not be feasible
  - or some may never be chosen by DM (and so they are not rationalizable)
- AT provide the following example

dtbpF3.378in2.2857in0ptFigure

- Suppose DM needs to allocate a fixed budget to “education,” “military spending,” and “healthcare”
- Suppose there are two perfectly informed experts, a left-wing E and a right-wing E
- Left-wing E has a bias towards spending more on education, while right-wing E has a bias towards spending more on the military, but both of them are unbiased with respect to healthcare
- The state space in this example is represented by triangle ABC
- At B it is optimal for DM to spend the whole budget on the military
- At C it is optimal to spend all money on education
- At A it is optimal to spend no money on either education or military

- Left-wing E's bias is orthogonal to AB in the direction of C and right-wing E's bias is orthogonal to AC in the direction of B
- Battaglini's solution would have left-wing E report along a line parallel to AC (like asking how much money to spend on the military), right-wing E to report along a line parallel to AB
- But here it is not true that any pair of such reports identifies a point in the state space!
- Look at state θ
- If left-wing E sends a truthful report, then the right-wing analyst can send reports that put you outside the state space
- ie. they say that expenditure should be larger than the budget
- Doesn't happen in equilm, but have to specify out of equilm beliefs and this can cause problems for the construction
- Key points:
  - With multiple senders, the amount of information that can be transmitted in equilm depends on fine details such as: the shape of the boundary of the state space, how similar preferences of the senders are,...
  - Also properties of the state space and sender preferences cannot be investigated independently if one allows state-dependent preferences

4.3 Good News and Bad News

- Milgrom (Bell, 1981)
- Monotonicity plays a crucial role in information economics: e.g. Spence-Mirrlees condition/single-crossing, monotonicity of the bidding function in auctions,...
- Milgrom: "...it is surprising that studies of rational expectations equilibria and of the problem of moral hazard make no use of any such property...Such results has, unfortunately, been out of reach because no device has been availabel for modelling 'good news."
- And he provides one
- Moreover, he provides a set of tools which are indispensable in modelling information problems in a *very* wide class (eg. Friedman and Holden (AER, 2007))
- Let $\Theta \subseteq \mathbb{R}$ be the set of possible values of a random parameter $\tilde{\theta}$
- The set of possible signals about $\tilde{\theta}$ is $X \subseteq \mathbb{R}^m$
- Let $f(x|\theta)$ be the conditional density

Definition 17. *A signal x is **more favorable** than the signal y if for every nondegenerate prior distribution G the posterior distribution $G(\cdot|x)$ first order stochastically dominates $G(\cdot|y)$*

- ie.
$$\int U(\theta)\, dG_1(\theta) > \int U(\theta)\, dG_2(\theta),$$
or equivalently $G_1(\theta) \le G_2(\theta)$ with at least one inequality strict.

- Suppose G is a prior that assigns probabilities $g(\theta)$ and $g\left(\bar{\theta}\right)$ to θ and $\bar{\theta}$. By Bayes' Rule
$$\frac{g\left(\bar{\theta}|x\right)}{g(\theta|x)} = \frac{g\left(\bar{\theta}\right)}{g(\theta)} \frac{f\left(x|\bar{\theta}\right)}{f(x|\theta)}, \tag{7}$$
and similarly for signal y.

Theorem 11. *x is more favorable than y if and only if for every $\bar{\theta} > \theta$*
$$f\left(x|\bar{\theta}\right) f(y|\theta) - f(x|\theta) f\left(y|\bar{\theta}\right) > 0.$$

Proof. Suppose $g(\theta) = g\left(\bar{\theta}\right) = 1/2$. The by the definition of more favorable and (7) we have
$$\frac{f\left(x|\bar{\theta}\right)}{f(x|\theta)} > \frac{f\left(y|\bar{\theta}\right)}{f(y|\theta)}.$$
Fix G and consider θ^* that $0 < G(\theta^*) < 1$. For $\theta \le \theta^*$ the above inequality implies
$$\frac{\int_{\bar{\theta}>\theta^*} f\left(x|\bar{\theta}\right) dG\left(\bar{\theta}\right)}{f(x|\theta)} > \frac{\int_{\bar{\theta}>\theta^*} f\left(y|\bar{\theta}\right) dG\left(\bar{\theta}\right)}{f(y|\theta)}$$
Flipping the inequality
$$\frac{f(x|\theta)}{\int_{\bar{\theta}>\theta^*} f\left(x|\bar{\theta}\right) dG\left(\bar{\theta}\right)} < \frac{f(y|\theta)}{\int_{\bar{\theta}>\theta^*} f\left(y|\bar{\theta}\right) dG\left(\bar{\theta}\right)}$$
Integrating over θ for $\theta \le \theta^*$ yields
$$\frac{\int_{\theta\le\theta^*} f(x|\theta)\, dG(\theta)}{\int_{\bar{\theta}>\theta^*} f\left(x|\bar{\theta}\right) dG\left(\bar{\theta}\right)} < \frac{\int_{\theta\le\theta^*} f(y|\theta)\, dG(\theta)}{\int_{\bar{\theta}>\theta^*} f\left(y|\bar{\theta}\right) dG\left(\bar{\theta}\right)},$$
which is equivalent to
$$\frac{G(\theta^*|x)}{1 - G(\theta^*|x)} < \frac{G(\theta^*|y)}{1 - G(\theta^*|y)}$$
which in turn implies that $G(\theta^*|x) < G(\theta^*|y)$ ■

