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Optimal Gerrymandering: Sometimes Pack, But Never Crack

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Standard intuitions for optimal gerrymandering involve concentrating one's extreme opponents in "unwinnable" districts ("packing") and spreading one's supporters evenly over "winnable" districts ("cracking"). These intuitions come from models with either no uncertainty about voter preferences or only two voter types. In contrast, we characterize the solution to a problem in which a gerrymanderer observes a noisy signal of voter preferences from a continuous distribution and creates N districts of equal size to maximize the expected number of districts she wins. Under mild regularity conditions, we show that cracking is never optimal—one's most ardent supporters should be grouped together. Moreover, for sufficiently precise signals, the optimal solution involves creating a district that matches extreme "Republicans" with extreme "Democrats," and then continuing to match toward the center of the signal distribution. (*JEL* D72)

One of the more curious features of American democracy is that electoral boundaries are drawn by political parties. In order to ensure a notion of equal representation, the Constitution of the United States provides that "Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers."[1] Since populations change over time, the Constitution also provides a time frame according to which representation shall be adjusted—"… within every subsequent Term of ten Years, in such Manner as they shall by Law direct"—where "they" represents the states. In practice, this leaves the process of redistricting to state legislatures and governors.

History has shown that political parties act in their own interests; redistricting is no exception, and the advantages gained can be large. From Massachusetts's Elbridge Gerry in 1812 (after whom the term "Gerrymander" was coined) to the recent actions of Texas Representative Tom DeLay, American politicians have used the redistricting process to achieve partisan political ends. Most recently, the much publicized Republican redistricting in Texas in 2003 caused four Democratic congressmen to lose their seats and would have been even more extreme but for the Voting Rights Act, which effectively protected nine Democratic incumbents. Other particularly stark current examples include Florida, Michigan, and Pennsylvania—states that are evenly divided, but whose delegations to the 109th Congress collectively comprised 39 Republicans and 20 Democrats. Democrats are also familiar with the practice; although President George W. Bush won Arkansas by more than 10 points in 2004, the state's delegation to the 109th Congress,

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[1] Article I, section 2, clause 3.

bolstered by the Democratic state legislature's redistricting in 2001, contained three Democrats and one Republican.

Although gerrymandering using unequal district sizes is unlawful, partisan gerrymandering remains legal, though controversial. In *Davis v. Bandemer* (1986), the Supreme Court declared partisan gerrymandering inimical to norms of fair and equal representation; but the majority was unable enunciate a workable test for where redistricting stops and gerrymandering begins. Nearly two decades later, despite numerous attempts to find such a standard, four members of the court (Chief Justice Rehnquist and Justices O'Connor, Scalia, and Thomas) found in *Vieth v. Jubelirer* (2004) (a 4-1-4 decision) that the test laid down in *Bandemer* was not practicable, in that it gave no guidance to legislatures and lower courts, and, absent such a test, partisan redistricting was not justiciable.[2]

In the wake of this decision and the controversial Texas redistricting in 2003, there has been renewed interest in legislative reform to change the partisan nature of redistricting. Currently, two states, Iowa (since 1980) and Arizona (since 2000), include nonpartisan commissions in their decennial redistricting processes, but only Arizona completely excludes political bodies. More than 20 states have considered similar amendments in the past decade, though, and movements advocating such changes seem to be gaining momentum.

Recently, three states, California, Florida, and Ohio, held referenda that proposed that panels of retired judges take charge of the redistricting process. None of these passed. But despite the great impact of gerrymandering on the American political system and the surge of recent interest in reform, few authors have attempted to understand the basic incentives at work.

In this paper, we view the issue of redistricting through the lens of an economist concerned with the endogenous formation of political institutions. In particular, we frame the issue as a maximization problem by the gerrymanderer where the choice variables are the allocations of voters to districts. In contrast, most previous analyses model the problem as a trade-off between "biasedness"—the degree to which an evenly divided population would elect an uneven slate of legislators—and "responsiveness"—the sensitivity of the share of seats held by a party to the share of voters supportive of that party (Guillermo Owen and Bernard Grofman 1988; Katerina Sherstyuk 1998; Gary W. Cox and Jonathan N. Katz 2002). In these models, the gerrymanderer optimally concentrates those least likely to vote for her in districts that are "thrown away" or "packed," and spreads remaining voters evenly over the other districts, which are "smoothed" or "cracked." A major limitation of these models is that they are not micro-founded; the gerrymanderer chooses properties of the redistricting plan, as a whole, rather than the placement of voters into districts. Since there is no one-to-one mapping from these aggregate characteristics to individual district profiles, there is no guarantee that the solution from these models is actually optimal.

Thomas W. Gilligan and John G. Matsusaka (1999) take an alternative approach, instead analyzing a micro-founded model in which individuals with known party affiliations vote for those parties with probability one. Since one party wins a district comprising $n + 1$ of its supporters and n opponents with certainty, the optimal strategy is to make as many districts like this as possible. Indeed, if one party holds a bare majority of the population, then they win all districts! Though the assumptions of observability and deterministic voting simplify the analysis greatly, they clearly do so at some cost.

Kenneth W. Shotts (2002) considers the impact of majority-minority districting. He develops a model with a continuum of voters whose identities are perfectly known to the gerrymanderer, and imposes a constraint he calls the "minimum density constraint." This requires the gerrymanderer to put a positive measure of all voter types in each district. This is a reduced form way of

[2] "… the legacy of the plurality's test is one long record of puzzlement and consternation," Scalia J.

modelling the constraint that districts be contiguous and the fact that in practice the gerrymanderer receives a noisy signal of voter preferences.

We analyze a model in which there is a continuum of voter preferences, and where the gerrymanderer observes a noisy signal of these preferences. We show that the optimal strategy always involves concentrating one's most ardent supporters together. Intuitively, since district composition determines the median voter, smoothing districts makes inefficient use of extreme Republicans as right-of-the-median voters in many districts, rather than having them be the median in some districts. This contrasts with the "cracking" intuition, which calls for the creation of identical profiles among districts the gerrymanderer expects to win. When the signal a gerrymanderer receives is sufficiently precise, we obtain a sharper characterization. The optimal strategy creates districts by matching increasingly extreme blocks of voters from opposite tails of the signal distribution. Intuitively, extreme Democrats can be best neutralized by matching them with a slightly larger mass of extreme Republicans.

This analysis is a first step toward a more complete understanding of the phenomenon of gerrymandering. There are important issues this paper does not address. Most notably, we abstract from geographical considerations, such as the legal requirement of contiguity (see Section I below, however), as well a preference for compactness or the recognition of communities of interest. Second, we focus exclusively on partisan incentives, to the exclusion of the motivations of incumbents (i.e., incumbent gerrymandering). Finally, we do not model the constraints imposed by the Voting Rights Act. Of course, this does not mean that racial and partisan gerrymandering are distinct phenomena. Given that race is a component of the signal of voter preference observed by the gerrymanderer, there may be circumstances where they are essentially the same practice. Ultimately this is an empirical question, which depends on the joint distribution of voter preferences and voter characteristics. (These issues are further explored in Section VI).

The remainder of the paper is organized as follows, Section I details the legal and institutional backdrop against which redistricting takes place. In Section II we present some basic examples to illustrate the primary intuitions of the solution to our more general model, which we present in Section III along with comparative statics. Section IV reports the result of a number of numerical examples of the model in order to illustrate further the optimal strategy and its comparative statics. In Section V we explore a number of extensions to the basic model, including alternative partisan objective functions, the effects of gerrymandering on policy outcomes, candidate specific advantages, and uncertain voter turnout. Finally, Section VI contains some concluding remarks and suggests directions for future work.

I. Institutional Background[3]

The process of redistricting was politicized in America as early as 1740 (in favor of the Quaker minority in the colony of Pennsylvania). Until the landmark Supreme Court decision *Baker v. Carr* in 1962, the major legal constraint on gerrymandering was that districts be contiguous. Many states, particularly in the South, had not redrawn Congressional districts after each decennial Census. Since population growth was much greater in urban areas, this inertia served to dilute the urban vote—often poor and black—and enhance the political power of rural white voters who traditionally supported the Democratic Party. After the 1960 Census, the population disparities between congressional districts had become as great as 3 to 1 in Georgia (and as extreme as 1,000 to 1 for state legislature seats in some states). The decision in *Baker* declared that challenges to such districting plans were justiciable, and two years later the Court clarified its

[3] This section details the legal and political backdrop against which gerrymandering occurs today. Readers uninterested in, or already familiar with, this material may wish to skip directly to the analysis in Section II.

position on the standard for unlawful redistricting plans, stating in *Wesberry v. Sanders* that only congressional districts with populations "as nearly equal as possible" were acceptable under the Equal Protection clause.[4] Furthermore, federal district courts were empowered, as part of their remedial discretion, to draw district boundaries themselves should a state prove either unable or unwilling to produce a satisfactory plan.

Consensus over the practical implications of the Court's decisions solidified over the next 15 years. Though federal district courts initially experimented with strict upper bounds on the maximum population deviation across districts, by the late 1970s states were subject to a more flexible set of criteria, in which concerns such as the compactness of districts or the preservation of "communities of interest" justified small deviations in representation. As of 1980, though, contiguity and population equality across districts were the principle constraints on redistricting.

In the 1990s, debates around gerrymandering shifted to the issue of "race conscious" redistricting. While it had long been clear that intentional dilution of the voting strength of racial minorities violated the Equal Protection clause, it was less clear that states could draw boundaries such that racial minorities could elect their preferred candidates (Samuel Issacharoff, Pamela S. Karlan, and Richard H. Pildes 2002). In a number of cases, culminating in *Shaw v. Reno* (1993), the Court found that redistricting plans would be held to the same strict scrutiny with respect to race as other state actions. In practice, this means that, once plaintiffs demonstrate that racial concerns were a "predominant factor" in the design of a districting plan, the plan is illegal unless the state can justify the use of race and show that such factors were considered only when necessary. This places a heavy burden on the states. Some federal courts initially interpreted these decisions as requiring states to ensure minority representation through the creation of majority-minority districts, but the Supreme Court declared that this practice would violate Section 2 of the Voting Rights Act. In more recent cases, the Court has continued to downplay the importance of racial considerations; for instance, litigation surrounding the 1991 North Carolina redistricting ended when the Court ruled, in *Easley v. Cromartie* (2001), that partisan concerns, not racial concerns, "predominated" in the construction of the heavily black and Democratic 12th district, and thus the plan was legal.

The history of attempts to ban partisan gerrymandering have proven less successful still. In *Davis v. Bandemer*, the Supreme Court attempted to limit the impact of partisan concerns in redistricting processes by stating that such claims were, in theory, justiciable (though they did not decide the merits). Though the years following this decision saw many attempts to define the level and shape of such a standard, there was little agreement, and no claim of partisan gerrymandering ever succeeded. In *Vieth v. Jubelirer*, four members of the Court found that such attempts were doomed. While *Bandemer* is still good law, the future justiciability of partisan gerrymandering claims seems far from assured.

The current reality of political redistricting reflects the past 40 years of case history. States now use increasingly powerful computers to aid in the creation of districts, and, accordingly, *Baker*'s "as nearly equal as possible" population requirement is extremely strict. A Pennsylvania redistricting plan was struck down in 2002 for having one district with 19 more people than another without justification! On the other hand, the law does allow for some slight deviations, provided there is adequate justification. In Iowa, for instance, congressional districts must comprise whole counties; the current maximum population deviation of the Iowa redistricting plan is 131 people, but the legislature rejected an earlier plan with a 483-person deviation. Such cases are not common, though. The current Texas districting plan is more representative and has,

[4] See *Wesberry v. Sanders* 376 US 1 (1964). The court applied a similar standard to districts for statewide legislative bodies in *Reynolds v. Sims* 377 US 533 (1964), and for general purpose local governments in *Avery v. Midland County* 390 US 474 (1968).

to integer rounding, equal population in each district.

As previously mentioned, districts must be contiguous. This requirement first appears in the Apportionment Act of 1842, though it was standard long before then. While technology has tightened the population equality constraint, computers have effectively loosened the contiguity requirement, as legislators can now draw districts more finely than ever before. In the 1970s, districting plans were laborious to create and difficult to change, as each required hours of drawing on large floor-maps using dry-erase markers; now lawmakers use Census TIGERLine files to create and analyze many alternative districting schemes both quickly and accurately. Contiguity has been stretched to the limit in such recent cases. Florida's 19th, 22nd, and 23rd districts, shown in Figure 1, are one such case. The 22nd comprises a coastal strip not more than several hundred meters wide in some places but 90 miles long, while tentacles from the 19th and 23rd intertwine to divide the voters of West Palm Beach and Fort Lauderdale. Even more striking is the shape of the Illinois 4th (shown in Figure 2), drawn to include large Hispanic neighborhoods in the North and South of Chicago but not much in between. Each of these districts is, in some places, no more than one city block wide, and such necks are often narrower than 50 meters.

