5 Appendix (To Be Published Online)

5.1 Proofs

Proof of Lemma 1. Consider the global game investors in set S play at time 4. In Section 3 of Morris and Shin (2003), it is shown that, in such a game, provided $\frac{\sigma^2}{\tau^4}$ is sufficiently small, there is a unique equilibrium in which investors follow strategies of the form: invest if and only if $\overline{\theta}_j > \kappa$, where $\overline{\theta}_j = \frac{\sigma^2 \cdot \mu + \tau^2 \cdot x_j}{\sigma^2 + \tau^2}$ denotes investor j's posterior on θ. We are focused on the case where $\sigma \to 0$, so $\frac{\sigma^2}{\tau^4}$ is indeed small. Furthermore, $\overline{\theta}_j \to x_j$ as $\sigma \to 0$, so in the limit, the cutoff rule becomes: invest if and only if $x_j > \kappa$.

Let us now solve for the cutoff (κ). If investor j invests, his expected payoff will be:

$$\begin{aligned}
\text{Payoff from investing} &= \beta_M \cdot E(R | x_j, \text{j invests}) \\
&= \beta_M \cdot E(\theta + vK | x_j, \text{j invests}) \\
&= \beta_M \cdot [\overline{\theta_j} + v(1 + (m-1) \cdot \Pr(\overline{\theta}_k > \kappa | x_j))] \\
&= \beta_M \cdot \left[\overline{\theta_j} + v \left(1 + (m-1) \cdot \Pr \left(x_k > \kappa + \left(\frac{\sigma^2}{\tau^2} \right) (\kappa - \mu) \;\Big|\; x_j \right) \right) \right].
\end{aligned}$$

Observe that investor j's posterior on θ is that it is distributed normally with mean $\overline{\theta_j}$ and with variance $\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}$. Since $x_k = \theta + \varepsilon_k$, j's posterior on x_k is that it is distributed normally with mean $\overline{\theta_j}$ and with variance $\frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} + \sigma^2$ (or, simplifying, with variance $\frac{2\sigma^2 \tau^2 + \sigma^4}{\sigma^2 + \tau^2}$). Consequently:

$$\begin{aligned}
\Pr \left(x_k > \kappa + \left(\frac{\sigma^2}{\tau^2} \right) (\kappa - \mu) \;\Big|\; x_j \right) &= \Pr \left(\frac{x_k - \overline{\theta}_j}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}} > \frac{(\kappa - \overline{\theta}_j) + \left(\frac{\sigma^2}{\tau^2} \right) (\kappa - \mu)}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}} \;\Big|\; x_j \right) \\
&= 1 - \Phi \left(\frac{(\kappa - \overline{\theta}_j) + \left(\frac{\sigma^2}{\tau^2} \right) (\kappa - \mu)}{\sqrt{\frac{2\sigma^2\tau^2 + \sigma^4}{\sigma^2 + \tau^2}}} \right),
\end{aligned}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution.

At the cutoff (that is, when $\overline{\theta}_j = \kappa$), investor j should be indifferent between investing and not. Hence, the payoff from investing should be equal to 1. This gives us the following formula:

$$\beta_M \cdot \left[\kappa + v\left(1 + (m-1)\cdot\left(1 - \Phi\left(\frac{\kappa-\mu}{\sqrt{\left(\frac{\tau^4}{\sigma^4}\right)\frac{2\sigma^2\tau^2+\sigma^4}{\sigma^2+\tau^2}}}\right)\right)\right)\right] = 1.$$

As $\sigma \to 0$, $\Phi\left(\frac{\kappa-\mu}{\sqrt{\left(\frac{\tau^4}{\sigma^4}\right)\frac{2\sigma^2\tau^2+\sigma^4}{\sigma^2+\tau^2}}}\right) \to \Phi(0) = \frac{1}{2}$. Hence, the formula simplifies to:

$$\beta_M \cdot \left[\kappa + v\left(\frac{m+1}{2}\right)\right] = 1.$$

Rearranging terms, we find:

$$\kappa = \frac{1}{\beta_M} - v\left(\frac{m+1}{2}\right).$$

This completes the proof.