Definition 18. *Let $X \subseteq \mathbb{R}$. The family of densitites $\{f(\cdot|\theta)\}$ have the (strict)* ***Monotone Likelihood Ratio Property (MLRP)*** *if for every $x > y$ and $\bar{\theta} > \theta$, $f\left(x|\bar{\theta}\right) f(y|\theta) - f(x|\theta) f\left(y|\bar{\theta}\right) > 0$.*

Remark 13. *This property holds for many families of densities, although it is trivial to construct counterexamples. It holds for, among others the: normal, uniform, exponential, Poisson and chi-squared distributions.*

Theorem 12. *The family of densities $\{f(\cdot|\theta)\}$ has the strict MLRP if and only if $x > y$ implies that x is more favorable than y.*

Proof. Immediate from the definition and the previous theorem. ■

Definition 19. *Two signals are* ***equivalent*** *if* $f\left(x|\bar{\theta}\right) f\left(y|\theta\right) - f\left(x|\theta\right) f\left(y|\bar{\theta}\right) = 0.$

Definition 20. *If two signals are equivalent or one is more favorable than the other then they are* ***comparable.***

- The following theorem establishes the very useful fact that information system which is comparable can be modelled as a real-valued variable with the MLRP

Theorem 13. *Suppose that two signals in X are comparable. Then there exists a function $H : X \to \mathbb{R}$ such that $H\left(\tilde{x}\right)$ is a sufficient statistic for $\tilde{x}$ and such that the densities of $H\left(\tilde{x}\right)$ have the strict MLRP.*

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded increasing function and let

$$H\left(x\right) = \int h\left(\theta dG\left(\theta|x\right)\right),$$

where G is a prior over θ. By comparability $H\left(x\right) > H\left(y\right) \Leftrightarrow x$ if more favorable than y. Then by the previous theorem the densities have the strict MLRP. And since $H\left(x\right) = H\left(y\right) \Leftrightarrow x$ and y are equivalent, $H\left(\tilde{x}\right)$ is a sufficient statistic. ■

- Now consider one application of this from Milgrom (1981)–persuasion games
- An interested party provides information to a decision maker in order to influence her decision
- Suppose there is a B(uyer) and a S(eller)
- Good is of unknown quality $\tilde{\theta}$
- B's payoff is $\tilde{\theta}F\left(q\right) - pq$, where q is quantity and p is price
- S's payoff is an arbitrary increasing function of q
- Suppose S has a vector of information about the product $\tilde{x} = \left(\tilde{x}_1, ..., \tilde{x}_N\right)$
- These can be concealed, but not misreported
- A report by S is a subset of $\mathbb{R}^N$ which asserts that $\tilde{x} \in \mathbb{R}^N$
- A reporting strategy is a function f from $\mathbb{R}^N$ to subsets of $\mathbb{R}^N$
- A precise report has $r(x) = \{x\}$ and a vague one has $r\left(x\right) = \mathbb{R}^N$
- Solution concept: sequential equilibrium
- Full disclosure is a report whereby S does not conceal any information which is payoff relevant to B
- Milgrom shows that in every equilbrium of this game the strategy chosen by S is one of full disclosure

- Key idea: any information which is withheld is interpreted as negative information– why didn't S report it if it wasn't?! (this deep idea also appears in Hart (1980), Grossman (1981)
- What if information is costly to communicate?
- Suppose that B can only listen to $k < N$ observations
- Also suppose that $\tilde{x}_1, ..., \tilde{x}_N$ are conditionally independent and that F has the MLRP
- In this setting S always reports the k most favorable observations in any sequential equilibrium
- Intuition: want to maximize B's expectation of the quality and by MLRP this is done by sending the most favorable messages
- Again, extreme skepticism/assume the worst

5 Incomplete Contracts

5.1 Real versus Formal Authority

- Inside the firm asset ownership doesn't matter
- Authority matters inside the firm – and this is not achieved through assets
- How is authority allocated inside a firm?
- Initial model: 2 parties, P and A – what is the optimal authority between P and A
- Assumption: authority can be allocated – this can be achieved contractually (eg. shareholders allocate authority to the board)
- Boards allocate authority to management – management to different layers of management
- AT call this stuff "Formal Authority" (legal / contractual)
- Distinction between this and "Real Authority" (which is what is the case if the person with Formal authority typically "goes along" with you)
- Asymmetric information important

Model:

- $\{P, A\}$
- Each can invest in "having an idea" – only 1 can be implemented
- P chooses prob E of having an idea at cost $g_p(E)$ with $E \in [0, 1]$

- A chooses prob e of having an idea at cost $g_a(e)$ with $e \in [0,1]$
- Assume $g_i(0) = 0, g_i'(0) = 0, g_i' > 0$ elsewhere, $g_i'' > 0, g_i'(1) = \infty \; \forall i \in \{A, P\}$, in order to ensure an interior solution
- If it exists, P's idea is implemented giving payoffs of B to P and αb to A where $\alpha \in [0,1]$ is a congruence parameter (their preferences are "somewhat" aligned)
- If A's idea is implemented the payoffs are b to A and αB to P