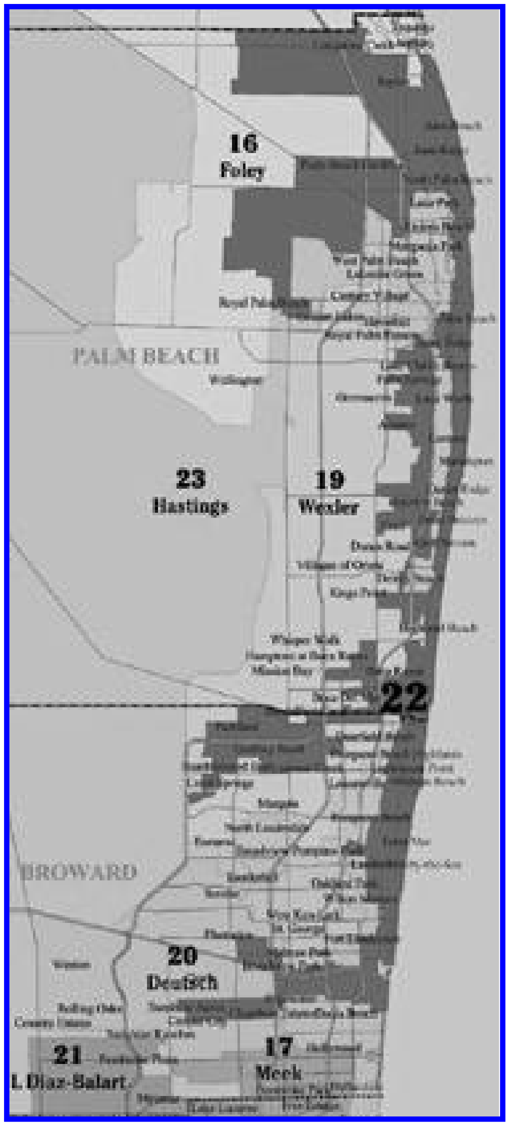

Figure 1.
Florida 16th to 23rd Congressional Districts

State law governs procedures for redrawing district boundaries. In most states, redistricting plans are standard laws, proposed by the members of the legislature and subject to approval by the legislatures and the governor. Arizona and Iowa delegate redistricting to independent commissions, though in Iowa legislators must still approve the plan and may edit proposed schemes after several have been rejected. In 2001, for instance, the legislature rejected the first proposed plan along partisan lines.[5] Arizona and Iowa also instruct their redistricting commissions to make districts "compact," respect the boundaries of existing "communities of interest," and use geographic features and existing political boundaries to delineate districts "to the extent practicable." Finally, Arizona mandates that "competitive districts should be favored where to do so would create no significant detriment" to other objectives.[6] No other states have explicitly defined redistricting goals along these lines.

[5] "Senate Rejects Districts," *Des Moines Telegraph Herald*, May 3, 2001.
[6] See Arizona Proposition 106, and 1981 Iowa Acts, 2nd Extraordinary Session, Ch. 1.

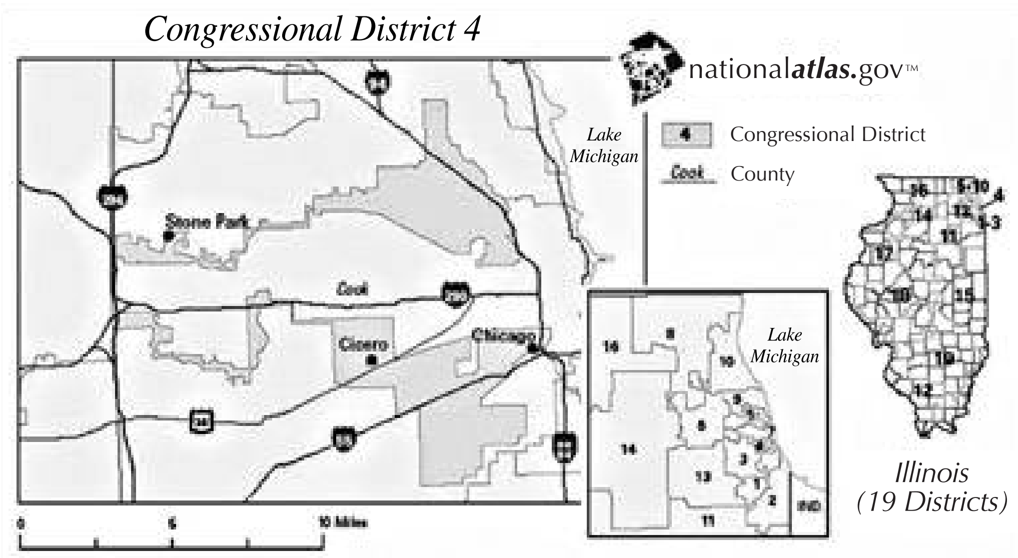

FIGURE 2. ILLINOIS 4TH CONGRESSIONAL DISTRICT

There are three key messages to understand from the backdrop against which gerrymandering takes place. First, contiguity may well not be a binding constraint because of the fine lines gerrymanderers use to create districts. Second, other spatial/geographic concerns such as compactness and communities of interest have found little legal traction. As such, they are really not constraints on gerrymanderers. Third, the Supreme Court has consistently considered partisan and racial gerrymandering to be analytically distinct—*Cromartie* even going so far as to allow racial gerrymandering if it is not deemed the predominant motive. The first two of these points suggest that spatial/geographic considerations are not first-order concerns. Accordingly, our model omits them. The third rests on the premise that signals of voting propensity and race are sufficiently uncorrelated that an optimal gerrymandering strategy does not conflate the two issues. This is a point to which we return later in the paper.

II. Some Simple Examples

In order to illustrate the intuition behind the theory in this paper, we now provide some very simple examples that capture the basic features of the more general model in Section III. In these examples, for simplicity, voters have single-peaked preferences. In the general model, voter preferences satisfy single-crossing—an arguably less restrictive condition. For instance, when voters have a convex loss function over the distance of their bliss point from the actual policy, then single-peakedness implies that single-crossing is satisfied.

A. *Example 1*

Consider the problem faced by a gerrymanderer in a state in which a population of voters has single-peaked preferences that are symmetric about a policy β, within a one-dimensional policy space. We assume that each voter has bliss point β, and that, across the population, β is distributed uniformly on $[-1,1]$. These assumptions imply that, in a two-party election, each voter supports the candidate located closest to her on the ideological spectrum. To begin, we assume that the gerrymanderer can directly observe β for each voter. We assume that all candidates—the right-wing "Republican" candidates and the left-wing "Democrats"—locate symmetrically about zero, and so the percent of votes captured by the Republican candidate in any election is simply the proportion of voters to the right of zero.

The gerrymanderer—suppose she is a Republican—must break up the population into equal-sized districts in which different elections take place with the goal of maximizing the expected number of seats won by her party. Since we abstract from geographic concerns here, the gerrymanderer can match any pieces of the population into a district. Suppose, for simplicity, that the gerrymanderer must form two districts, so that each district must comprise a one-half mass of voters. Since all voters for whom $\beta \geq 0$ support the Republican candidate with certainty, Republicans win all districts containing one-quater or more mass of such voters.[7] From Gilligan and Matsusaka (1999), the optimal gerrymander makes exactly half of the voters in each district have preferences $\beta \geq 0$; in this basic setup, Republicans win each district with certainty. It does not matter which right-wing voters go into each district.

B. *Example 2*

We now add some noise to the preferences in Example 1. Suppose that, after candidates are positioned, an aggregate preference shock A affects the population so that preferences are now single-peaked about $\hat{\beta} = \beta - \mathrm{A}$. The gerrymanderer observes only β and not A or $\hat{\beta}$. Suppose that A is distributed uniformly on $[-1,1]$. While voters for whom $\beta > 0$ now vote for the right-wing candidate in expectation, only those for whom $\beta = 1$ support the Republican candidate with certainty; a voter with $\beta = 0.5$, for instance, prefers the right-wing candidate only if $A < 0.5$, which happens 75 percent of the time.

In this example we can make a sharper prediction about the form of the optimal gerrymander. Half of the voters in each district should have $\beta > 0$, but it now matters which of these voters go into which district. The optimal gerrymander groups all extreme voters for whom $\beta \in [0.5,1]$ into one district (denoted as District 1) and more moderate right wingers with $\beta \in [0,0.5]$ into District 2. These blocks of right-wing voters are then grouped with any mass of voters for whom $\beta < 0$; since the preference of the median voter in each district ($\mu_1 = 0.5$ in District 1 and $\mu_2 = 0$ in District 2) is already determined, the composition of the left-wing voters does not matter. Republican candidates now win District 1 with probability 0.75 and District 2 with probability 0.5. Any other distribution of right-wing voters between the two districts (with one-quarter mass to each) would dilute the power of the extreme right-wing voters by wasting some in District 2, since that median voter would still have $\beta = 0$ while the preferences of the median voter of District 1 would fall. Only by concentrating the most extreme right-wing voters together can the gerrymanderer make the most effective use of her supporters.

C. *Example 3*

Finally, suppose (in addition to the setup in the second example) that individual preferences are measured with noise by the political parties. That is, let the gerrymanderer observe only s, a signal of preferences, instead of β itself. Across the population, let s be distributed uniformly on $[-1,1]$, and let $\beta|s$ be distributed uniformly on $[s-0.5, s+0.5]$, with an independent draw

[7] For the sake of simplicity, we resolve all "ties" in this example in favor of the Republican candidate. Voters with $\beta = 0$ support the right-wing candidate, and if the candidates have equal vote shares, the Republican wins.

of β for each voter with a given signal s. Suppose the gerrymanderer creates districts as above (grouping voters for whom $s \in [0.5, 1]$ into District 1 and $s \in [0, 0.5]$ into District 2), and, furthermore, groups the most extreme left-wing voters into District 1 and the others in District 2. Because the measurement of preferences is noisy, the median voter in District 1 falls to $\mu_1 = 0$; the Republicans gain no advantage over proportional representation. Intuitively, the Republicans are "cutting it too close" in District 1. Although District 1 contains the most extreme right-wing voters, there are only one-quarter mass of them, and so the most left-wing voter with a right-wing signal is the median voter. Since some of those right-wing voters end up with more moderate preferences than their signal suggested, the median voter in the district is a moderate.

Instead, consider a gerrymander who groups all voters with $s \in [p, 1]$ into District 1 and $s \in [0, p]$ into District 2. Because of the intuition developed in the second example, this districting scheme still keeps the most extreme right-wing voters together. Now, though, the Republicans have more than just a bare majority of supporters in District 1, reducing the problem caused by preference mismeasurement above.

To complete this optimal districting, the gerrymanderer must allocate the left-wing voters. Her problem here is exactly opposite that faced with the right-wing voters: she must decide how best to *neutralize* the voting power of the extreme left-wingers. The key to this problem is that, since the majority of District 2 voters are left-wingers (assuming $p < 1/2$), μ_2 is far more sensitive to the allocation of these voters than μ_1. Thus, the optimal gerrymanderer should concentrate those least likely to vote for the Republican candidate into District 1, where they affect the median voter least.

Combining these insights, consider a districting plan such that voters for whom $s \in [-1, -1 + p] \cup [p, 1]$ make up District 1 and the rest are placed in District 2. The particular distributional assumptions made above imply that

$$\mu_1 = p + \sqrt{1 - 2p} - \frac{1}{2} \text{ and } \mu_2 = p - \frac{1}{2}.$$

The optimal gerrymander sets $p^* = 3/8$; Republican candidates win $1\frac{1}{8}$ seats in expectation. By including more right-wingers in District 1, μ_1 becomes less sensitive to the mismeasurement of preferences, and thus increases quite a bit, while μ_2, which depended less on the precision of the signal, does not decrease by as much. Furthermore, the right-wing voters of District 1 determine that $\mu_1 = \sqrt{1/4} - 1/8 = 3/8$, and so the inclusion of the most extreme left-wingers has no effect. If, for instance, the gerrymanderer had included these least favorable voters into District 2 and placed voters with $s \in [-1 + p, -1 + 2p]$ into District 1, μ_2 would fall while μ_1 would not change.

These three simple examples illustrate how key features of an optimal partisan gerrymander differ from the standard "throwing away" and "smoothing" intuitions. First, it is not best to "smooth" extreme and moderate right-wing voters across many districts; rather, one should concentrate the most extreme right-wingers into a single district in order to not waste them all as right-of-median voters. Second, it is not efficient to "pack" those least likely to vote for one's candidate into a district that is "thrown away"; instead, these extreme left-winger voters are best countered by matching them with a greater number of extreme right-wingers.

We now turn to our model, which provides a more general characterization of the optimal partisan gerrymander, but the intuitions brought out in our examples are still prominent. Indeed, under certain regularity conditions, the optimal districting scheme has exactly the same form as in the final example above, matching increasingly extreme slices of voters from opposite sides of the signal distribution for the population.