□

Proof of Lemma 2. $\Pi^*_M(n)$ must be weakly increasing in n since M is less constrained in his choice of m as n increases. To prove the remainder of the lemma, it is sufficient to show that $\Pi_M(m, \beta^*(m))$ is strictly increasing in m for v large. Recall that:

$$\Pi_M(m,\beta) = \mu + v\cdot m(1-\beta m)\cdot \Pr(\theta > \kappa) - \beta m \cdot E(\theta|\theta > \kappa)\cdot \Pr(\theta > \kappa),$$

where $\kappa = \frac{1}{\beta} - v\left(\frac{m+1}{2}\right)$.

Observe that $E(\theta|\theta > \kappa)\cdot \Pr(\theta > \kappa)$ is bounded between 0 and $E(\theta|\theta > 0)\cdot \Pr(\theta > 0)$. Therefore, $\frac{1}{v}\Pi_M(m,\beta) \to m(1-\beta m)\cdot \Pr(\theta > \kappa) \equiv \tilde{\Pi}_M(m,\beta)$ as $v \to \infty$. To show $\Pi_M(m,\beta^*(m))$ is strictly increasing in m for v large, it will be sufficient to show $\tilde{\Pi}_M(m,\tilde{\beta}^*(m))$ is strictly increasing in m for v large, where $\tilde{\beta}^*(m) = \arg\max_\beta \tilde{\Pi}_M(m,\beta)$.

We can apply the Envelope Theorem to differentiate $\tilde{\Pi}_M(m,\tilde{\beta}^*(m))$ with respect to m:

$$\frac{d\tilde{\Pi}_M(m,\tilde{\beta}^*(m))}{dm} = m(1-\tilde{\beta}^*(m)m)\cdot\frac{vf(\kappa)}{2} + (1-2\tilde{\beta}^*(m)m)\cdot \Pr(\theta > \kappa),$$

where $\kappa = \frac{1}{\tilde{\beta}^*(m)} - v\left(\frac{m+1}{2}\right)$ and $f(\cdot)$ denotes the pdf of the $N(\mu, \tau^2)$ distribution. We see that, provided $\tilde{\beta}^*(m)m < \frac{1}{2}$, $\frac{d\tilde{\Pi}_M(m, \tilde{\beta}^*(m))}{dm} > 0$.

We can show, by contradiction, that $\tilde{\beta}^*(m)m < \frac{1}{2}$ for v large. In fact, we can show $\tilde{\beta}^*(m)m < \rho$ for any $\rho > 0$. Suppose $\tilde{\beta}^*(m)m \geq \rho$. The first-order condition for $\tilde{\beta}^*$ is:

$$m(1 - \tilde{\beta}^* m) f(\kappa) \left(\frac{1}{\tilde{\beta}^*}\right)^2 - m^2 \Pr(\theta > \kappa) = 0,$$

where $\kappa = \frac{1}{\tilde{\beta}^*} - v\left(\frac{m+1}{2}\right)$. Observe that $\kappa \to -\infty$ as $v \to \infty$ if $\tilde{\beta}^* m \geq \rho$. But, when $\kappa \to -\infty$, the left-hand side of the first-order condition converges to $-m^2$. The first-order condition is consequently violated, which is a contradiction. This proves the lemma.

□

Proof of Lemma 3. For now, let us take it as given that $G(\mathbf{d}_M, e_M)$ is weakly increasing in e_M (in a first-order stochastic dominance sense) and strictly increasing in e_M whenever $\mathbf{d}_M > \mathbf{0}$. Let us also take it as given that $G(\mathbf{d}_M, e_M)$ is weakly increasing in $\mathbf{d}_M$ and strictly increasing in $\mathbf{d}_M$ whenever $e_M > 0$.

From Lemma 2, we know that $\Pi_M^*(n)$ is weakly increasing in n. Since $\Pi_M^*(n)$ is weakly increasing in n and $G(\mathbf{d}_M, e_M)$ is weakly increasing in $\mathbf{d}_M$, $E[\Pi_M^*(n)|n \sim G(\mathbf{d}_M, e_M)] - c(e_M) - b_{(2)}$ is weakly increasing in $\mathbf{d}_M$. It follows from the Envelope Theorem that $V(\mathbf{d}_M)$ is weakly increasing in $\mathbf{d}_M$. This establishes part (1) of the lemma.