Case I: P has formal authority

$$
\begin{aligned}
U_P &= EB + (1-E)e\alpha B - g_p(E) && (8)\\
U_A &= E\alpha b + (1-E)eb - g_a(e) && (9)
\end{aligned}
$$

- P maximizes (8) by choosing E and A maximizes (9) by choosing e
- The FOCs are:

$$
\begin{aligned}
B(1-e\alpha) &= g_p'(E)\\
b(1-E) &= g_a'(e)
\end{aligned}
$$

- If we assume $B = b$ then there are no gains from renegotiationdbpF174.875pt157.5pt0ptFigure
- Under a stability assumption you get a unique Nash Equilibrium
- P and A effort are substitutes – whereas in Hart-Moore they are complements
- Higher effort from P crowds-out effort from A
  - May want to "overstretch"
  - May want to find an agent with more congruent preferences

Case II: A has formal authority

- P solves:

$$\max_E \{e\alpha B + (1-e)EB - g_p(E)\}$$

- A solves:

$$\max_e \{eb + (1-e)E\alpha b - g_a(e)\}$$

- The FOCs are:

$$
\begin{aligned}
B(1-e) &= g_p'(E)\\
b(1-\alpha E) &= g_a'(e)
\end{aligned}
$$

- Which implies $E \uparrow, e \downarrow$ (effort levels are strategic substitutes)

- Comparing the FOCs with the P formal authority shows that A effort increases when A has formal authority
- If there is a P with several Agents then the P may "want to be overstretched" to give good innovation incentives to subordinates – just "puts out fires"

Comments:

1. Seems to have quite a nice flavor – sounds like the right setup
2. Ignores ex post renegotiation (since $B = b$) – imposes an ex post inefficiency.
   (i) Perhaps authority is ex post non-transferable and implementing ideas is ex post non-contractable

   (ii) But this opens another door – lead to ex post inefficiency
3. Inside a firm, what gets allocated? Formal or Real authority?

5.2 Financial Contracting

- An important, pervasive, high-stakes form of contract
- Many different types of financial contracts
  - Debt
  - Equity
  - Debt with warrants
  - Options of many different types
  - Convertible preferred stock
  - ...
- Want to explain the existence of different types of contracts and understand the economic drivers on the particular form
  - eg. what role is the conversion option playing?
- Also look at the design of financial insitutions–most notably the public company
  - How can we understand different forms of organizations, voting arrangements, etc.
- We will act like financial anthropologists
  - Think: "the natives pay dividends on stock. why is that?"

5.2.1 Incomplete Contracts & Allocation of Control

- Aghion-Bolton (Restud '92)
- Basic idea: incomplete contracts plus wealth constraints make allocation of control an important part of financial contracts
- Entrepreneur is risk-neutral (with no wealth) – but has a project
- Capitalist has wealth and is also risk-neutral
- Project costs K
- No relationship specific investments – but private benefits ex post
- Date 1: E & C contract, date 2: action taken which leads to the realization of a monetary benefit and a private benefit
- Assume that the future is too complicated for the parties to contract on action in advance - but at the end of the period the uncertainty is resolved and can contract perfectly on the action
- Action $a \in A$
- $a \to y(a)$ monetary benefits which are verifiable and contractible
- $a \to b(a)$ private benefits which are non-verifiable and non-transferable so that E gets them
- C cares only about money–E cares about both types of benefit
- Two things you can do ex ante: (i) divide up $y(a)$, (ii) allocate the right to decide a (ie. RCRs)
- $b(a)$ is *measured* in monetary units even though it is non transferable
- For simplicity, suppose that all of $y(a)$ is allocated to C
- FB:
$$\max_{a \in A} \{b(a) + y(a)\} \to a^*$$
- SB: Case I – E owns and controls the project:
$$\max_{a \in A} \{b(a)\} \to a_E$$
- Only maximizes private benefits because the contract allocated all the pecuniary benefits to C
- Assume that E has all the bargaining power ex post – they will negotiate to a^* and E demands $y(a^*) - y(a_E)$ from C
- C still gets $y(a_E)$ (because they have no bargaining power)
- E gets $b(a^*) + y(a^*) - y(a_E) \geq b(a_E)$ (by the definition of efficiency)

- SB: Case II – C has control

$$\max_{a \in A} \{y(a)\} \to a_C$$

- That is, maximizes only monetary returns
- E would like to get C to take action a^*
- If they tried to get a^* then C would demand $y(a_C) - y(a^*) > 0$
- But they have no wealth!
- Can't move away from the inefficient a_C because of the wealth constraint
- *There are potential gains from trade that go unexploited because of the wealth constraint*
- C's payoff is $y(a_C) > K$ (if not it was a doomed project from the beginning)
- Optimal Contract: Could have E have control with probability π and C with prob $1 - \pi$ such that $\pi y(a_E) + (1 - \pi)y(a_C) = K$
  - This is a bit of an odd contract
  - But we can add some ingredients to the model to get a contract which is not at all odd, and does the same thing
- Embellishment: Introduce a verifiable state θ, realized after the contract is signed but before a is chosen