III. The Model

A. *Overview*

There are two parties, D and R, which can be interpreted as the Democratic Party and the Republican Party. One of these parties (without loss of generality, we assume it to be R) is the gerrymanderer and creates districts. There is a unit mass of voters with preferences over a one-dimensional policy space. The gerrymanderer does not observe a voter's preferences, but, instead, receives a noisy signal of them. Also, she observes the posterior distribution of policy preferences conditional on the signal. We will sometimes refer to the marginal distribution of the signal as the "signal distribution." Thus, her problem is to create N voting districts by allocating voters from the signal distribution. Her objective is to maximize the expected number of districts won. The probability that each party wins a district is determined by the median voter in that district. The only constraints we place on the gerrymanderer are that: (a) each voter must be allocated to one and only one district; and (b) all districts must contain an identical mass of voters.

B. *Statement of the Problem*

There is a unit mass of voters who differ in their political preference over two candidates who locate on the real line such that $D < R$. We assume that this location happens prior to observing any signals about voters preferences. Denote the payoff to voter i of candidate x being elected as $u_i(x)$.

DEFINITION 1: *Voter preferences satisfy Single-Crossing if, for any two voters i and j such that $i < j$ and any two candidate locations $D < R$, the following hold: (i) $u_j(D) > u_j(R) \Rightarrow u_i(D) > u_i(R)$ and (ii) $u_i(R) > u_i(D) \Rightarrow u_j(R) > u_j(D)$.*

We assume voters have preferences satisfying single-crossing. Let $\beta_i = u_i(R) - u_i(D)$, for each voter type $i \in \mathbb{R}$. Without loss of generality, we reorder the voters so that β is monotonic. From this point on, the indexing of voters will reflect this reordering.[8]

These preferences are not observed by the gerrymanderer, who instead receives a noisy signal, $s \in \mathbb{R}$. Let the joint distribution of β and s be given by $F(\beta,s)$, which is assumed to have full support on $\mathbb{R}^2$. Let player R be the gerrymanderer. Let R have a Bayesian posterior $\mathrm{G}(\beta|s)$ for the distribution of preferences given an observed signal. We refer to this distribution as the "conditional preference" distribution. We assume that both F and G are absolutely continuous. Define the marginal distribution of s as

$$h(s) = \int f(\beta,s)\,d\beta.$$

Since there is a continuum of voters, we can interpret h not only as characterizing a single draw from the population of voters, but also the mass of voters in the population. We refer to h as the "signal distribution." R allocates mass from this distribution in order to form districts. Normalize the median of s in the population to zero.

Since preferences satisfy single-crossing, the median voter determines a Condorcet winner (Paul Rothstein 1991). As a reduced form representation of electoral uncertainty, we assume that, in each election, *after* R observes the signal s, there is an aggregate shock decreasing all

[8] In the Appendix, we offer a result, which is of independent interest, that under single-crossing preferences the probability that a voter votes Republican is increasing in her type.

preferences by A. Thus, if the median voter in district n has preferences such that $\beta = \mu_n$, she votes for the Republican candidate if and only if $A \leq \mu_n$, which occurs with probability $B(\mu_n)$, where $B(\cdot)$ denotes the c.d.f of A. We assume that A can take any value in $\mathbb{R}$ with positive probability, so that B is strictly increasing.[9] One can also think of A as an "electoral breakpoint" such that voters positioned above (to the right) of the realization of the breakpoint vote for the Republican candidate, while those on the left vote democratic. Importantly, once the breakpoint is determined, all uncertainty is resolved and the position of voters relative to A determines for whom they vote with certainty. The uncertainty about whom a particular voter will vote for comes from the fact that A is stochastic.

Our assumptions about the location of candidates imply: (a) that all candidates of a given party and state locate in the same place; and (b) that this location takes place before receiving signals of voter preferences. In essence, these assumptions imply that there is nothing "local" about an election. Though perhaps counterintuitive, research suggests that this may not be far from the truth. Stephen Ansolabehere, James M. Snyder, and Charles Stewart III (2001) argue that, while district-to-district competition may exert some influence on the candidate platforms, the effect is "minor compared to the weight of the national parties." Allowing for state-to-state differences would surely leave even less variation in local platforms. Similarly, David S. Lee, Enrico Moretti, and Matthew J. Butler (2004) demonstrate that exogenous shifts in electoral preferences do not affect the menu of candidates offered to voters, perhaps because politicians have no way to credibly commit to campaign promises. We discuss the effects of certain departures from this assumption in Section V.

R divides the population into N equal-sized districts to maximize the expected number of seats won in the election. Let $\psi_n(s)$ denote the mass of voters from the population placed in district n. Formally, R solves the program

$$(1) \qquad \max_{\{\psi_n(s)\}_{n=1}^{N}} \left\{ \frac{1}{N} \sum_{n=1}^{N} B(\mu_n) \right\}$$

$$\text{s.t.} \int_{-\infty}^{\infty} \psi_n(s)\, ds = \frac{1}{N}, \forall n \qquad \sum_{n=1}^{N} \psi_n(s) = h(s), \forall s \qquad 0 \leq \psi_n(s) \leq h(s), \forall n, s,$$

where

$$(2) \qquad \mu_n = \hat{\beta} \qquad \text{s.t.} \int_{-\infty}^{\infty} G(\hat{\beta}|s)\psi_n(s)\, ds \equiv \Gamma_n(\hat{\beta}) = \frac{1}{2N}.$$

It will be useful to define the following for notational purposes:

$$(3) \qquad \gamma_n(\beta) = \frac{\partial \Gamma_n(\beta)}{\partial \beta}.$$

Given a district profile $\psi_n(s)$, equation (2) determines μ_n with certainty. Though R could not identify any single voter as the median voter in a district, there is nothing stochastic about the preference parameter of the median voter.[10]

[9] This implies that the shock is independent of voter type. It may be the case that more "extreme" types are less affected by such shocks. This could be explored in future work.

[10] This model structure is isomorphic to the inclusion of further levels of uncertainty between signals and preferences. For instance, suppose that the gerrymanderer believed that, with 50 percent probability, preferences had a conditional distribution $G_1(\beta|s)$, and otherwise they were conditionally distributed as $G_2(\beta|s)$. Equation (2) would then

C. *Characterization of the Optimum*

No Cracking.—In order to analyze the problem, it is necessary to place some structure on the conditional distribution of preferences. The first restriction we require is that the signal be informative in the following sense.

CONDITION 1 (Informative Signal Property): *Let* $\partial G(\beta|s)/\partial s = z(\beta|s)$. *Then,*

$$\frac{z(\beta|s')}{z(\beta|s)} < \frac{z(\beta'|s')}{z(\beta'|s)}, \forall s' > s, \beta' > \beta.$$

This property is similar to the Monotone Likelihood Ratio Property (MLRP) due to Samuel Karlin and Herman Rubin (1956) (see also Paul R. Milgrom 1981). In fact, if a higher signal simply shifts the mean of the conditional preference distribution, then this property is equivalent to MLRP.[11] When this is the case, the condition essentially states that higher and higher signals (more right-wing) are more and more likely to come from voters who have underlying preferences that are farther to the right. Many common distributions satisfy it, including: the normal, exponential, uniform, chi-square, Poisson, binomial, noncentral t, and noncentral *F*. If a higher signal also changes the shape of the conditional distribution, then this property, like MLRP, becomes less intuitive. Condition 1 does, in general, imply first-order stochastic dominance,[12] and as such rules out cases where observing a higher signal makes *both* the probability of the voter being extreme left-wing *and* the probability of being extreme right-wing increase.

The second condition we require is a form of unimodality.

CONDITION 2 (Central Unimodality): $g(\beta|s)$ *is a unimodal distribution where the mode lies at the median.*

Also note that, without loss of generality, we can "rescale" s such that $s = \max_\beta g(\beta|s)$. Though many distributions that satisfy Condition 1 are unimodal, some are not, and we rule these out. Furthermore, Condition 1 implies that the mode of $g(\beta|s)$ must lie below the mode of $g(\beta|s')$ if $s < s'$. We can thus "relabel" the signals such that the mode of $g(\beta|s)$ lies at s. The two properties in Condition 2, taken together, intuitively imply that, conditional on signal s, preferences are distributed "near" s and not elsewhere.

Step 1: Slicing

LEMMA 1: *Suppose Condition 1 holds, and consider two districts, i and j, such that* $\mu_i < \mu_j$. *Consider any two voter types,* $s_1', s_2' \in \psi_i$ *(i.e., in district i). Then, any districting plan such that* $s \in \psi_j$ *for any* $s \in [s_1', s_2']$ *cannot be optimal, except perhaps on a set of measure zero.*

become $\int_{-\infty}^{\infty} \frac{1}{2}[G_1(\mu_n|s) + G_2(\mu_n|s)]\psi_n(s)\, ds = \frac{1}{2N}$, which is isomorphic to our original problem, if instead $G(\beta|s) = \frac{1}{2}[G_1(\mu_n|s) + G_2(\mu_n|s)]$.

[11] To see this, note that, if changing s shifts only the mean of the conditional preference distribution, then $G(\beta|s) = G(\beta|(s' - s))$. Therefore, $z(\beta|s) = -g(\beta|s)$, and hence Condition 1, imply MLRP.

[12] MLRP always implies this as well.

Lemma 1 shows that we can restrict attention, without loss of generality, to a much smaller strategy space. Districts are constructed from vertical slices of h—either whole slices (as in districts 1, 2, and 3 in the figure below), or a slice shared between districts that have the same median ("parfaits") (as in districts 4 and 5). Furthermore, in the optimal gerrymander, the voters in higher-median districts must lie outside—that is, have more extreme preferences—those in lower-median districts. The intuitions here are very similar to those discussed in the examples above. Extreme right-wing voters should be concentrated to maximize their voting strength—that is, the optimal districting scheme places an unbroken mass of voters with higher signals into the higher-median district rather than alternate smaller slices into all districts.

Step 2: No Parfaits

LEMMA 2: *Suppose that Conditions 1 and 2 hold. If $j \neq i$, then $\mu_j \neq \mu_i$.*

This penultimate step rules out parfaits, as defined above. Parfaits appeared stable above because the split equated both the medians and the sensitivity of the median to changes across the two districts. But this is not so. One can reallocate mass between two such districts to maintain the equality of medians and make one district more sensitive to change than the other. Then, a profitable deviation exists which lowers the less sensitive median by some but increases the other by more. Hence, parfaits cannot be optimal.

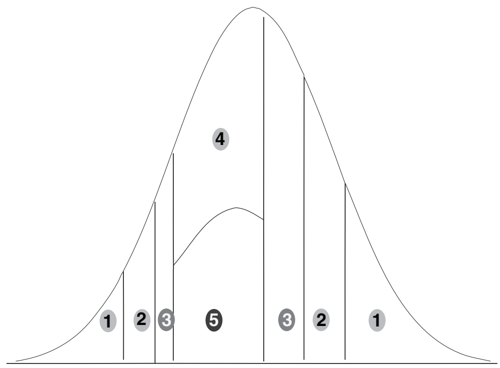

FIGURE 3. SLICES AND PARFAITS

Once again, the driving intuition in this case in that of concentrating extreme voters together to maximize their electoral power. In a way, parfaits are the least efficient use of extreme voters, and so it cannot be surprising that they are not optimal. Thus, the optimal gerrymander must contain *only* vertical slices of the signal distribution h that do not violate the ordering restriction from Lemma 1.

Step 3: No Intermediate Slices

LEMMA 3: *Suppose Condition 1 holds and consider three districts j, i, and k such that $\mu_j > \mu_i > \mu_k$. Now, fix $h(s)$ and N. Then, for a sufficiently precise signal, there does not exist a voter type $s^* \in \psi_j$ such that $s' > s^* > s''$ where $s' \in \psi_i$ and $s'' \in \psi_k$, except perhaps on a set of measure zero.*

This final step expands Lemma 2 by showing that voters in a higher-median district cannot lie within the set of all voters in lower-median districts. That is, by ruling out cases like that in Figure 4, it shows that optimal districts must comprise either a single slice or two slices matching mass from opposite tails of the distribution. The intuition is very similar to that of Lemma 2, that lower medians (such as those

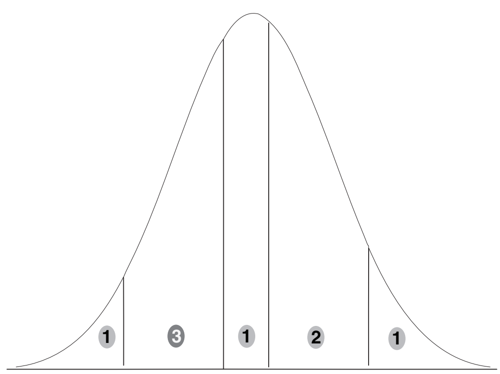

FIGURE 4.
AN EXAMPLE OF A STRATEGY RULED OUT BY LEMMA 3

in Districts 2 and 3 in Figure 4) are more positively affected by the inclusion of moderate instead of extreme left-wing voters. On the other hand, the higher medians (such as that of District 1) are hardly lowered by the substitution of extreme left-wingers. In order for these arguments to hold, though, the signal distribution must have high enough quality. If it does not, then intermediate slices are possible.