Now, assume $v > \hat{v}$. From Lemma 2, we know that $\Pi_M^*(n)$ is strictly increasing in n. Since $\Pi_M^*(n)$ is strictly increasing in n and $G(\mathbf{d}_M, e_M)$ is strictly increasing in $\mathbf{d}_M$ when $e_M > 0$, $E[\Pi_M^*(n)|n \sim G(\mathbf{d}_M, e_M)] - c(e_M) - b_{(2)}$ is strictly increasing in $\mathbf{d}_M$ when $e_M > 0$. It follows from the Envelope Theorem that $V(\mathbf{d}_M)$ is strictly increasing in $\mathbf{d}_M$ provided $e^*(\mathbf{d}_M) > 0$. While it remains to be shown, $e^*(\mathbf{d}_M) > 0$ whenever $\mathbf{d}_M > \mathbf{0}$; this establishes that $V(\mathbf{d}_M)$ is strictly increasing in $\mathbf{d}_M$.

Now, let us show that $e^*(\mathbf{d}_M) > 0$ when $\mathbf{d}_M > \mathbf{0}$ and $v > \hat{v}$. If $\mathbf{d}_M > \mathbf{0}$ and $v > \hat{v}$, $G(\mathbf{d}_M, e_M)$ is strictly increasing in e_M and $\Pi_M^*(n)$ is strictly increasing in n. It follows that $E[\Pi_M^*(n)|n \sim G(\mathbf{d}_M, e_M)]$ is strictly increasing in e_M. Therefore, the marginal benefit of exerting effort is greater than zero; at $e_M = 0$, the marginal cost of exerting effort is equal to zero ($c'(0) = 0$). Hence, it must be the case that $e^*(\mathbf{d}_M) > 0$.

It remains to show that $G(\mathbf{d}_M, e_M)$ is stochastically increasing in its arguments. Let B_{lk} denote a random variable that equals 1 when an investor l, connected to M with degree k, becomes aware of the project and 0 otherwise. We can write n, the total number of aware investors, as follows:

$$n = \sum_{k=1}^{\infty} \sum_{l=1}^{d_M^k} B_{lk}$$

Observe that the B_{lk}'s are independent random variables; B_{lk} follows a Bernoulli distribution with parameter $\delta^{k-1} \cdot e_M$. Hence n, which follows a $G(\mathbf{d}_M, e_M)$ distribution, is the sum of independent Bernoulli distributions.

The Bernoulli distribution is strictly increasing in its parameter (in a first-order stochastic dominance sense). An increase in e_M increases $\delta^{k-1} \cdot e_M$: the parameter of B_{lk}'s distribution. Therefore, provided $\mathbf{d_M} > \mathbf{0}$, $G(\mathbf{d}_M, e_M)$ is strictly increasing in e_M. When $\mathbf{d_M} = \mathbf{0}$, $n = 0$; so we can say that $G(\mathbf{d}_M, e_M)$ is weakly increasing in e_M for all $\mathbf{d}_M$.

Now, let us show that if $\mathbf{d}_i > \mathbf{d}_j$ and $e > 0$, $G(\mathbf{d}_i, e) >_{FOSD} G(\mathbf{d}_j, e)$. This will show that $G(\mathbf{d}_M, e_M)$ is strictly increasing in $\mathbf{d}_M$ when $e_M > 0$. When $e_M = 0$, $n = 0$; so we can say that $G(\mathbf{d}_M, e_M)$ is weakly increasing in $\mathbf{d}_M$ for all e_M.

Let us define $\mathbf{d}_{i1}$ as follows: $\mathbf{d}_{i1} = (d_j^1, [d_i^1 - d_j^1] + d_i^2, d_i^3, d_i^4, ...)$. Notice that $\mathbf{d}_i > \mathbf{d}_j$ implies $d_i^1 - d_j^1 \geq 0$. Observe that, if n follows a $G(\mathbf{d}_{i1}, e)$ distribution rather than a $G(\mathbf{d}_i, e)$ distribution, the difference is that $(d_i^1 - d_j^1)$ of the Bernoullis switch from parameter e to parameter $\delta \cdot e$. Hence $G(\mathbf{d}_i, e) \geq_{FOSD} G(\mathbf{d}_{i1}, e)$ and $G(\mathbf{d}_i, e) >_{FOSD} G(\mathbf{d}_{i1}, e)$ if $d_i^1 - d_j^1 > 0$.