$$y(a, \theta) = \alpha(\theta)z(a) + \beta(\theta)$$

- where $\alpha > 0$, $\alpha' < 0$, $z > 0$
- $\left|\frac{\partial y}{\partial a}\right| = \alpha(\theta)|z'|$ decreasing in $\theta \Rightarrow y$ is less sensitive when θ is high
- Can show that the optimal contract has cutoff $\theta^* \to$ if $\theta > \theta^*$ E has control and if $\theta < \theta^*$ C has control
- Just a more refined version of the stochastic contract
- If $\alpha'(\theta)z(a) + \beta(\theta) > 0$ the high θ states are high profit states $\Rightarrow$ E has control in good states and C has control is bad states
- This looks a lot like securities which we see

Summary:

1. Non-voting equity always leads to the *ex post* efficient action choice but may violate C's PC
2. E control is most likely to satisfy C's PC but may impose inefficient action choices in too many states of nature - and these may not be able to be renegotiated around because of the wealth constraint
3. Debt or contingent control of some kind may allocate control to the wrong agent in the wrong state since the signal and the state may not be perfectly correlated - but as the correlation coefficient $\to 1$ and/or the probability of such a misallocation is small then contingent control becomes the optimal contract

5.2.2 Costly State Verification

- Townsend '78, Gale-Hellwig '85
- Shows circumstances in which debt can be the optimal contract
- Idea: debt is less informationally sensitive than equity
- An Entrepreneur and an Investor: information asymmetry can be undone for a cost c (paid by E)
  - Both risk-neutral
- E needs to raise K for a project with return a random variable $x \geq 0$ with density $f(x)$
- No ex ante information asymmetry
- Ex post only E obersves x
- Want a contract which allows I to get some of x but without "too much" costly auditing
- Let $B(x)$ be the auditing dummy (=1 if audit) – this is a restriction on the contracting space
- Let $r(x)$ be amount paid to I
- Want to minimize the deadweight costs of auditing

$$\int cB(x)f(x)dx$$

  subject to (i) $\int r(x)f(x)dx \geq K$ (I's breakeven constraint), (ii) $B(x) = 0 \Rightarrow r(x) = F$ (payment can't depend on x if there is no auditing otherwise E will choose the lower payment), (iii) $r(x) + c \leq F$ when $B(x) = 1$ (gross payment bounded above by F when there is auditing–otherwise E will lie and pay F)

- Truth-telling requires that E reveal that she should be audited when $B(x) = 1$ – ie. $B(x) = 1 \Rightarrow r(x) + c \leq x$ and $B(x) = 0 \Rightarrow r(x) \leq x$
- Define a Straight Debt Contract ("SDC") as follows:
  - If $x > p$ then E pays p
  - If $x < p$ then E defaults and I pays c – and takes all of the y in this event
- This has the "Maximal Recovery Property"

Proposition 7. *The SDC is the optimal contract*

Proof. Consider an arbitrary contract $\{B_A(x), r_A(x)\}$ and suppose $B_A(x) = 0 \Rightarrow r_A(x) = F_A$. An SDC can be fully represented by its face value F_D. Consider $F_D = F_A$. 4 cases. Case (i) $B_A(x) = 0, B_D(x) = 0$. Contracts are equivalent since $F_D = F_A$ if no audit. Case (ii) $B_A(x) = 1, B_D(x) = 1$. SDC weakly dominates because of the maximal recovery property. Case (iii) $B_A(x) = 1, B_D(x) = 0$. SDC strictly dominates here since SDC gets F_D but the alternative contract pays out less because of auditing costs and incentive compatibility. Case (iv) $B_A(x) = 0, B_D(x) = 1$. $B_D(x) = 1 \Rightarrow x < F_D \Rightarrow x < F_A$ and hence violates the resource constraint $B(x) = 0 \Rightarrow r(x) \leq x$. So if one audits at state x under the SDC one also audits in the arbitrary contract. Hence the region in case (iv) is empty. ■

- Intuition: SDC does weakly better for I in all the relevant states and auditing costs are no higher because case (iv) is the empty set – SDC minimizes auditing costs
- A more informationally sensitive contract involves more (costly) auditing

Comments:

1. No obvious role for equity here
2. Unclear what c really refers to
3. Perhaps more a theory of monitoring
4. Recall: Innes – wealth constrained risk-neutral agent basically led to a debt contract
5. Ex post renegotiation ruled out – but could be optimal for I not to do the audit

5.2.3 Voting Rights

- 2 questions
  - How to allocate voting rights to securities – when is one-share/one-vote optimal?
  - What determines the value of corporate votes – why is the voting premium sometimes high and sometimes low
- Focus on role of votes as determinants of takeover battles in a setting with private benefits
- Grossman & Hart (Bell, 1980)
- Charter designed to maximize the value of securities issued
- Two classes of shares: A and B
- Share of cashflows s_A, s_B and votes v_A, v_B
- Assume $v_A \geq v_B$
- One-share/one-vote means $s_A = v_A = 1$
- 2 control candidates: incumbent (I) and rival (R)
- R needs α of the votes to get control with $1/2 \leq \alpha \leq 1$ to gain control
- If I has control then public cashflows of y_I accrue evenly to all claimants and private benefit of z_I accrues to I – symmetric if R has control
- Private benefits could be synergies, perks, diverted cashflows – might be bigger for some parties than others