PROPOSITION 1: *Suppose that Conditions 1 and 2 hold, and that the signal distribution is of sufficiently high quality (as defined in Lemma 3). Consider a districting plan with N districts labelled such that $\mu_j > \mu_i$ if and only if $j < i$. This plan is optimal if and only if it can be characterized by "breakpoints" $\{u_n\}_{n=1}^{N-1}$ and $\{l_n\}_{n=1}^{N-1}$ (ordered such that $u_1 > u_2 > \cdots > u_{N-1} > l_{N-1} \geq l_{N-2} \geq \cdots l_1 \geq -\infty$) such that*

$$\psi_1 = \begin{cases} h(s) & \text{if } s < l_1 \text{ or } s > u_1 \\ 0 & \text{otherwise} \end{cases},$$

$$\psi_n = \begin{cases} h(s) & \text{if } l_{n-1} < s < l_n \text{ or } u_{n-1} > s > u_n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 < n < N,$$

and

$$\psi_N = \begin{cases} h(s) & \text{if } s > l_{N-1} \text{ or } s < u_{N-1} \\ 0 & \text{otherwise} \end{cases}.$$

At this point, we have established that cracking is not optimal, although some form of packing may still be. That is, we have not yet ruled out the type of strategy depicted in Figure 5. We will now provide conditions under which packing is not optimal—and show that matching of extreme supporters with extreme opponents is.

No Packing.—We now offer a result which shows that if the signal quality is sufficiently high, the optimal strategy cannot involve packing, by which we mean concentrating one's most ardent opponents into a single district—a notion we immediately make precise.

PROPOSITION 2: *Suppose Conditions 1 and 2 hold and the signal is of sufficiently high quality. Then, there exists n, and $s < s'$, such that $\mu_n > \mu_N$ and $s \in \psi_n$, $s' \in \psi_N$.*

To understand the intuition for this result, first consider a potential deviation from a districting plan that "packs," as in Figure 5: *R* could take the most left-wing voters from District 3 into District 1, and then "slide" Districts 2 and 3 to the right, thereby gaining in Districts 2 and 3 but losing ground in District 1. Now, consider how this strategy changes in value as we remove noise from the signal. As the signal becomes more precise, the cost of the proposed change in District 1 decreases, since the voters *R* removes from District 1 are less likely to be actually right-of-median. (The voters *R*

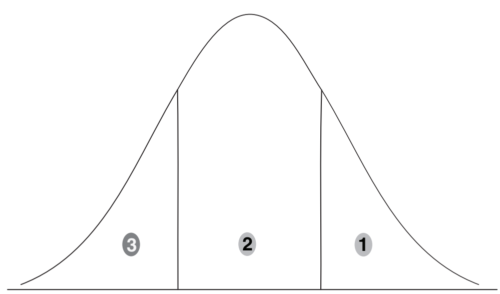

Figure 5. Proposition 2 Rules Out This Strategy

adds can be so far to the left that they are *always* left-of-median.) But the gains in Districts 2 and 3 stay roughly constant, since the entire districts are sliding to the right. At some point, when the signal is precise enough, the steady gains must begin to outweigh the shrinking loss. In the limit, as the signal becomes perfect, there is no cost to R in District 1 from this deviation, and R seeks to match an infinitesimally larger slice of right-wing voters with left-wing voters in each district, as in Example 2 in Section II.

Figure 6 is an example of a potentially optimal strategy. District 1 comprises a slice of extreme Republicans and a slice of extreme Democrats, and this slicing proceeds toward the center of the signal distribution. The slices from the right tail of the signal distribution contain more mass than the matched slice from the left tail, lest Republicans "cut it too close" in accounting for the noisy measurement of preferences. This follows the intuition developed in the third example in Section II.

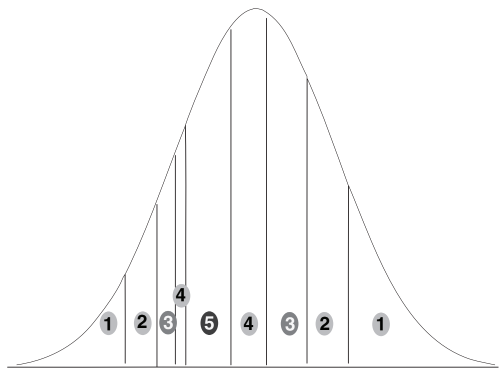

FIGURE 6. AN EXAMPLE OF THE OPTIMAL STRATEGY

We are unable to offer an analytical solution for the "breakpoints" $\{u_i\}_{n=1}^{N-1}$ and $\{l_i\}_{n=1}^{N-1}$. However, they are easily computed numerically, given a signal distribution (as Section IV demonstrates). We also conjecture that as the spread of the noise distribution increases, the ratio of mass in upper slices to lower slices increases—limiting to the case where districts are comprised of whole slices, rather than matching ones. This is certainly the case in a wide variety of numerical examples we have explored, and we are yet to find a counterexample. It does, however, remain a conjecture.

D. *Comparison with Received Literature*

Previous work has considered two types of models which are both special cases of our model. The approach most similar to ours is that of Gilligan and Matsusaka (1999), in which voters always vote for a given party and their preferences are known with certainty to the gerrymanderer. Our model simplifies to this case (as shown in the first example in Section II) if the conditional preference distribution limits to a point-mass at the true preference (so that preferences are observable) and if the breakpoint distribution $B(\cdot)$ is a point mass (so that voters are either Democrats or Republicans). As such, our model is more general and captures an important intuition—that more noise leads the gerrymanderer to create a larger buffer. Furthermore, our model has a continuum of preferences, and therefore is instructive not only as to the optimal number of Republicans and Democrats in a district, but also as to which types of Republicans and Democrats should be combined.

The second approach to modelling gerrymandering—one perhaps more popular than that of Gilligan and Matsusaka—is a binary signal model with noise. In such a model (e.g., Owen and Grofman 1988), the optimal strategy involves "packing" some districts and "cracking" others. Owen and Grofman refer to this as a "bipartisan gerrymander," since there are Democratic districts (those thrown away) and Republican districts (the others). For instance, if 60 percent of the population have signal r and 40 percent signal d, then the optimal strategy involves creating a certain number of districts that contain only those with signal d, and spreading the r voters uniformly over the remaining districts. This result is also a special case of our model, with additional assumptions, as shown in Proposition 3.

PROPOSITION 3: *Suppose $s \in \{d, r\}$ and that Conditions 1 and 2 hold. Suppose further that $B(\cdot)$ is unimodal, with mode greater than d and less than r. Then, the optimal gerrymander involves creating some districts with all voters of type d, and others with a constant proportion of r and d, and possibly one "odd district" with a nonzero but less-than-half proportion of r (from integer rounding problems). When $N \rightarrow \infty$, the optimal solution is a pure "bipartisan gerrymander."*

Thus, our model nests the solution of "bipartisan gerrymandering," but the conclusions of such a model are very sensitive to several extreme assumptions. Furthermore, the intuitions this special case highlights are very misleading. For instance, suppose that there are three signals: *r*, *d*, and *i* (Independents). As Proposition 2 shows, the optimal strategy matches increasingly extreme segments from the right and left tails (in this case Republicans and Democrats) into the same districts. The district where Republicans have the lowest chance of winning is not one that contains many Democrats, but rather one that contains many Independents. That is, these least Republican districts contain voters from the middle of the signal distribution, not the extreme left tail. It is also clear that "smoothing" is not a robust intuition. It is true only in the special case of a binary signal, because there is no heterogeneity among potential Republican voters.

E. *Comparative Statics*

In this subsection, we consider how the value of being the gerrymanderer responds to changes in the underlying distribution of voter preferences and signals. We also consider how this value changes as the number of districts to be created changes.

Our first comparative static shows that more precise signals are always better for the gerrymanderer.

DEFINITION 2: *Consider two conditional preference distributions g and g'. The distribution g provides a More Precise signal than g' if there exists a conditional distribution $c(s'|s)$ such that*

$$\int g(\beta|s')c(s'|s)\,ds' = g(\beta|s).$$

PROPOSITION 4: *The expected number of districts won by the gerrymanderer is increasing in the precision of the signal.*

This result shows that the gerrymanderer wins more districts in expectation as the signal received becomes more precise. Intuitively, as the gerrymanderer receives a better signal, the need for a large "buffer" of voters in a district declines. Instead, she can construct districts of a given median with a smaller proportion of voters from the right hand tail, leaving more right-wingers for other districts. Mathematically, the gerrymanderer could always lower the quality of the signal, while the reverse operation is not possible. Thus, it cannot be that a lower quality signal is better.

Our second comparative static result shows that the gerrymanderer does better as the distribution of voters becomes more spread out.

PROPOSITION 5: *Consider two joint distributions $F(\beta, s)$ and $\hat{F}(\beta, s)$, with marginal distributions of β given by $F(\beta)$ and $\hat{F}(\beta)$, such that $\hat{F}(\beta)$ is a symmetric spread of $F(\beta)$. Then, the expected number of districts won by the gerrymanderer is higher for $\hat{F}$ than for F.*

Intuitively, suppose that all signals have the same variance of preferences conditional on the signal. But, if the breakpoint is more likely to be near the center of the preference distribution, there is less uncertainty as to the voting patterns of extreme voters. For instance, suppose the breakpoint is normally distributed. If a voter has either $\beta = -0.5$ or $\beta = 0.5$, she will vote Republican either 31 percent or 69 percent of the time, quite a bit of uncertainty; but if a voter has either $\beta = 1.5$ or $\beta = 2.5$, she will vote Republican either 93 percent or 99 percent of the time. Extreme voters are thus more valuable to the gerrymanderer. Since an increase in the variance of the voter preference distribution increases the share of extreme voters in the population, the expected number of seats won increases.

The final comparative static concerns the number of districts.

PROPOSITION 6: *Suppose that the number of districts increases by an integer multiple (that is, doubles or triples). Then, the expected percentage of districts won by the gerrymanderer strictly increases.*

In previous analyses in this literature, proportional increases in the number of districts has little import; if twice the number of districts are required, the existing districts are split into equal parts, and so the voter profiles of the districts do not change. Our model implies that such parfaits are inefficient. Instead, the gerrymanderer can do better by slicing within previous districts, grouping the most and least Republican voters from an old district into one new district, and giving the all less extreme voters to the other.

IV. Numerical Examples

In order to illustrate the characterization of the optimal gerrymandering strategy and its comparative statics, we report the results of a number of numerical examples in this section. The examples all assume that there are five districts and that the gerrymanderer is Republican. In these examples, we assume that the joint distribution of preferences and signals, $F(\beta, s)$, is multivariate normal with parameters $\mu_\beta = \mu_s = 0$ and covariance matrix $\sum$, with

$$\sum = \begin{pmatrix} \sigma_\beta^2 & \rho\sigma_\beta\sigma_s \\ \rho\sigma_\beta\sigma_s & \sigma_s^2 \end{pmatrix}.$$

This implies that both the signal distribution and the conditional preference distribution are themselves normal. Note that this assumption satisfies Conditions 1 and 2. In this base case, we assume a distribution of $F(\beta, s)$ such that $\beta \sim N(0,5)$ and $\rho = 0.5$. Furthermore, we assume that $\sigma_s = \rho\sigma_\beta$ so that $G(\beta|s) \sim N\big(s, \sigma_{\beta|s}^2 = \sigma_\beta^2(1-\rho)\big)$. In all examples, we let $B \sim N(0,1)$ and set $N = 5$. Note that these assumptions imply that, nominally, half the voters are Republicans and half are Democrats—without gerrymandering, each party would win 2.5 seats, in expectation.

Panel A of Table 1 highlights a number of features of the optimal strategy. First, the highest median district (District 1) consists of 62 percent from a slice from the right tail of the distribution and 38 percent from a slice from the left tail. These upper slices get progressively larger for the lower median districts. While District 4 comprises a whole slice, Districts 1 through 3 are formed by matching slices from the right and left tails. (District 5 consists of a whole slice containing those voters remaining after removing the first four districts from the signal distribution, and so the fraction in the upper and lower slice is not relevant.) Second, note that the probability of winning District 1 is very high—87.5 percent. This means that those in the left-most part of the distribution have very little chance of gaining representation. Third, no districts are "thrown

TABLE 1—NUMERICAL EXAMPLES OF OPTIMAL GERRYMANDERING

Panel A. Baseline example

			District				
			1	2	3	4	5
		Upper Slice	0.62	0.73	0.91	1	n/a
		Lower Slice	0.38	0.27	0.09	0	n/a
		Prob (win)	87.5%	74.8%	65.7%	41.7%	13.7%

Panel B. Signal coarseness

			Probability of winning district				
Signal variance	E[Districts won]		1	2	3	4	5
0.50	3.46		97.4%	86.9%	74.3%	56.6%	30.9%
2.50	2.83		87.5%	74.8%	65.7%	41.7%	13.7%
4.50	2.53		68.2%	61.9%	55.7%	41.8%	25.9%

Panel C. Spread of voter preferences

			Probability of winning district				
Preference variance	E[Districts won]		1	2	3	4	5
3.0	2.55		71.0%	62.3%	55.6%	41.2%	25.1%
5.0	2.83		87.5%	74.8%	65.7%	41.7%	13.7%
25.0	3.78		100.0%	97.1%	90.6%	73.9%	16.4%

Panel D. Partisan bias of the population

			Probability of winning district				
% Republican	E[Won]	"Value"	1	2	3	4	5
30%	2.04	0.58	49.4%	47.0%	40.7%	27.8%	10.2%
40%	2.44	0.48	87.0%	73.0%	52.3%	25.1%	6.2%
50%	2.83	0.33	87.5%	74.8%	65.7%	41.7%	13.7%
60%	3.24	0.20	87.8%	76.1%	67.3%	58.6%	34.5%
70%	3.67	0.12	90.2%	79.6%	71.7%	65.0%	59.1%

away"; the gerrymanderer has more than a 13 percent chance of winning even the district least favorable to her. If she had "thrown away" the district—that is, put those with the lowest signal into it—then, in this example, she would win it only 1.4 percent of the time. Finally, the number of districts won in expectation in this case is 2.8, compared with a non-gerrymandered equal representation benchmark of 2.5. Hence, in this case, the ability to be the gerrymanderer leads to a 13 percent increase in the expected number of districts won.