We can similarly define $\mathbf{d}_{i2}$ as follows: $\mathbf{d}_{i2} = (d_j^1, d_j^2, [(d_i^1 + d_i^2) - (d_j^1 + d_j^2)] + d_i^3, d_i^4, d_i^5, ...)$. Notice that $\mathbf{d}_i > \mathbf{d}_j$ implies $[(d_i^1 + d_i^2) - (d_j^1 + d_j^2)] \geq 0$. Observe that, if n follows a $G(\mathbf{d}_{i2}, e)$ distribution rather than a $G(\mathbf{d}_{i1}, e)$ distribution, the difference is that $[(d_i^1 + d_i^2) - (d_j^1 + d_j^2)]$ of the Bernoullis switch from parameter $\delta \cdot e$ to parameter $\delta^2 \cdot e$. Hence $G(\mathbf{d}_{i1}, e) \geq_{FOSD} G(\mathbf{d}_{i2}, e)$ and $G(\mathbf{d}_{i1}, e) >_{FOSD} G(\mathbf{d}_{i2}, e)$ if $[(d_i^1 + d_i^2) - (d_j^1 + d_j^2)] > 0$.

More generally, define $\mathbf{d}_{il} = (d_j^1, ..., d_j^l, [\sum_{k=1}^l d_i^k - \sum_{k=1}^l d_j^k] + d_i^{l+1}, d_i^{l+2}, d_i^{l+3}, ...)$. By the same logic, $G(\mathbf{d}_{il}, e) \geq_{FOSD} G(\mathbf{d}_{i(l+1)}, e)$ and $G(\mathbf{d}_{il}, e) >_{FOSD} G(\mathbf{d}_{i(l+1)}, e)$ if $\sum_{k=1}^l d_i^k > \sum_{k=1}^l d_j^k$.

Since $\mathbf{d}_i > \mathbf{d}_j$, we know that $\sum_{k=1}^{l} d_i^k > \sum_{k=1}^{l} d_j^k$ for some l. So, $G(\mathbf{d}_{il}, e) >_{FOSD} G(\mathbf{d}_{i(l+1)}, e)$ for some l. We also know that $\lim_{l \to \infty} \mathbf{d}_{il} = \mathbf{d}_j$. It follows that $G(\mathbf{d}_i, e) >_{FOSD} G(\mathbf{d}_j, e)$. This completes the proof.

□

Proof of Proposition 2. As a first step in the proof, observe that if investor j does not receive an equity offer, he earns a payoff of 1 for sure. An investor j who does receive an equity offer earns an expected payoff strictly greater than 1. The reason is as follows. Recall from the proof of Lemma 1 that, when $x_i \leq \kappa$, j earns a payoff of 1 but when x_i is strictly greater than κ, j earns an expected payoff strictly greater than 1. Since we assumed $\sigma \to 0$, $\Pr(x_i > \kappa) = \Pr(\theta > \kappa)$. $\Pr(\theta > \kappa) > 0$ since θ is distributed $N(\mu, \tau^2)$.

Uniqueness. Suppose there is an investor j who does not link at time 0 to the eventual auction winner, Y, but instead links to another manager. Investor j earns a payoff of 1 since there is no chance of receiving an equity offer. Now suppose j deviates and links to Y. Investor j's deviation makes Y more connected than he was previously, so Y still wins the auction. Consequently, in deviating, j has a nonzero chance of receiving an equity offer so receives an expected payoff strictly greater than 1. Since the deviation is profitable, it follows that all players must link to the eventual auction winner in equilibrium. This establishes uniqueness.

Existence. To prove existence, we need to show that, in linking to manager Y at time 0, investors are best-responding. Observe that, when all investors link to Y, they receive an expected payoff strictly greater than 1 since there is a chance of receiving an equity offer. Suppose investor j deviates at time 0 and links to another manager. There are at least 3 investors, so even when j deviates, Y is still the most connected manager and wins the auction. Hence, in deviating, j links to a manager who loses the auction. Investor j receives a payoff of 1 from this deviation since there is no chance of receiving an equity offer. Therefore, j's deviation is not profitable. We conclude that players are indeed best responding in linking to Y.

□

Proof of Proposition 3. We can use backward induction to analyze the game.