- ys and zs not know when charter written but common knowledge at time of bidding contest
- Assume shareholders behave atomistically (this is important to rule out strategic effects)
- Bid form: unconditional and restricted (partial) offer for shares of a class
- Case 1: Restricted Offers not allowed so must pay for *all* shares tendered – consider 4 sub-cases
- eg1. z_I small relative to y_I, y_R, z_R. Let $y_I = 200$, $y_R = 180$, $s_A = s_B = 1/2$, $v_A = 1$ (class A shares have all the votes)
  - Suppose R tenders for all of class A at 101 – profitable for R if $z_R > 11$ since get 1/2 of cashflows and all private benefits
  - If no counteroffer A class holders get 101 if tender, get 90 if don't tender and R wins, get 100 if don't tender and R loses
  - So they tender and I can't top the bid because they have small private benefits
  - Total value of A+B shares under R is 191, but 200 under I control – value reduced by takeover
  - Key: B class shares devalued by R control but since they are non-voting there's no point in R buying them
  - Suppose one-share/one-vote
  - Now R must buy all stock – so must bid 200 or I will top the bid
  - A+B jointly better off under one-share/one-vote
  - Shareholders can extract more from R if she faces competition – when shares and votes are separated competition is reduced because here I has no control benefits
  - With one-share/one-vote α doesn't matter but with asymmetric voting it can
- eg2: z_R insignificant. Let $y_I = 180$, $y_R = 200$, $s_A = s_B = 1/2$, $v_A = 1$
  - With no private benefits R offers 100 for both A and B shares – I offers $90 + z_I$ for A shares (since $v_A = 1$)
  - If $z_I > 10$ then I wins and A+B get 190 jointly – but get 200 if R wins
  - Under one-share/one-vote I can only beat R by buying all shares for 200 and will only do this if $z_I > 20$
- eg3: z_I, z_R both insignificant $\rightarrow$ bidder with higher y wins independent of voting structure
- eg4: z_I, z_R both significant. Let $y_I = 90$, $y_R = 100$, $z_I = 4$, $z_R = 5$
  - Now one-share/one-vote might not be optimal
  - With one-share/one-vote R buys all shares for $100 + \varepsilon$ and wins
    - If R offered less the the shareholders who expect the bid to succeed would not sell–preferring to be minority shareholders

  - But if A shares are voting with no cashflow rights and B shares are non-voting with all the cashflows then R must pay 4 for the votes to outbid I so A+B shares worth 104
  - Intuition: make I and R compete over something for which they have very similar reservation values (here votes) in order to extract lots of R's private benefit
  - In general can get an interior solution where the optimum lies b/w pure votes and one-share/one-vote
  - Overall: if ex ante probability of both parties having large private benefits is small then one-share/one-vote is approximately optimal

- Now consider restricted offers
- Can allow inferior offers to win
- eg. $y_I = 60$, $y_R = 40$, $z_I = 0$, $z_R = 15$, $\alpha = 1/2$
  - R wins with a restricted offer for 1/2 of shares for a total of $30 + \varepsilon$ since I values 1/2 shares at 30 but R values them at 35
  - In equilm shareholders are better off tendering to R because if you don't you get a claim on 20 if R wins
- Restricted offer is only valuable to a party with large private benefits
- Conclusions: if only z_I is large then set $\alpha = 1/2$ and make I buy a lot of profit stream to keep control, if only z_R is large then set $\alpha = 1$ and make R buy a lot of profit stream to get control, intermediate values of α depend on which party is more likely to have the larger z, maintain one-share/one-vote

5.2.4 Collateral and Maturity Structure

- Hart-Moore (QJE, 1998)
- Entrepreneur is risk-neutral and has wealth $W < I$ where I is the cost of a project
- Competitive supply of risk-neutral investors
- $t = 0$: invest, $t = 1$: cash of R_1 comes out and can also liquidate for value L, $t = 2$: if not liquidated get R_2
- Interest rate = 0
- Assume that the asset is worthless at date 2
- Ignore here the reinvestment option which exists in the paper
- R_1, R_2, L are ex ante uncertain – resolved at date 1
- Assume symmetric information throughout
- R_1, R_2, L are observable but not verifiable
- Assume R_1, R_2 can be diverted by E, but the assets cannot be