Panel B illustrates how a change in the spread of the conditional preference distribution affects the gerrymanderer. In accordance with our comparative static results, the gerrymanderer does worse as the quality of her signal deteriorates. This is reflected in a lower probability of winning each district, and hence a lower overall value to being the gerrymanderer. For instance, note that when the signal is very coarse, $\sigma^2_{\beta|s} = 4.5$, the gerrymanderer wins only 2.54 districts in expectation—barely more than the 2.5 won under proportional representation. Also, in the $\sigma^2_{\beta|s} = 0.5$ case, the gerrymanderer has a 31 percent chance of winning district 5—if she "threw it away" that would be just 0.2 percent. Finally, although the expected districts won, and hence the value function is monotonic in $\sigma^2_{\beta|s}$ (as we have shown analytically), the probability of winning each district is not monotonic. Intuitively, as the signal becomes more informative, the gerrymanderer can cut the districts finer, but the probability of winning the votes of those with the lowest signals decreases. These two effects work in opposite directions, which leads to the potential nonmonotonicity of the probability of winning districts with "low" medians (here Districts 4 and 5).

Panel C shows how a change in the spread of the voter preferences affects the gerrymanderer. As voter preferences become more spread out, the gerrymanderer does better, as our comparative static results showed. There is a monotonic increase in the probability of winning Districts 1–4 as voter preferences become more spread out, since fewer extreme voters are necessary to provide a solid margin of victory (in expectation). A similar nonmonotonicity, as discussed above, is at work here with the probability of winning District 5.

Panel D reports how changes in the mean affect gerrymandering. A natural interpretation of a change in the mean is that it is a change in the number of nominal Republicans/Democrats. With the mean at zero, there are 50 percent nominal Republicans. As the mean increases, the share of nominal Republicans increases, and vice versa. Note that as the proportion of nominal Republicans increases, the expected number of seats won increases, and the value to being the gerrymanderer decreases. This value represents the difference in expected seats won compared to proportional representation.

V. Extensions

In this section, we discuss some extensions to the basic model.

A. *Majority Power, Risk Aversion, and District-Specific Objectives*

Our analysis thus far has considered a gerrymanderer whose payoff function is equal to the expected number of districts won. This is likely a good approximation for congressional districting, where the uncertainty over the eventual party balance in the House of Representatives makes each district in a given state equally important. But in state legislatures, other objectives may play an important role. For instance, a party might derive great benefit from remaining in the majority, in which case the gerrymanderer's value function would include a positive discontinuity at 50 percent of the seats. The marginal benefit to the gerrymanderer from each seat won might also be diminishing as she wins more seats, in which case the objective function would become concave. Finally, some districts may be more important than others, since different incumbents may be more valuable to the party than others. The next proposition shows that Propositions 1 and 2 characterize the optimum in all of these cases.

PROPOSITION 7: *Suppose that the gerrymanderer constructs districts so as to maximize*

$$E\left[V\left(\frac{1}{N}\sum_{n=1}^{N} w_n d_n\right)\right],$$

where $d_n = 1$ *if the Republicans win district n and* $d_n = 0$ *otherwise; V is any strictly increasing function; and* $\{w_n\}_{n=1}^{N}$ *are a strictly positive set of weights which add to* 1. *Then Propositions* 1 *and* 2 *characterize the optimal partisan gerrymander.*

Proposition 7 shows that our earlier analysis is robust to most any plausible gerrymanderer objective function. The key to this result is the fact that the domain of the underlying objective function comprises only a discrete subset of values, since one of the parties must actually win each seat in the election. Taking an expectation over this underlying function smooths out the problem, so that increasing the probability of winning any one district, holding the others constant, has a linear impact on the expected value of the redistricting scheme. Our earlier assumption of a linear objective function made this marginal impact the same across all districts. Extending our results to this broader case, where the slope of each impact may vary across districts, merely adds a constant in our proofs, but the linearity ensures the proofs still go through.

The only restriction we must place on the objective function is that the gerrymander must gain from winning another district. If, at some point, V were flat or decreasing, so that the gerrymanderer was indifferent or averse to winning, our result would not hold. Similarly, we require that the weights $\{w_n\}_{n=1}^{N}$ be bounded away from zero, lest the gerrymanderer not care at all about a certain district.

Though Propositions 1 and 2 still hold, the effect of the optimal redistricting plan will vary as the underlying objective function changes. For instance, suppose the objective function were linear but for a positive discontinuity at winning a majority. Under normal circumstances, where the gerrymander possesses a commanding popular majority in the state, redistricters would now be risk averse and thus seek to win fewer districts but hold the majority with greater probability. Practically, such a change would mean grouping larger numbers of Republican voters (the right-hand "slice") into a small majority of the districts. On the other hand, if the gerrymanderer faces a hostile population (perhaps due to the inequities of gerrymanders past), the party would become risk-loving. The other two alternative objective functions we mentioned above—concavity and unequal weighting among districts—manifest themselves in more straightforward ways in district composition, with incumbents making some districts more secure at the expense of others.

Risk-aversion also provides a simple rationale for ruling out cracking. As previously noted, a districting plan determines the probability of winning each district; and in the previous sections we have considered the mean of these probabilities. However, a celebrated theorem of Siméon-Denis Poisson (1837) allows us to analyze the variance as well. Substantially generalizing the work of Bernoulli, Poisson showed that the variance of nonidentical independent trials $p_1,\ldots,p_n$ is

$$Var(x) = n\bar{p}(1-\bar{p}) - n\sigma_p^2,$$

where $\bar{p} = (\sum_{i=1}^{n} p_i)/n$ and σ_p^2 is the variance of $p_1,\ldots,p_n$. It is immediate that, fixing $\bar{p}$, the variance is reduced by "spreading out" $(p_1,\ldots,p_n)$. That is, the *maximum* variance of the number of successes (i.e., districts won) is achieved when $p_1 = p_2 = \cdots = p_n$. Further, Wassily Hoeffding (1956) showed that, fixing $\bar{p}$, any increasing concave function of the number of successes is minimized when $p_1 = p_2 = \cdots = p_n$. These theorems show that cracking is suboptimal for a risk-averse gerrymanderer, since cracking involves making a number of districts have the same median voter type, and hence the same probability of winning. Under a pack-and-crack strategy, probabilities of winning districts are as follows:

$$p_1^c = \cdots = p_k^c > p_{k+1}^p > \cdots > p_N^p, \tag{4}$$

where superscripts p and c denote packed districts and cracked districts, respectively. The district winning probabilities under the strategy of Propositions 1 and 2 is

$$p_1 > \cdots > p_N. \tag{5}$$

Now, consider a deviation toward (5) from the pack-and-crack strategy which generates (4). In particular, suppose two cracked districts are altered so that $\hat{p}_1^c > p_1^c$ and $\hat{p}_2^c < p_2^c$, with $\hat{p}_1^c + \hat{p}_2^c = p_1^c + p_2^c$. Proposition 2 tells us that there exists such a deviation with $\hat{p}_1^c + \hat{p}_2^c > p_1^c + p_2^c$, but to apply combinatoric theorems with the expected number of successes constant, we address the case where $\hat{p}_1^c + \hat{p}_2^c = p_1^c + p_2^c$. By Poisson's Theorem the variance of the number of districts won under pack-and-crack is $N\bar{p}(1-\bar{p}) - N \cdot Var(p_1^c,\ldots,p_N^p)$. Under the proposed deviation, the variance is $N\bar{p}(1-\bar{p}) - N \cdot Var(\hat{p}_1^c,\ldots,\hat{p}_N^p)$. To show that the number of districts won under the deviation is lower, we require $Var(p_1^c,\dots,p_N^p) < Var(\hat{p}_1^c,\dots,\hat{p}_N^p)$. That is, $1/N \sum_{i=1}^N (p_i - \bar{p})^2 < 1/N \sum_{i=1}^N (\hat{p}_i - \bar{p})^2$. Removing common terms, this becomes $(p_1^c - \bar{p})^2 + (p_2^c - \bar{p})^2 < (\hat{p}_1^c - \bar{p})^2 + (\hat{p}_2^c - \bar{p})^2$, or, equivalently, $(p_2^c - \bar{p})^2 - (\hat{p}_2^c - \bar{p})^2 < (\hat{p}_1^c - \bar{p})^2 - (p_1^c - \bar{p})^2$. Since $\hat{p}_1^c > p_1^c = p_2^c > \hat{p}_2^c$, the inequality holds.

Cracking, therefore, not only lowers the mean number of districts won, it also increases the risk borne by the gerrymanderer.

It is important to note that, since the aggregate shock affects all districts, the probabilities of winning districts are not independent trials. As we show, however, in Proposition 8 below, the analysis leading to Propositions 1 and 2 applies to the case where there are district-specific shocks. Therefore, treating the trials as we have here as independent is arguably a more general approach.

Applying Hoeffding's Theorem to the kind of deviational argument just made, a deviation such as the one above is preferred by a gerrymanderer whose payoff function is increasing and concave in the number of districts won. Thus, pack-and-crack is suboptimal for any gerrymanderer whose payoff is an increasing concave function of the number of districts won.

B. *Policy Consequences*

Our analysis has thus far considered only a districting scheme's impact on party representation in the legislature. In this section, we consider the potential distance between the median voter's preference and the actual outcome under the optimal partisan gerrymander.[13] We have in mind a setting where district medians determine the preferences of legislators, who then vote on policy alternatives. To illustrate this, we consider the case where voter preferences are perfectly observable (i.e., $\beta = s$). Let each voter have a most preferred policy given by the c.d.f. $H(s)$ with continuous p.d.f. $h(s)$. Assume that the median voter is given by $H(s_m) = 1/2$. Let the ideal policy of the median voter in district d be s_m^d. Ordering these median voters within a district as $s_m^1 \geq \cdots \geq s_m^{(N+1)/2} \geq \cdots s_m^N$, we have what we will refer to as the "representative median voter" $s_m^{(N+1)/2}$. We take this to be the preference of the median legislator. For simplicity, we assume that N is odd—although nothing important hinges on this.

The question we ask here is: what is the difference in preferences between the representative median voter and the population median voter under the optimal gerrymander? That is, what is the magnitude of $|H(s_m^{(N+1)/2}) - H(s_m)|$?

If the gerrymanderer maximizes $H(s_m^{(N+1)/2})$, then—since the signal is perfect—Proposition 2 tells us that this is achieved by combining a mass of voters with the highest bliss points with an (infinitesimally smaller) mass of voters with the lowest bliss points, and then continuing to match into the center of the distribution. Under this gerrymander, the median voter in the median district is the left-most voter in the right-hand slice of district $(N + 1)/2$. It is immediate that, under this gerrymander, $\lim_{N\to\infty} H(s_m^{(N+1)/2}) = \lim_{N\to\infty} (N + 1)/4N = 1/4$, and hence $|H(s_m^{(N+1)/2}) - H(s_m)| = 1/4$. Therefore (for states with large numbers of districts[14]), under the optimal gerrymander, a minority constituting just 25 percent of the population can constitute a winning coalition.

Interestingly, the "dominance of the 25 percent majority" under representative systems was conjectured in the seminal work of James M. Buchanan and Gordon Tullock (1962, 221–22).

This analysis of policy consequences could be extended to the case of a noisy signal. We conjecture that the "buffer" of voters required by the gerrymanderer to equate median-like outcomes becomes larger as the signal quality decreases, and hence $|H(s_m^{(N+1)/2}) - H(s_m)|$ decreases

[13] We are grateful to an anonymous referee for suggesting this, as well as details of the approach.

[14] For a state with 53 districts (e.g., California), $H(s_m^{(N+1)/2}) = 0.255$, and for a state with 5 districts is 0.3.

monotonically in the quality of the signal. We have found this to be the case in a large number of numerical examples—but it remains a conjecture.