Consider time 4. Let $\Delta_M = \alpha_M + v s_M$. Observe that the project yields a return

$R = \tilde{\theta} + v \cdot \sum_{j \in S} a_j$, where $\tilde{\theta} = \theta + \Delta_M$. The time 4 game is therefore analogous to the time 4 game in the baseline model, with $\tilde{\theta}$ substituted for θ. Consequently, we conclude that investors invest with probability 1 when $\tilde{\theta} > \kappa$; investor do not invest with probability 1 when $\tilde{\theta} < \kappa$. As before, $\kappa = \frac{1}{\beta_M} - v\left(\frac{m+1}{2}\right)$.

Now, consider time 3. Previously, we wrote M's expected share of the project's return as $\Pi_M(m, \beta_M)$. In this case, M's share also depends upon Δ_M. Therefore, we will write M's expected share of the project's return as $\Pi_M(m, \beta_M, \Delta_M)$. The following is a formula for $\Pi_M(m, \beta_M, \Delta_M)$ (which we obtain by substituting $\tilde{\theta}$ for θ in our previous formula for $\Pi_M(m, \beta_M)$):

$$\Pi_M(m, \beta_M, \Delta_M) = (\mu + \Delta_M) + vm(1 - \beta_M m) \Pr(\tilde{\theta} > \kappa) - \beta_M m E(\tilde{\theta} | \tilde{\theta} > \kappa) \Pr(\tilde{\theta} > \kappa),$$

where $\kappa = \frac{1}{\beta_M} - v\left(\frac{m+1}{2}\right)$ and $\tilde{\theta} = \theta + \Delta_M$.

As before, M will choose β_M and m to maximize Π_M subject to the constraint that $m \leq n$. We will write the value of Π_M when β_M and m are optimally chosen as $\Pi^*(n, \Delta_M)$. Observe that Lemma 2 carries over, since all we have done is substitute $\tilde{\theta}$ for θ. So, $\Pi^*(n, \Delta_M)$ is weakly increasing in n and strictly increasing in n for v large.

There are two further results that are useful to establish: (1) $\Pi^*(n, \Delta_M)$ is strictly increasing in Δ_M, and (2) $\frac{\partial \Pi^*(n, \alpha_M + v s_M)}{\partial s_M} > 1$ for v large.

First, let us establish (1). We would like to show that $\Pi^*(n, \Delta_1) > \Pi^*(n, \Delta_0)$ when $\Delta_1 > \Delta_0$. Let m_0^* and β_0^* denote the optimal choices of m and β_M when $\Delta_M = \Delta_0$. Let $m_1 = m_0^*$ and let $\beta_1 = \frac{1}{\frac{1}{\beta_0^*} + (\Delta_1 - \Delta_0)}$. It will be sufficient to show that $\Pi_M(m_1, \beta_1, \Delta_1) > \Pi^*(n, \Delta_0)$.

Three observations will be useful. First, notice that κ_1 – the value of κ corresponding to m_1 and β_1 – is equal to $\kappa_0 + (\Delta_1 - \Delta_0)$. Second, $\beta_1 < \beta_0^*$. Finally, we know that $m_0^* \beta_0^* < 1$. The reason is that it is never optimal to choose $m\beta_M \geq 1$: one could always do better by choosing $m\beta_M = 0$ instead. Consequently:

$$\begin{aligned}\Pi_M(m_1, \beta_1, \Delta_1) =&(\mu + \Delta_1) + vm_1(1 - \beta_1 m_1) \Pr(\tilde{\theta}_1 > \kappa_1)\\ &- \beta_1 m_1 E(\tilde{\theta}_1 | \tilde{\theta}_1 > \kappa_1) \Pr(\tilde{\theta}_1 > \kappa_1)\\ =&(\mu + \Delta_1) + vm_0^*(1 - \beta_1 m_0^*) \Pr(\tilde{\theta}_0 > \kappa_0)\\ &- \beta_1 m_0^* E(\tilde{\theta}_0 | \tilde{\theta}_0 > \kappa_0) \Pr(\tilde{\theta}_0 > \kappa_0) - \beta_1 m_0^*(\Delta_1 - \Delta_0) \Pr(\tilde{\theta}_0 > \kappa_0)\\ >&(\mu + \Delta_1) + vm_0^*(1 - \beta_0^* m_0^*) \Pr(\tilde{\theta}_0 > \kappa_0)\\ &- \beta_0^* m_0^* E(\tilde{\theta}_0 | \tilde{\theta}_0 > \kappa_0) \Pr(\tilde{\theta}_0 > \kappa_0) - (\Delta_1 - \Delta_0)\\ =&\Pi_M(m_0^*, \beta_0^*, \Delta_0) = \Pi_M^*(n, \Delta_0).\end{aligned}$$

This establishes (1).