- $R_2 > L$ with probability 1
- $E[R_1 + R_2] > I$ (ie. it's a good project in the FB)
- Partial / fractional liquidation is allowed and the production technology is CRS
- Natural to look at a debt contract
- Let E be called D and the Investor who is chosen be called C
- D borrows $B = I - W + T$ and promises fixed payments p_1 and p_2 at dates 1 and 2
- $T \geq 0$
- If D fails to pay then C can seize all the project assets
- wlog assume that $p_2 = 0$ (any payment promised at date 2 is not credible)
- But may be willing to pay something at $t = 1$ – doesn't want to lose control of the project
- Debt contract is just represented by (P, T) where $P = p_1$
- T goes in a private, bankruptcy remote, savings account
- At date 1: R_1, R_2, L all realized
- $T + R_1$ is in the private account
- D can liquidate assets to repay C (a last resort as it turns out, since $R_2 > L$)–but can't divert this
- C may not choose to exercise her liquidation rights–renegotiation may take place
- Renegotiation
  - We have wealth constrained renegotiation (different than with no such constraint)
  - Assume that with probability $1 - \alpha$ D makes a TIOLIO to C and that with probability α C makes a TIOLIO to D
  - Nice modelling trick where one party has all the bargaining power, but who that party is is stochastic
  - C's payoff without renegotiation is L
  - If $\alpha = 1$ C gets: Case I: $T + R_1 > R_2 \rightarrow$ C gets R_2 and $f = 1$ (the fraction of assets left in place), Case II: $T + R_1 < R_2 \rightarrow$ sell some fraction $1 - \frac{T+R_1}{R_2}$, $f = \frac{T+R_1}{R_2}$, C gets $T + R_1 + L\left(1 - \frac{T+R_1}{R_2}\right)$
  - Combining these C gets:

$$\min\left\{R_2, T + R_1 + L\left(1 - \frac{T + R_1}{R_2}\right)\right\}$$

- Back to the $\alpha = 0$ case

- D pays $P \Leftrightarrow P \leq L$ (need the self-liquidation assumption here - would get awkward discontinuities in C's payoff otherwisedtbpF229.5pt165.5pt0ptFigure

- If $\min\{P, L\} < T + R_1$ then no inefficiency

- If $\min\{P, L\} > T + R_1$ then inefficiency because of asset liquidation

- C's payoff is $\min\{P, L\}$

- Let $N = \min\{P, L\} - T$

$$f = \min\left\{1, 1 - \left(\frac{N - R_1}{L}\right)\right\}$$

- since $T + R_1 + (1 - f)L = Min\{P, L\}$

- D's date 1 payoff is
$$T + R_1 + (1 - f)L - \min\{P, L\} \equiv \Pi$$

- Optimal debt contract at date 0:

$$\max E[\Pi]$$
$$s.t. E[N] \geq I - W$$

- The constraint will hold with equality because of the competitive capital market assumption

- $\Rightarrow \Pi + N = R_1 + fR_2 + (1 - f)L$

- The optimal contract solves:

$$\max_{P,T} \{E[f(R_2 - L)]\}$$
$$s.t. E[N] = I - W$$

- 2 instruments with different roles: P $\downarrow$ makes C worse off and must be balanced by $T \downarrow$

- P $\downarrow \Rightarrow$ pay less in solvency states

- T $\downarrow \Rightarrow$ D has less in all states

- Define: (i) Fastest debt contract has $P = 0$, (ii) Slowest debt contract has $P = \infty$

- Note that $P = \infty \Rightarrow$ C control in Aghion-Bolton, $P = 0 \Rightarrow$ D control in Aghion-Bolton, $P > 0 \Rightarrow$ Mix of D & C control in Aghion-Bolton

Proposition 8. *Suppose R_1, R_2, L are non-stochastic, then any debt contract satisfying C's break-even constraint with equality is optimal*

Proof. $E[N] = N = I - W$. Objective function is $f(R_2 - L)$, $f = \min\left\{1, 1 - \left(\frac{N-R_1}{L}\right)\right\}$ which is simply $f = \min\left\{1, 1 - \left(\frac{I-W}{L}\right)\right\}$ ■

Example:

- $I = 90, W = 50$
- State 1: $R_1 = 50, R_2 = 100, L = 80$ (this state occurs with probability 1/2)
- State 2: $R_1 = 80, R_2 = 100, L = 30$ (this state occurs with probability 1/2)
- Consider $T = 0, P = 50 \rightarrow S1$: No default and C gets 50, $f = 1$, $S2$: D defaults, renegotiation occurs and C gets 30, no liquidation and $f = 1$
- $\Rightarrow$ First-Best: all assets are left in place and the expected return to C is 40–so willing to lend
- Suppose $T > 0$ then in $S1$ C gets P, in $S2$ C gets 30
- To break even $\frac{P+30}{2} = 40 + T \Rightarrow P = 50 + 2T$
- Liquidation in $S1$ unless $T = 0$

Comments:

1. More general contracts are possible, eg. an option contract: give C an option to buy the project for $\$K$ – will only exercise if it has positive net value, which is effectively a transfer from E to C. This works well if L is stochastic (if L very high then R_2 also high and $R_2 \simeq L$). The paper provides sufficient conditions for this NOT to be the optimal contract – have to assume that re-invested funds earn $s \equiv \underline{R_2} \Rightarrow$ CRS beyond the project value AND R_1, R_2, L, s are "positively correlated"
2. Dynamic version in Hart ch. 5 under perfect certainty – can analyze maturity structure considerations. See also Hart-Moore QJE '94 (actually written after the 98 paper!)
3. Collateral becomes important here – unlike in the Costly State Verification literature
4. Macroeconomics applications: (i) Shleifer-Vishny, (ii) Kiyotaki-Moore (can amplify business cycles)
5. Several Outsiders: (i) wealth constraints, (ii) risk-aversion, (iii) multiple creditors may harden the budget constraint, even though there are negotiation problems - committing not to renegotiate (but only good in some states), (iv) different types of claims may be good (Dewatripont & Tirole)