C. *Candidate Effects*

Another empirical regularity of congressional races is the seemingly large electoral advantage enjoyed by incumbents—fewer than 3 percent of incumbents are defeated in the typical election cycle. There are three possible causes for this edge. First, an incumbent may simply reflect the preferences of her constituents, or may generally be of high quality. In this case, incumbency is simply a proxy for match quality between a representative and her district, and one can say that incumbency, per se, has no effect. Second, the incumbent may be more well known to her constituents in a variety of ways, and thus more easily elected; a (Republican) gerrymanderer would respond to this type of incumbent advantage by maintaining Republican incumbent districts as constant as possible, while matching Democratic incumbents to new and unfamiliar (though not necessarily different, from a signal profile perspective) districts. Indeed, such tactics were a key part of the Republican gerrymander of Texas in 2003. This effect is primarily a geographic concern, though, and is thus somewhat orthogonal to the predictions of our model.

A third source of advantage for an incumbent may be, broadly speaking, her résumé of congressional experience and the resulting low quality of opponents, an edge which would follow her no matter the make-up of her district. Stephen Ansolabehere, James M. Snyder, and Charles Stewart III (2000) use the decennial redrawing of district boundaries to estimate that this third channel accounts for one-third to one-half of the incumbency advantage, on average, though there is surely much individual heterogeneity in the magnitude of the effect. The conclusions of our model would change in the presence of large incumbent effects of this third type, which would, in effect, make the distribution of the electoral breakpoint district-specific. For instance, suppose that a particular Democratic incumbent was universally well liked and assured of election regardless of the composition of her district. It would then be optimal for a Republican gerrymanderer to "throw away" her district by including in it the most extreme Democrats.

We can model this extension by assuming that incumbent n (from district n) has an electoral advantage ζ_n such that voters support the incumbent if $\beta - A + \zeta_n > 0$. Republicans have positive ζ's, and Democratic incumbents have negative ζ's. Furthermore, suppose that this advantage is independent of the voters in the incumbent's district. As the intuition above suggests, our Lemma 3, and thus Proposition 1, fail with this addition. But, as the following proposition shows, Lemmas 1 and 2 still hold.

PROPOSITION 8: *Suppose that incumbent n in district n has an additional electoral advantage ζ_n, and that $F(\beta, s)$ satisfies Conditions 1 and 2. Then, Lemmas 1 and 2 hold, while Lemma 3, in general, does not.*

Though the ordering of the slices would be somewhat different, the main force of our results still hold. Optimal districts comprise only vertical slices, and such slices may not "interlock," as in Lemma 2. This model does generate the familiar prescription of districts that are "thrown away," but it does not generate "smoothing" across Republican voters, as in standard model. Of course, such a deviation depends on the magnitude of a quite particular effect of incumbency which, in practice, may be quite limited. Even the most well-liked politicians may have trouble attracting votes from affiliates of the opposite party; would Rep. Tom Delay still get elected if his district contained the poor inner cities of Houston instead of Sugarland? Nevertheless, this is the only extension from our model we discuss that does generate "throwing away" districts, and it perhaps deserves further study.

D. *Voter Turnout*

In our model, we have implicitly assumed that everyone votes; obviously, in a system with non-compulsory voting, voter turnout is a real and important issue. In theory, voter turnout could vary with any aspect of the individual or district; research on electoral participation suggests two sets of factors that might affect turnout. First, the literature has identified a number of individual attributes—including education, age, marriage status, occupation, and ideological extremism—which affect the probability of voting (see Orley Ashenfelter and Stanley Kelly, Jr., 1975; Raymond Wolfinger and Steven Rosenstone 1980; John G. Matsusaka and Filip Palda 1993; Edward L. Glaeser, Giacomo A. M. Ponzetto, and Jesse Shapiro 2005). These factors do not have a direct impact on our results, since voter turnout exogenous to the creation of districts will not affect the predictions.

The political science literature has also found a number of district-specific effects. For instance, Kamhon Kan and C. C. Yang (2001) find that turnout is higher when the perceived differences between candidate ideological platforms are higher and when voters "fear" one candidate more than the other. But this type of effect will not change our characterization of the optimal strategy either, since all voters in a district would turn out more or less, depending on the particulars of district construction. Similarly, Ebonya Washington (2006) finds that black candidates increase turnout both among black and white voters, and the difference is not statistically significant.

A final class of models of endogenous turnout allows the probability of voting to depend on district-specific characteristics, but affects different voters within a district in different ways. For instance, people might be more or less likely to vote if their policy bliss point is closer to one of the candidate's platform. Alternatively, moderate voters might be more or less likely to turn out if grouped in the same district with extreme voters from their own party, or extreme voters from different parties. Such models can change the structure of the optimal gerrymander; for instance, if extreme voters of one party make moderate voters from the other party less likely to vote, the matching of extreme democrats with extreme republicans may fail. Of course, the structure of the optimal strategy in our model could just as easily be reinforced if the opposite were true, and incensed Republican moderates turned out to oppose the more extreme Democrats with whom our strategy would match them. Since there is little evidence of either the presence or the direction of these effects, we do not explicitly model these factors here, but such efforts might be a plausible direction for future work.

VI. Conclusion

This paper shows that existing intuitions for optimal partisan gerrymandering are rather misleading—and are the consequence of simplifying assumptions. We have analyzed a more general model with a continuum of voter preferences and noisy signals of those preferences. The model nests major models in the literature as special cases. Smoothing supporters evenly is always suboptimal. When the signal the gerrymanderer receives is precise enough, the optimal strategy involves matching extreme Republicans with extreme Democrats. This characterization of the optimal partisan gerrymander is robust to a number of extensions, including alternative partisan objective functions.

The primary import of our paper is to suggest a reexamination of widely held intuitions about the effects of partisan gerrymandering. These intuitions are not simply academic speculations, but give rise to conventional wisdom about partisan gerrymandering which is not wholly accurate. For instance, traditional models imply that groups that have very different preferences from the gerrymanderer do not fare so badly—that is, although gerrymandering makes them worse off than proportional representation, they are assured of a lower bound of representation due to the gerrymanderer's "throwing away" some districts. Our model has very different implications. Instead, because of the "matching slices" strategy, they are combined into districts with a larger group of voters who have extremely different preferences from them, and so they have *very little* representation as a result of gerrymandering. Thus, our model suggests that the negative consequences of partisan gerrymandering for minority representation in government may be far worse than currently thought.

A natural question that follows from this analysis is to ask: who are the voters in the opposite tail of the distribution to the gerrymanderers? To illustrate this connection, suppose that the gerrymanderer is a Republican and that African Americans are highly represented in the far-left tail of the signal distribution (i.e., they have characteristics that make them very likely to vote for Democrats). In this case, under the optimal gerrymander, African Americans would be placed in districts such that they receive very little representation. Data from the 2000 US Census and the 2000 presidential election suggest that African Americans do, indeed, constitute the far-left tail, and so an implementation of the optimal strategy, as characterized in this paper, would be severely disadvantageous to that population. The unmistakable implication of these facts is that partisan gerrymandering (when practiced by Republicans) and racial gerrymandering are basically synonymous *in effect*. Since the 1960s, however, the Supreme Court has adopted a test based on intent, rather than effect.

A further implication of our analysis is that gerrymandering can be very valuable, and indeed is more valuable today than ever before. Technological advances have allowed gerrymanderers to gain better information about voters—in our model, a less coarse signal distribution in the sense of Blackwell—and draw boundaries with a finer pen. One would therefore expect parties to use an increasingly large amount of resources in order to become the gerrymanderer. Since the practice itself probably lowers social welfare (see Stephen Coate and Brian Knight (2006) for an illuminating analysis of socially optimal districting), spending resources on it merely exacerbates the social loss associated with partisan gerrymandering. This implies that the welfare loss from gerrymandering is linked to such technologies, and has grown over time.

There are two clear directions for future work. The first involves empirical investigations of gerrymandering in light of the theory developed here. The structure provided by our characterization of the optimal gerrymandering strategy is important for such empirical work. Previous empirical work on gerrymandering (see, for instance, Andrew Gelman and Gary King 1990, 1994) assumes a nonmicrofounded structural model which may give inaccurate estimates of the degree of gerrymandering. The second set of open issues involves the regulation of gerrymandering. Enriching the model to capture spatial considerations would make it possible to analyze the impact of constraints such as compactness. Although there is a body of work that attempts to deal with spatial considerations, the underlying models of gerrymandering they employ are, as we have discussed, insufficiently rich to capture the core intuitions of the optimal strategy.

Ultimately, the effect of gerrymandering is an empirical question. As our model highlights, the impact of it depends on the particulars of the signal and preference distribution. One thing this paper demonstrates, however, is that empirical investigations alone can be misleading. Without understanding the optimal strategy for a gerrymanderer, one cannot properly assess the impact of partisan gerrymandering.

Appendix

A. *Monotonicity of Voting*

We remarked in a footnote in the text that, under the assumption of single-crossing preferences, the probability that a voter votes Republican is increasing in her type. This is not of direct relevance to the other results in the paper, but may be of independent interest.

DEFINITION 3: *Let X and Y be subsets of* $\mathbb{R}$, *and let* $K : X \times Y \to \mathbb{R}$. *We say that K is Totally Positive of order n ("*TP_n*") if* $x_1 < \cdots < x_n$ *and* $y_1 < \cdots < y_n$ *imply*

$$\begin{vmatrix} K(x_1,y_1) & \cdots & K(x_1,y_m) \\ \vdots & & \vdots \\ K(x_m,y_1) & \cdots & K(x_m,y_m) \end{vmatrix} \geq 0$$

for each $m = 1,\ldots,n$.

THEOREM 1 (Karlin 1968): *Let K be* TP_n *on* $X \times Y$ *and let* μ *be a* σ*-finite measure on X. If* $f : \mathbb{R} \to \mathbb{R}$ *has at most* $k \leq n - 1$ *sign changes, then for* $y \in Y$,

$$f^*(y) = \int f(x) K(x,y)\, d\mu(x)$$

has at most k sign changes. Furthermore, if f^* *has exactly k sign changes, then f and* f^* *have the same pattern of sign changes.*

Total positivity of order two is familiar in economics and has had wide applications in the theory of moral hazard, as well as mechanism and market design.

REMARK 1: *Suppose that* $K(x,y)$ *is a probability density function, denoted* $f(x|y)$, *with respect to a* σ*-finite measure* μ *such that* $\int f(x|y)\mu(dx) = 1$. *Then, if* $f(x|y)$ *is* TP_2, *then* $f(x|y)$ *satisfies the MLRP.*[15]

Karlin's Theorem (commonly referred to as the Variation Diminishing Property (VDP)) allows us to observe that voters of higher type (higher *i*) are more likely to vote Republican provided $g(\beta|s)$ is TP_2. To see this, recall that, since voter preferences satisfy single-crossing (combined with our reordering), $\beta_i = u_i(R) - u_i(D)$ is a monotonic function with one sign change. The stochastic objective $f^*(s) = \int \beta g(\beta|s)\, d\beta$ is then also monotonic. Let k be an arbitrary constant and consider $f^*(s) - k = \int(\beta - k) g(\beta|s)\, d\beta$. Since $\beta - k$ has only one sign change, the VDP implies that $f^*(s) - k$ has only one sign change. This immediately implies monotonicity of $f^*(s)$. Monotonicity of $f^*(s)$ implies that for any two signals of voter types, $i > j$, the probability that type i votes Republican is greater than the probability that type j does.

B. *Proofs*

PROOF OF LEMMA 1:

The maximization problem can be described by the Lagrangian

$$L = \sum_{n=1}^{N} B(\mu_n) - \sum_{n=1}^{N} \lambda_n \left[\int_{-\infty}^{\infty} \psi_n(s)\, ds - \frac{1}{N} \right], \tag{6}$$

[15] For the classic reference to likelihood ratios and their applications to economics, see Milgrom (1981).

in addition to the boundary constraints. Note that the first-order necessary conditions imply

$$\Delta\psi_n(s)\left(b(\mu_n)\frac{\partial\mu_n}{\partial\psi_n(s)} - \lambda_n\right) = 0 \text{ for } n = i,j, \qquad s = s_1, s_2. \tag{7}$$

Now, consider districts i and j, and suppose that $\mu_i < \mu_j$.