Now, let us show (2):

$$\begin{aligned}\Pi_M(m, \beta_M, \alpha_M + vs_M) =&(\mu + \alpha_M + vs_M) + vm(1 - \beta_M m) \Pr(\tilde{\theta} > \kappa)\\ &- \beta_M m E(\tilde{\theta} | \tilde{\theta} > \kappa) \Pr(\tilde{\theta} > \kappa)\\ =&(\mu + \alpha_M + vs_M) + [vm(1 - \beta_M m)\\ &- \beta_M m(\alpha_M + vs_M)] \Pr(\theta > \kappa - \alpha_M - vs_M)\\ &- \beta_M m E(\theta | \theta > \kappa - \alpha_M - vs_M) \Pr(\theta > \kappa - \alpha_M - vs_M),\end{aligned}$$

where $\kappa = \frac{1}{\beta_M} - v\left(\frac{m+1}{2}\right)$.

Observe that $E(\theta | \theta > \kappa - \alpha_M - vs_M) \Pr(\theta > \kappa - \alpha_M - vs_M)$ is bounded between 0 and $E(\theta | \theta > 0) \Pr(\theta > 0)$. Therefore, $\frac{1}{v}\Pi_M(m, \beta_M, \alpha_M + vs_M) \to s_M + m(1 - \beta_M m - \beta_M s_M) \cdot \Pr(\theta > \kappa - \alpha_M - vs_M)$ as $v \to \infty$. So, in the limit as $v \to \infty$, M's problem becomes one of choosing m and β_M to maximize $\tilde{\Pi}_M(m, \beta_M) = s_M + m(1 - \beta_M m - \beta_M s_M) \cdot \Pr(\theta > \kappa - \alpha_M - vs_M)$. Let $\tilde{m}^*$ and $\tilde{\beta}_M^*$ denote the maximizing choices. Applying the Envelope Theorem, we find:

$$\begin{aligned}\frac{\partial \tilde{\Pi}_M(\tilde{m}^*, \tilde{\beta}_M^*)}{\partial s_M} = 1 + \tilde{m}^* v(1 - \tilde{\beta}_M^* \tilde{m}^* - \tilde{\beta}_M^* s_M) f(\kappa - \alpha_M - vs_M)\\ - \tilde{\beta}_M^* \tilde{m}^* \cdot \Pr(\theta > \kappa - \alpha_M - vs_M),\end{aligned}$$

where $\kappa = \frac{1}{\beta_M^*} - v\left(\frac{\tilde{m}^*+1}{2}\right)$ and f denotes the pdf of the $N(\mu, \tau^2)$ distribution. Recall from the proof of Lemma 2 we showed, for v sufficiently large, $\tilde{\beta}_M^* \tilde{m}^* < \rho$ for any $\rho > 0$. Or, put another way, $\tilde{\beta}_M^* \tilde{m}^* \to 0$ as $v \to \infty$. The same argument applies here. Therefore, the second term is positive for v large. The third term converges to zero as $v \to \infty$. Hence, for v large, $\frac{\partial \tilde{\Pi}_M(\tilde{m}^*, \tilde{\beta}_M^*)}{\partial s_M} \geq 1$.

We conclude that, for v large, $\frac{\partial \Pi^*(n, \alpha_M + v s_M)}{\partial s_M} \geq v$. So, it is certainly true that $\frac{\partial \Pi^*(n, \alpha_M + v s_M)}{\partial s_M} > 1$ for v large. This establishes (2).

Now, consider time 2 of the game. M's expected payoff at time 2 is: $E[\Pi^*(n, \alpha_M + v s_M) | n \sim G(\mathbf{d_M}, e_M)] - c(\frac{e_M}{\gamma_M}) - s_M - b_{(2)}$. M will choose e_M and s_M to maximize this expression subject to the constraint that $s_M \leq k_M$. We can write M's resulting payoff as: $V(\mathbf{d}_M, \alpha_M, \gamma_M, k_M) - b_{(2)}$.