5.3 Public v. Private Ownership

- Economists generally agree that there are some public goods (eg. military expenditure, prisons) – that have to be *paid for by the government*
- But that does not mean that the government has to own the production technology - they can contract for these goods
- Just a make-or-buy decision

Schmidt (JLEO, 1996):

- Manager puts in effort which affects production cost (could be low or high)
- Government is buying stuff from this firm
- Under outside contracting the Government doesn't observe cost – procurement under asymmetric information
- Optimal to have high cost firm produce too little – make it unattractive for a low cost firm to mimic them – satisfy IC
- Manager is an empire builder who doesn't like this so they put in effort to try and be low cost
- Under public ownership G observes cost – get a better ex post efficient allocation – but effort goes down because the empire building manager is less disciplined

Hart-Shleifer-Vishny (QJE, 1998)

- Consider prisons and other things
- In a world of complete contracting it doesn't matter who owns what because you just write a perfect contract
- Introducing asymmetric information or moral hazard doesn't change anything because now you just have some optimal second-best contract or mechanism
- Contractual incompleteness implies that ownership does matter because the allocation of RCRs matters
- First paper to take such an approach was by Schmidt (above) – but he does it through asymmetric information - owner has better information
- Consider a G(overnment) and a M(anager)

Case I: Prison is Private, owned by M

- G & M contract on how the prisoners are going to be looked after – the "Basic good", with price p_0 – this is a complete contract
- Basic good yields benefit B_0 and costs C_0 to produce
- Then the "Actual good", which produces a social benefit of $B_0 - b(e) + \beta(i)$ at cost $C_0 - c(e)$
- e and i are chosen by the manager
- e is an investment in cost – more e reduces cost, but quality also deteriorates
- i is an investment in innovation – more i means higher quality
- Date 1: Contract written and ownership structure chosen, Date 2: M chooses e, i, Date 3: Renegotiation and payoffs (if they can't agree then the basic good gets provided)

- Benefit enjoyed by society, cost incurred by M
- Assume e and i investment consequences can be implemented without violating the terms of the contract
- $b(e) \geq 0, c(e) \geq 0, b(0) = c(0) = 0, c' - b' > 0, \beta' > 0$
- The last two imply that quality reduction from cost innovation does not offset the cost reduction and the cost increase from a quality innovation does not offset the quality increase
- Also assume $b'(\cdot) > 0, c'(\cdot) > 0, c'(\cdot) > b'(\cdot) \Rightarrow c(e) - b(e) \geq 0, \forall e$
- FB:
$$\max_{e,i} \left\{ B_0 - b(e) + \beta(i) - (C_0 - c(e)) - e - i \right\}.$$
- Which is equivalent to:
$$\max_{e,i} \left\{ -b(e) + c(e) + \beta(i) - e - i \right\}$$
- The FOCs are:
$$\begin{aligned} -b'(e) + c'(e) &= 1 \\ \beta'(i) &= 1 \end{aligned}$$
- Under private ownership (absent renegotiation) the cost innovation is implemented (M has RCRs) but quality innovation is not (because G won't pay for it)
- Because M doesn't have to ask permission to implement innovations we have–assuming 50:50 Nash Bargaining
$$U_G = B_0 - p_0 - b(e) + \frac{1}{2}\beta(i)$$
- And M's payoff is
$$U_M = p_0 - C_0 + c(e) + \frac{1}{2}\beta(i) - e - i$$
- There is only renegotiation over the quality innovation
- The FOCs for M are:
$$\begin{aligned} c'(e) &= 1 \\ \frac{1}{2}\beta'(i) &= 1 \end{aligned}$$
- Let the solutions to these be e_M and i_M
- $S_M = B_0 - C_0 - b(e_M) + c(e_M) + \beta(i_M) - e_M - i_M$

Case II: Public Ownership

- An At-Will employment contract (in the formal legal sense)

- Now the e idea is not implementable because G has RCRs
- G has a veto but can renegotiate
- Default payoffs are:

$$\begin{aligned} U_G &= B_0 - p_0 \\ U_M &= p_0 - C_0 - e - i \end{aligned}$$

- In the asbsence of renegotiation both innovations are implemented becuase G has RCRs
- Under 50:50 Nash Bargaining

$$U_G = B_0 - p_0 + \frac{1}{2}\left[-b(e) + c(e) + \beta(i)\right]$$

$$U_M = p_0 - C_0 + \frac{1}{2}\left[-b(e) + c(e) + \beta(i)\right] - e - i$$

- More generally ($\lambda = 1$ is like M being irreplaceable)

$$U_G = B_0 - p_0 + \left(1 - \frac{\lambda}{2}\right)\left[-b(e) + c(e) + \beta(i)\right]$$

$$U_M = p_0 - C_0 + \frac{\lambda}{2}\left[-b(e) + c(e) + \beta(i)\right] - e - i$$