Throughout, whenever we speak of removing voters of type s, we refer to an interval $[s - \varepsilon/2, s + \varepsilon/2]$. Denote the derivative of the objective function with respect to a switch of voters of type s from district j to district i as $\phi_{ji}(s)$. Then, for any ε, the change in the value of the objective function is

$$\Delta V(s) = \int_{s-\varepsilon/2}^{s+\varepsilon/2} \phi_{ji}(s')\,ds'.$$

Note that, as $\varepsilon \to 0$, the change in the value of the objective function from such a move approaches the derivative of the objective function at s multiplied by ε, since

$$\lim_{\varepsilon\to 0}\int_{s-\varepsilon/2}^{s+\varepsilon/2} \phi_{ji}(s')\,ds' = \phi_{ji}(s)\varepsilon.$$

The derivative of the objective function from moving voters of type s from district j and adding them to district i is

$$\phi_{ji}(s) = \left(b_i\frac{\partial\mu_i}{\partial\psi_i(s)} - \lambda_i\right) - \left(b_j\frac{\partial\mu_j}{\partial\psi_j(s)} - \lambda_j\right).$$

Implicitly differentiating (2), which determines the medians, yields

$$0 = \int_{-\infty}^{\infty} g(\mu_i|s)\psi_i(s)\,ds\partial\mu_i + G(\mu_i|s)\partial\psi_j(s);$$

$$\frac{\partial\mu_i}{\partial\psi_i(s)} = -\frac{G(\mu_i|s)}{\int_{-\infty}^{\infty} g(\mu_i|s)\psi_i(s)\,ds} \tag{8}$$

$$\equiv -\frac{G(\mu_i|s)}{\gamma_i(\mu_i)}. \tag{9}$$

Hence, the change in the value of the objective function is

$$\Delta V(s) = \varepsilon\left(\frac{b(\mu_j)}{\gamma_j(\mu_j)}G(\mu_j|s) - \frac{b(\mu_i)}{\gamma_i(\mu_i)}G(\mu_i|s) + \lambda_j - \lambda_i\right). \tag{10}$$

Note that if $\phi_{ji}(s) > \phi_{ji}(s')$, then $\Delta V(s) > \Delta V(s')$ for any $\varepsilon > 0$. While equation (10) need not be positive for all s in district n, it must be, $\forall s' \in \psi_j$ and $s \in \psi_i$, that $\phi_{ji}(s) \geq \phi_{ji}(s')$. Note that $\partial\phi_{ji}(s)/\partial s > 0$ is equivalent to $z(\mu_j|s)/z(\mu_i|s) < b(\mu_i)\gamma_j(\mu_j)/b(\mu_j)\gamma_i(\mu_i)$, and since the left-hand side is monotonically increasing in s from Condition 1, $\phi_{ji}(s)$ cannot be convex. If $s_1, s_2 \in \psi_i$, then, for any point $s' \in [s_1, s_2]$, $\phi_{ji}(s') > \min[\phi_{ji}(s_1), \phi_{ji}(s_2)]$. Thus, $s' \notin \psi_j$, if $\varepsilon > 0$.

This implies that any two districts j and i (where, without loss of generality $\mu_j > \mu_i$) cannot share voters of the same type except on a set of measure zero.

This also implies that districts must comprise vertical slices. Suppose that there exists an interval voter of types $[s - a, s + a]$ such that all voters of type $s' \in [s - a, s + a]$ are in both districts j and i. This contradicts the statement above that if $s_1, s_2 \in \psi_i$, then, for any point $s' \in [s_1, s_2]$, $s' \notin \psi_j$, if $\varepsilon > 0$.

PROOF OF LEMMA 2:

Suppose, by way of contradiction, that there exist districts j and i such that $\mu_j = \mu_i$, and that there exist intervals of positive measure about types s_1 and s_2 (with $s_1 > s_2$), which are in both districts. Consider moving a small mass from an interval about s_1 into district j and a comparable mass of voters around s_2 back into district i. The first-order conditions imply that the net gain, which must equal zero, is proportional to

$$\frac{b(\mu_j)}{\gamma_j(\mu_j)}[\mathrm{G}(\mu_j|s_2) - G(\mu_j|s_1)] - \frac{b(\mu_i)}{\gamma_i(\mu_i)}[G(\mu_i|s_2) - G(\mu_i|s_1)] \tag{11}$$

for $\varepsilon > 0$. Since $\mu_i = \mu_j$, we know that $b(\mu_j) = b(\mu_i)$ and $G(\mu_i|s_2) - G(\mu_i|s_1) = G(\mu_j|s_2) - G(\mu_j|s_1)$. Therefore, it must be that $\gamma_i(\mu_i) = \gamma_j(\mu_j)$.

Consider, again, the districts j and i with $\mu_i = \mu_j$. By Lemma 1, those voters in districts j and i must make up one or two complete vertical slices of $h(s)$. Since F has full support and the two aforementioned slices contain a positive interval of voter types, there must exist four voter types $s_1 < s_2 < \mu_j < s_3 < s_4$ such that $G(\mu_j|s_1) - G(\mu_j|s_2) = G(\mu_j|s_3) - G(\mu_j|s_4)$ and $\psi_i(s_1) > 0$, $\psi_i(s_4) > 0$, $\psi_j(s_2) > 0$, and $\psi_j(s_3) > 0$. In words, one district contains some of the inner type of voters, while the other district contains some of the more extreme types of voters relative to the district medians.

Now, consider a perturbation in which an equal mass of voters around type s_1 and around type s_4 are transferred to district j from district i, and similarly an equal mass of voters around type s_2 and around type s_3 are transferred from district j to i. By construction, both μ_j and μ_i remain unchanged, as does the value function; but $\gamma_i(\mu_i)$ and $\gamma_j(\mu_j)$ have changed. By definition,

$$\frac{\partial \gamma_i(\mu_i)}{\partial \psi(s)} = g(\mu_i|s),$$

and so the derivative of $\gamma_i(\mu_i)$ for perturbations of this type is

$$\partial \gamma_i(\mu_i) = \varepsilon \left(\frac{\partial \gamma_i(\mu_i)}{\partial \psi(s_2)} - \frac{\partial \gamma_i(\mu_i)}{\partial \psi(s_1)} + \frac{\partial \gamma_i(\mu_i)}{\partial \psi(s_3)} - \frac{\partial \gamma_i(\mu_i)}{\partial \psi(s_4)} \right)$$

$$= \varepsilon (\mathrm{g}(\mu_i|s_2) - g(\mu_i|s_1) + g(\mu_i|s_3) - g(\mu_i|s_4)).$$

But, by Condition 2, the modes of the lower signals lie below μ_i. Thus, we know that $\mathrm{g}(\mu_i|s_2) > \mathrm{g}(\mu_i|s_1)$, and similarly that $\mathrm{g}(\mu_i|s_3) > \mathrm{g}(\mu_i|s_4)$, and so $\partial\gamma_i(\mu_i) > 0$, for $\varepsilon > 0$. By similar reasoning, $\partial\gamma_j(\mu_j) < 0$. After performing such a perturbation, the new districting arrangement has $\mu_j = \mu_i$, while $\gamma_i(\beta) \neq \gamma_j(\beta)$. This now violates the condition above, which holds that for two districts that share a positive mass of voters and for which $\mu_j = \mu_i$, it must be that $\gamma_i(\beta) = \gamma_j(\beta)$. This new arrangement is not optimal, but the value function is unchanged from the old districting plan, and so the old plan cannot be optimal either—a contradiction.

PROOF OF LEMMA 3:

Suppose, by way of contradiction, that such a case existed. Without loss of generality, from Lemma 1, we can assume that districts i and k each comprise one whole slice. It also must be that $s^* < s'$ for all $s' \in \psi_i$ and that $s^* > s''$ for all $s'' \in \psi_k$. Denote $\bar{s}_i = sup\{s \in \psi_i\}$, $\bar{s}_k = sup\{s \in \psi_k\}$, $\underline{s}_i = inf\{s \in \psi_i\}$, and $\underline{s}_k = inf\{s \in \psi_k\}$. Of course, $\bar{s}_i > \underline{s}_i > s^* > \bar{s}_k > \underline{s}_k$.

The Lagrangian from equation (6) implies that, if $s \in \psi_j$, then

$$\varepsilon(-a_j G(\mu_j|s) - \lambda_j) \geq \varepsilon \max_n(-a_n G(\mu_n|s) - \lambda_n)$$

for all districts n, and, hence,

$$-a_j G(\mu_j|s) - \lambda_j \geq \max_n(-a_n G(\mu_n|s) - \lambda_n), \forall \varepsilon > 0,$$

where $a_n = b(\mu_n)/\gamma_n(\mu_n)$. These a_n coefficients represent the sensitivity of the median of district n to changes. For each district n, denote these expressions by η_n. We know that

$$\eta_i(\bar{s}_i) \geq \eta_j(\bar{s}_i) \text{ and } \eta_j(s^*) \geq \eta_i(s^*),$$

which implies that

$$a_j \leq a_i \frac{G(\mu_i|s^*) - G(\mu_i|\bar{s}_i)}{G(\mu_j|s^*) - G(\mu_j|\bar{s}_i)}. \tag{12}$$

Equation (12) states that district j must not be too sensitive compared to district i. Were this so, a profitable deviation would exist by shifting district i down to include s^* and giving voters of type $\bar{s}_i$ to district j. Similar arguments imply that

$$a_j \geq a_k \frac{G(\mu_k|\underline{s}_k) - G(\mu_k|s^*)}{G(\mu_j|\underline{s}_k) - G(\mu_j|s^*)}, \tag{13}$$

which has the interpretation that district j must be sensitive enough relative to district k so that shifting district k up to include s^* is not profitable. Of course, (12) and (13) can hold simultaneously only if the right-hand side of (12) is greater than or equal to the right-hand side of (13). This requires

$$\frac{a_k}{a_i} = \frac{b(\mu_k)\gamma_i(\mu_i)}{b(\mu_i)\gamma_k(\mu_k)} \leq \frac{G(\mu_i|s^*) - G(\mu_i|\bar{s}_i)}{G(\mu_k|\underline{s}_k) - G(\mu_k|s^*)} \frac{G(\mu_j|\underline{s}_k) - G(\mu_j|s^*)}{G(\mu_j|s^*) - G(\mu_j|\bar{s}_i)}. \tag{14}$$

Now, consider what happens to this ratio as we increase the precision of the signal (which can be thought of here as shrinking the conditional preference distribution G into the median). Since district i contains voters closer in signal to the median of district j, the ratio $[G(\mu_j|\underline{s}_k) - G(\mu_j|s^*)]/[G(\mu_j|s^*) - G(\mu_j|\bar{s}_i)]$ will shrink, going to 0 in the limit. On the other hand, both $G(\mu_i|s^*) - G(\mu_i|\bar{s}_i)$ and $G(\mu_k|\underline{s}_k) - G(\mu_k|s^*)$ rise to 1, since $\underline{s}_k < \mu_k < s^* < \mu_i < \bar{s}_i$. Thus, the right-hand side of (14) shrinks to 0 as the precision of the signal increases. Note, however, that the ratio a_k/a_i is bounded away from 0, since $\gamma_i(\mu_i)/\gamma_k(\mu_k)$ will limit to 1 (by the definition of $\gamma(\mu)$) and $b(\mu_k)/b(\mu_i)$ is bounded away from 0 since the medians μ_i and μ_k are bounded and the c.d.f. B is strictly increasing. Thus, for sufficiently high signal quality, the inequality in (14) cannot hold—a contradiction.

PROOF OF PROPOSITION 1:

Apply Lemmas 1–3.

PROOF OF PROPOSITION 2:

Suppose not. Consider the districting plan that entirely packs. That is, consider the districting plan described by $N - 1$ cutoffs $\{\tau_n\}_{n=1}^{N-1}$ (where $\tau_1 > \tau_2 > \cdots > \tau_{N-1}$) such that $s \in \psi_n$ if and only if $s \in [\tau_n, \tau_{n-1}]$. (For notational ease, suppose that $\tau_0 = \infty$ and $\tau_N = -\infty$.) Consider the marginal gain from moving voters of type τ_n from district n to district $n + 1$ and moving voters from the far-left tail to district 1. Following the first-order condition in equations (7) and (9) (contained in the Appendix in the proof of Lemma 1 p. 137), the impact on μ_n for $n > 1$ is

$$\Delta\mu_n = \varepsilon\left(\frac{b(\mu_n)}{\gamma_n(\mu_n)}[G(\mu_n|\tau_n) - G(\mu_n|\tau_{n-1})]\right) > 0,$$

since $\tau_n < \mu_n < \tau_{n-1}$ and, therefore, $G(\mu_n|\tau_n) > 0.5 > G(\mu_n|\tau_{n-1})$. We use ε here to denote the small positive mass of voters moved in each shift, as we discuss in detail in the proof of Lemma 1. The impact on μ_1 will be

$$\Delta\mu_1 = \varepsilon\left(\frac{b(\mu_1)}{\gamma_1(\mu_1)}[G(\mu_1|\tau_1) - G(\mu_1|\tau_N)]\right) < 0,$$

where, for these purposes,

$$G(\mu_1|\tau_N) = \lim_{s\to-\infty} G(\mu_1|s) = 1.$$

Note further that, by the definition of τ_1 and μ_1, $G(\mu_1|\tau_1) > 0.5$.