Lemma 3 clearly carries over. So, V is weakly increasing in $\mathbf{d_M}$ and strictly increasing in $\mathbf{d_M}$ provided v is large ($v > \hat{v}$). Furthermore, $e_M^* > 0$ whenever v is large ($v > \hat{v}$) and $\mathbf{d_M} > \mathbf{0}$.

We showed that $\Pi^*(n, \alpha_M + v s_M)$ is strictly increasing in α_M. So, by the Envelope Theorem, V is strictly increasing in α_M. The Envelope Theorem also implies that V is weakly increasing in γ_M and strictly increasing provided $e_M^* > 0$.

As k_M increases, this simply makes M less constrained in his choice of s_M. So, it is clear that V is weakly increasing in k_M. We showed that, for v large ($v > \hat{v}$), $\frac{\partial \Pi^*(n, \alpha_M + v s_M)}{\partial s_M} > 1$. Hence, when $v > \hat{v}$, the marginal benefit of increasing s_M always exceeds the marginal cost (which is equal to 1). So, it is optimal for M to choose $s_M = k_M$. Additionally, an increase in k_M strictly increases M's payoff, V.

Finally, turning back to time 1, we know that managers will bid their valuations of asset A in the auction: $b_i = V(\mathbf{d_i}, \alpha_i, \gamma_i, k_i)$. This completes the proof.

□

Proof of Proposition 4. As in the baseline model, if an investor j does not receive an equity offer, he earns a payoff of 1 for sure. An investor j who does receive an equity offer earns an expected payoff strictly greater than 1.

Uniqueness. Suppose there is an investor j who does not link at time 0 to the eventual auction winner, Y, but instead links to another manager. Investor j earns a payoff of 1 since there is no chance of receiving an equity offer. Now suppose j deviates and links to Y. Investor j's deviation makes Y more connected than he was previously, so Y still wins the auction. Consequently, in deviating, j has a nonzero chance of receiving an equity offer so receives an expected payoff strictly greater than 1. Since the deviation is profitable, it follows that all players must link to the eventual auction winner in equilibrium. This establishes uniqueness.

Existence. Suppose $V(\mathbf{d}_{max} - \hat{\mathbf{d}}, \alpha_i, \gamma_i, k_i) > \max_{j \in N_M} V(\hat{\mathbf{d}}, \alpha_j, \gamma_j, k_j)$. We need to show that an equilibrium exists in which $i = Y$. One condition that must be met for such an equilibrium to exist is that manager i outbids other managers in the auction when he is connected to all the investors: $V(\mathbf{d}_{max}, \alpha_i, \gamma_i, k_i) > \max_{j \in N_M} V(\mathbf{0}, \alpha_j, \gamma_j, k_j)$. Observe that this follows from V being an increasing in $\mathbf{d}$ and $V(\mathbf{d}_{max} - \hat{\mathbf{d}}, \alpha_i, \gamma_i, k_i) > \max_{j \in N_M} V(\hat{\mathbf{d}}, \alpha_j, \gamma_j, k_j)$. The second condition that must be met for existence is that there is no profitable deviation at time 0 for investors. Observe that, when all investors link to manager i, they receive an expected payoff strictly greater than 1 since there is a chance of receiving an equity offer. Suppose investor j deviates at time 0 and links to another manager. Since $V(\mathbf{d}_{max} - \hat{\mathbf{d}}, \alpha_i, \gamma_i, k_i) > \max_{j \in N_M} V(\hat{\mathbf{d}}, \alpha_j, \gamma_j, k_j)$, manager i still wins the auction. Hence, in deviating, j links to a manager who loses the auction. Investor j receives a payoff of 1 from this deviation since there is no chance of receiving an equity offer. Therefore, j's deviation is not profitable. We conclude that an equilibrium indeed exists in which manager $i = Y$.

Suppose $V(\mathbf{d}_{max}, \alpha_i, \gamma_i, k_i) > \max_{j \in N_M} V(\mathbf{0}, \alpha_j, \gamma_j, k_j)$. An equilibrium does not exist in which $i = Y$ since, when all investors link to manager i, he is outbid in the auction. This completes the proof.

□

5.2 Distributions of K and R

This section provides more detail regarding how to calculate the distributions of K and R.