- The FOCs are:

$$\frac{\lambda}{2}\left(-b'(e_G) + c'(e_G)\right) = 1$$

$$\frac{\lambda}{2}\beta'(i_G) = 1$$

- Social Surplus is:

$$S_G = B_0 - C_0 - b(e_G) + c(e_G) + \beta(i_G) - e_G - i_G$$

- Conclusion: Privatize $\Leftrightarrow S_M > S_G$
- Under G ownership we get underinvestment for the usual reason - in fact there is a further deterrent
- Under private ownership there is over investment because there is an externality to do with quality $e_M > e^* > e_G$
- $i_G < i_M < i^*$
- Private ownership: e too high and i too low but not as bad as under G ownership
- Public ownership: e too low and i too low
- Prisons: use of force very hard to contract on, quality of personnel a big issue

5.4 Markets and Contracts

5.4.1 Overview

- A lot of what we have done thus far considers bi-lateral (or sometimes multilateral) relationships
- But in some/many contexts, contracts between agents exist in market settings
- This has been recognized for a long time–Rothschild and Stiglitz (1976) analyze screening in such a context
- But there are a number of other issues of interest
- We will only touch on a few of them here

5.4.2 Contracts as a Barier to Entry

- There is a long traditional in legal scholarship/law and economics which argues that contracts can be anti-competitive in effect
- Sellers may be able to "lock up" buyers with long-term contracts which prevent or at least deter entry to some degree
- Key reference is Aghion and Bolton (1987)
- Contracts that specify penalties for early termination can be used to extract rents from future entrants who may be lower cost than the current provider
- Suppose there are two time periods $t = 1$ and $t = 2$
- At $t = 1$ there is an incumbent who can sell a product at cost $c_I \leq 1/2$ and a buyer has reservation value $v = 1$ for this widget
- At $t = 2$ a potential entrant has cost c_E which is uniformly distributed on $[0, 1]$
- Obviously $p_1 = 1$ in period 1
- Assume that if entry occurs there is Bertrand competition at $t = 2$
- So entry occurs if $c_E \leq c_I$
- If there is no contract / a spot contract then if entry occurs $p_2 = \max\{c_E, c_I\} = c_I$ and if no entry then $p_2 = 1$
- So under the spot contract the expected payoff of the buyer is

$$\begin{aligned} V_B &= (1 - \Pr(entry))\,0 + \Pr(entry)(1 - c_I) \\ &= c_I\,(1 - c_I) \end{aligned}$$

- And the incumbent firm's payoff is

$$\begin{aligned} V_I &= p_1 - 1 + (1 - \Pr(entry))\,(1 - c_I) + \Pr(entry)\,(1 - c_I) \\ &= 1 - c_I + (1 - c_I)^2 \end{aligned}$$

- Now consider the case where the incumbent and the buyer sign a contract at $t = 1$ which specifies as price for each period and a penalty d for breach / termination
  - The contract is a triple (p_1, p_2, d)
- So the buyer will only breach the contract if the entrants price p_E is such that

$$1 - p_E \geq 1 - p_2 + d$$

i.e. surplus under the new contract compensates for the surplus under the old including damages

- The probability of entry given this contract is

$$\Pr(c_E < p_2 - d) = p_2 - d$$

- The buyer's expected payoff under the contract is

$$\begin{aligned} V_B^L &= (1 - p_1) + (1 - p_E) + d \\ &= (1 - p_1) + (1 - (p_2 - d)) + d \\ &= (1 - p_1) + (1 - p_2) \end{aligned}$$

- The incumbent's expected payoff is

$$\begin{aligned} V_I^C &= p_1 - c_I + (1 - \Pr(entry))(p_2 - c_I) + \Pr(entry)d \\ &= p_1 - c_I + (1 - p_2 + d)(p_2 - c_I) + (p_2 - d)\, d \end{aligned}$$

- The buyer will only accept the contract if

$$(1 - p_1) + (1 - p_2) \geq c_I (1 - c_I)$$

- So the incumbent solves

$$\max_{p_1, p_2, d} \{p_1 - c_I + (1 - p_2 + d)(p_2 - c_I) + (p_2 - d)\, d\}$$
$$\text{subject to}$$
$$(1 - p_1) + (1 - p_2) \geq c_I (1 - c_I)$$

i.e. maximize the payoff under the contract subject to the buyer being willing to accept

- The incumbent can always set $p_1 = 1$, so the problem is

$$\max_{p_2, d} \{1 - c_I + (1 - p_2 + d)(p_2 - c_I) + (p_2 - d)\, d\}$$
$$\text{subject to}$$
$$(1 - p_2) \geq c_I (1 - c_I)$$

- Noting that the constraint binds we have $1 - c_I (1 - c_I) = p_2$

- So the program is

$$\max_d \{1 - c_I + (1 - (1 - c_I (1 - c_I)) + d) ((1 - c_I (1 - c_I)) - c_I) + ((1 - c_I (1 - c_I)) - d) d\}$$

- The solution is

$$d^* = \frac{1 + (1 - c_I)(1 - 2c_I)}{2} > 0$$

- So the probability of entry is

$$p_2 - d^* = \frac{c_I}{2}$$

- The incumbent always wants to sign the contract
- This contract is competition reducing since the probability of entry is $\frac{c_I}{2}$ instead of c_I
- Markets with contracts may not be as efficient as spot contract markets!
- Robust to certain extensions
  - Renegotiation
  - Multiple buyers