Now, consider increasing the signal quality, which is to say decreasing the spread of the conditional distribution of β given s about the center of that distribution. Note that $G(\beta|s)$ is centered around s by Condition 2, and so, if $G(\mu_n|s) > 0.5$, then $\partial G(\mu_n|s)/\partial\sigma^2_{\beta|s} < 0$, so that $G(\mu_n|s)$ increases as the signal quality increases. (When we shrink $\sigma^2_{\beta|s}$, we refer to a reduction in the spread of the distribution around the median and mode of s, rather than the mean, so as to maintain Condition 2.) If $G(\mu_n|s) < 0.5$, then $\partial G(\mu_n|s)/\partial\sigma^2_{\beta|s} > 0$. The term $\gamma_1(\mu_1)$ will also increase, but it is (by definition) bounded above by the marginal distribution of β in the population. Thus, we know that, at least for high enough signal quality,

$$\frac{\partial\Delta\mu_n}{\sigma^2_{\beta|s}} > 0 \quad \forall n,$$

which implies that

$$\frac{\partial}{\sigma^2_{\beta|s}} \sum_{n=1}^{N}\Delta\mu_n = \varepsilon\frac{b(\mu_1)}{\gamma_1(\mu_1)} > 0.$$

The aggregate impact on the expected number of seats won from the proposed deviation becomes more positive or less negative as the signal quality increases. Finally, note that

$$\lim_{\sigma^2_{\beta|s}\to 0} G(\mu_1|\tau_1) = 1,$$

so that

$$\lim_{\sigma^2_{\beta|s}\to 0} \Delta\mu_1 = 0,$$

while

$$\lim_{\sigma^2_{\beta|s}\to 0} \Delta\mu_n > 0,\ n \neq 1,$$

and therefore

$$\lim_{\sigma^2_{\beta|s}\to 0} \sum_{n=1}^{N} \Delta\mu_n > 0.$$

Since the sum converges to the limit as $\sigma^2_{\beta|s}$ decreases, we know that there exists $\underline{\sigma}^2$ such that $\sum_{n=1}^{N}\Delta\mu_n > 0$ whenever $\sigma^2_{\beta|s} < \underline{\sigma}^2$.

PROOF OF PROPOSITION 3:

Suppose not. The choice variable for each district can be summarized by ψ_n, the proportion of R in the district. Then, there exist two districts j and i such that $\psi_j \neq \psi_i$ and $\psi_n > 0$ for $n = \{j, i\}$. Without loss of generality, let $\mu_j > \mu_i$. By Condition 1, $G(\beta|r)$ first-order stochastically dominates $G(\beta|d)$, and so $\psi_j > \psi_i$.

In order that there be no profitable deviations, it must that $\partial\mu_i/\partial\psi_i = \partial\mu_j/\partial\psi_j$. But, in general,

$$\frac{\partial^2\mu}{\partial\psi_n^2} = \frac{\dfrac{\partial\mu_n}{\partial\psi_n}}{\gamma(\mu)^2}\left\{\begin{aligned}&\big([g(\mu|d) - g(\mu|r)]b(\mu) + [G(\mu|d) - G(\mu|r)]b'(\mu)\big)\\ &\qquad \times [\psi_n(g(\mu|r) - g(\mu|d)) + g(\mu|d)]\\ &\qquad - b(\mu)[G(\mu|d) - G(\mu|r)][\gamma'(\mu) + g(\mu|r) - g(\mu|d)]\end{aligned}\right\},$$

which is positive when $\mu < 0$ and negative when $\mu > 0$. Since $\mu > 0 \Leftrightarrow \psi > 0.5$, the concavity of μ implies that one could never have $\psi_j > \psi_i \geq 0.5$, since then $\partial\mu_i/\partial\psi_i > \partial\mu_j/\partial\psi_j$, and so R could do better by increasing i and decreasing j. It also implies that there cannot be $0.5 > \psi_j \geq \psi_i$, since then $\partial\mu_i/\partial\psi_i < \partial\mu_j/\partial\psi_j$ and the opposite deviation would improve R's representation. Thus, there can be only one "odd district" with $0 < \psi < 0.5$, and all districts with $\psi > 0.5$ must have equal proportions of r and d.

Suppose that $N \to \infty$. Note that there can be only one odd district. Let the mass of voters in this district have Lebesgue measure τ. Since each district must have an equal mass of voters, $\tau = 1/N$. Clearly, $\lim_{N\to\infty}\tau = 0$.

PROOF OF PROPOSITION 4:

First, note that signal precision provides a partial ordering on conditional preference distribution. If the signal contains no information, the expected number of seats won by the gerrymanderer is the population share. If the signal is perfectly precise such that $s = \beta$, it is possible (see Proposition 1) to create districts such that only the lowest median district has a median equal to the population median, while all others lie above. Hence, the gerrymanderer wins more seats in expectation with a perfect signal. Now, consider any two conditional preference distributions g and g' such that g provides a more precise signal than g'. The gerrymanderer must win at least as many seats in expectation under g than g' since the value function has the Blackwell Property. That is, she could construct a distribution c such that from g she could generate g'.

PROOF OF PROPOSITION 5:

Fix the optimal districting plan under $F(\beta, s)$ and consider the construction of the highest median district (without loss of generality, District 1) with median μ_1 given by $\int_{s \in \psi_1} G(\mu_1 | s) h(s)\, ds = 1/2N$, comprising an upper and lower slice. Let the upper slice contain w_1 share of the voters in the district. Suppose that, under $\hat{F}(\beta, s)$, the gerrymanderer sets $\hat{\mu}_1 = \mu_1$. This can be achieved with at least as small an upper slice $\hat{w}_1 \leq w_1$, since the Republican voters (who make up more than half of the district) are at least as likely to vote Republican as before. If $\hat{w}_1 < w_1$, then note that all other districts $2, \ldots, N$ have a higher medians even if we set $\hat{w}_i = w_i$ for all i, that is, without reoptimizing their construction. If $\hat{w}_1 = w_1$, then repeat this procedure until finding a district n^* such that $\hat{w}_{n^*} < w_{n^*}$. By assumption that $\hat{F}$ has greater symmetric spread than F, this must be true for at least one district. Hence the value function under $\hat{F}(\beta, s)$ is higher than under $F(\beta, s)$. This reasoning must hold for any such pair of distributions.

PROOF OF PROPOSITION 6:

Consider an increase from N districts to mN, where m is an integer. By replication, the gerrymanderer could do at least as well with mN districts as with N—but this replication involves creating parfaits. From Lemma 2, this is a suboptimal strategy. Hence, the value function under the optimal strategy must be higher.

PROOF OF PROPOSITION 7:

Suppose that the objective function is now

$$E\left[V\left(\frac{1}{N}\sum_{n=1}^{N} w_n d_n\right)\right],$$

and suppose that V is a strictly increasing function. We can rewrite this expression as the sum of $V(D)$, where $D = 0, \ldots, N$, weighted by the combinatorial probability that the Republicans win exactly D districts. Note that this expression can be factored into two parts: those outcomes where R wins some district n, and those where R loses district n. Since the probability of winning a district is just $B(\mu_n)$, this expression is just

$$B(\mu_n)K_n + (1 - B(\mu_n))L_n,$$

where $K_n = \mathrm{E}[V | d_n = 1]$, the expected value if the Republican candidate wins in district n; and $L_n = \mathrm{E}[V | d_n = 0]$, the expected value if the Democrat wins in district n. Now, fix the districting scheme and consider the marginal benefit from a small deviation x in district n, which is

$$\frac{\partial E[V]}{\partial x} = b(\mu_n)(K_n - L_n)\frac{\partial \mu_n}{\partial x}.$$

The conditions from (7) must still hold for these new first-order conditions, but note that this expression is identical to the value derived in equation (7) but for the term $(K_n - L_n)$, which is fixed for all deviations from a districting plan. Thus, the "sensitivities" $\{a_n\}_{n=1}^{N}$ (as in Lemma 3) are now differently scaled, but the constant does not affect any proofs. Propositions 1 through 6 hold.

PROOF OF PROPOSITION 8:

Suppose candidates are each associated with an electoral benefit ζ_n such that voters support them if $\beta - A + \zeta_n > 0$. In this case, the Republican candidate wins district n if and only if $\mu_n + \zeta_n > A$, which occurs with probability $B(\mu_n + \zeta_n)$. The marginal benefit to R from a small deviation x in district n would be

$$\frac{\partial V}{\partial x} = b(\mu_n + \zeta_n)\frac{\partial \mu_n}{\partial x}.$$

Since the district-specific constant $b(\mu_n + \zeta_n)$ cancels out in Lemma 1, the proof still holds. Lemma 2 is similarly unaffected, as the constant does not affect the proofs. In Lemma 3, the ratio $a_k/a_i = b(\mu_k + \zeta_k)\gamma_i(\mu_i + \zeta_i)/b(\mu_i + \zeta_i)\gamma_k(\mu_k + \zeta_k)$ is no longer bounded away from 0, because $\gamma_i(\mu_i + \zeta_i)$ need not limit to 1 as the precision of the signal increases.

REFERENCES

Ansolabehere, Stephen, James M. Snyder, and Charles Stewart III. 2000. "Old Voters, New Voters, and the Personal Vote: Using Redistricting to Measure the Incumbency Advantage." *American Journal of Political Science*, 44(1): 17–34.

Ansolabehere, Stephen, James M. Snyder, and Charles Stewart III. 2001. "Candidate Positioning in the U.S. House Elections." *American Journal of Political Science*, 45(1): 36–59.

Ashenfelter, Orley, and Stanley Kelly, Jr. 1975. "Determinants of Participation in Presidential Elections." *Journal of Law and Economics*, 18(3): 695–733.

Buchanan, James M., and Gordon Tullock. 1962. *The Calculus of Consent: Logical Foundations of Constitutional Democracy*. Ann Arbor: University of Michigan Press.

Coate, Stephen, and Brian Knight. 2007. "Socially Optimal Districting: A Theoretical and Empirical Exploration." *Quarterly Journal of Economics*, 122(4): 1409–71.

Cox, Gary W., and Jonathan N. Katz. 2002. *Elbridge Gerry's Salamander: The Electoral Consequences of the Apportionment Revolution*. Cambridge, MA: Cambridge University Press.

Downs, Anthony. 1957. *An Economic Theory of Democracy*. New York: Harper and Row.

Gelman, Andrew, and Gary King. 1990. "Estimating the Electoral Consequences of Legislative Redistricting." *Journal of the American Statistical Association*, 85(410): 274–82.

Gelman, Andrew, and Gary King. 1994. "Enhancing Legislative Redistricting Through Legislative Redistricting." *American Political Science Review*, 88(3): 541–59.

Gilligan, Thomas W., and John G. Matsusaka. 1999. "Structural Constraints on Partisan Bias under the Efficient Gerrymander." *Public Choice*, 100(1–2): 65–84.

Glaeser, Edward L., Giacomo A. M. Ponzetto, and Jesse M. Shapiro. 2005. "Strategic Extremism: Why Republicans and Democrats Divide on Religious Values." *Quarterly Journal of Economics*, 120(4): 1283–1330.

Hoeffding, Wassily. 1956. "On the Distribution of the Number of Successes in Independent Trials." *Annals of Mathematical Statistics*, 27(3): 713–21.

Issacharoff, Samuel, Pamela S. Karlan, and Richard H. Pildes. 2002. *The Law of Democracy: Legal Structure of the Political Process*. New York: Foundation Press.

Kan, Kamhon, and C. C. Yang. 2001. "On Expressive Voting: Evidence from the 1988 U.S. Presidential Election." *Public Choice*, 108(3-4): 295–312.

Karlin, Samuel, and Herman Rubin. 1956. "The Theory of Decision Procedures for Distributions with the Monotone Likelihood Ratio." *Annals of Mathematical Statistics*, 27(2): 272–99.

Lee, David S., Enrico Moretti, and Matthew J. Butler. 2004. "Do Voters Affect or Elect Policies? Evidence from the U.S. House." *Quarterly Journal of Economics*, 119(3): 807–59.

Matsusaka, John G., and Filip Palda. 1993. "The Downsian Voter Meets the Ecological Fallacy." *Public Choice*, 77(4): 855–918.

Milgrom, Paul R. 1981. "Good News and Bad News: Representation Theorems and Applications." *Bell Journal of Economics*, 12(2): 380–91.

Owen, Guillermo, and Bernard Grofman. 1988. "Optimal Partisan Gerrymandering." *Political Geography Quarterly*, 7(1): 5–22.

Poisson, Siméon-Denis. 1837. *Recherches sur la probabilité des jugements en matière criminelle et en matière civile, précedées des regles générales du calcul des probabilités.* Paris: Bachelier.

Rothstein, Paul. 1991. "Representative Voter Theorems." *Public Choice*, 72(2–3): 193–212.

Sherstyuk, Katerina. 1998. "How to Gerrymander: A Formal Analysis." *Public Choice*, 95(1–2): 27–49.

Shotts, Kenneth W. 2002. "Gerrymandering, Legislative Composition, and National Policy Outcomes." *American Journal of Political Science*, 46(2): 398–414.

Washington, Ebonya. 2006. "How Black Candidates Affect Turnout." *Quarterly Journal of Economics*, 121(3): 973–98.

Wolfinger, Raymond, and Steven Rosenstone. 1980. *Who Votes?* New Haven: Yale University Press.