Recall from Section 2.3 that:

$$K = m^*(n) \cdot \mathbb{1}_{\{\theta > \kappa^*(n)\}} \text{ almost surely,}$$

where $\kappa^*(n) = \frac{1}{\beta^*_M(m^*(n))} - v\left(\frac{m^*(n)+1}{2}\right)$. Since θ and n are independent random variables, distributed $N(\mu, \tau^2)$ and $G(\mathbf{d}_M, e^*(\mathbf{d}_M))$ respectively, we conclude that:

$$\Pr(K = k) = \sum_{\{\hat{n}:m^*(\hat{n})=k\}} \Pr(n = \hat{n}) \cdot (1 - F(\kappa^*(\hat{n})) \text{ for all } k \geq 1.$$

$$\Pr(K = 0) = 1 - \sum_{k \geq 1} \Pr(K = k).$$

F denotes the cdf of the $N(\mu, \tau^2)$ distribution. $\Pr(n = \hat{n})$ is the probability that $n = \hat{n}$ given that n is distributed $G(\mathbf{d}_M, e^*(\mathbf{d}_M))$.

We can calculate $\Pr(n = \hat{n})$ from the observation that:

$$n = \sum_{k=1}^{\infty} \sum_{l=1}^{d_M^k} B_{lk},$$

where the B_{lk}'s are independent random variables with B_{lk} following a Bernoulli distribution with parameter $\delta^{k-1} \cdot e^*(\mathbf{d}_M)$.

In the numerical example from Figure 1, we assume M only has direct connections. When M has d direct connections and no indirect connections, n simply follows a Binomial distribution with parameters d and e^*_M. Hence, in this case: $\Pr(n = \hat{n}) = \binom{d}{\hat{n}}(e^*_M)^{\hat{n}}(1 - e^*_M)^{d-\hat{n}}$.

Now, let us characterize the distribution of R. We can denote the cdf and pdf of R by $H(r)$ and $h(r)$ respectively. Observe that:

$$\begin{aligned} H(r) &= \Pr(R \leq r) \\ &= \Pr(\theta \leq r - vK) \\ &= \sum_k F(r - vk) \Pr(K = k). \end{aligned}$$

Differentiating, we find that:

$$h(r) = \sum_k f(r - vk) \Pr(K = k).$$

5.3 Distributions of K and R (More General)

This section shows how to calculate the distributions of K and R for the generalized version of the model presented in Section 3.

In Section 2.3, we showed that for the baseline model:

$$K = m^*(n) \cdot \mathbb{1}_{\{\theta > \kappa^*(n)\}} \text{ almost surely,}$$

where $\kappa^*(n) = \frac{1}{\beta_M^*(m^*(n))} - v\left(\frac{m^*(n)+1}{2}\right)$.

From the proof of Proposition 3, we obtain the following generalization:

$$K = s_M^* + m^*(n, \alpha_M, s_M^*) \cdot \mathbb{1}_{\{\theta + \alpha_M + v s_M^* > \kappa^*(n, \alpha_M, s_M^*)\}} \text{ almost surely,}$$

where $\kappa^*(n, \alpha_M, s_M^*) = \frac{1}{\beta_M^*(n, \alpha_M, s_M^*)} - v\left(\frac{m^*(n, \alpha_M, s_M^*)+1}{2}\right)$.

Since θ and n are independent random variables, distributed $N(\mu, \tau^2)$ and $G(\mathbf{d}_M, e^*(\mathbf{d}_M))$ respectively, we conclude that:

For all $k \geq 1$,

$$\Pr(K = k + s_M^*) = \sum_{\{\hat{n} : m^*(\hat{n}, \alpha_M, s_M^*) = k\}} \Pr(n = \hat{n}) \cdot (1 - F(\kappa^*(\hat{n}, \alpha_M, s_M^*) - \alpha_M - v s_M^*)).$$

$$\Pr(K = s_M^*) = 1 - \sum_{k \geq 1} \Pr(K = k + s_M^*).$$

F denotes the cdf of the $N(\mu, \tau^2)$ distribution. $\Pr(n = \hat{n})$ is the probability that $n = \hat{n}$ given that n is distributed $G(\mathbf{d}_M, e^*(\mathbf{d}_M))$.

As before, we can calculate $\Pr(n = \hat{n})$ from the observation that:

$$n = \sum_{k=1}^{\infty} \sum_{l=1}^{d_M^k} B_{lk},$$

where the B_{lk}'s are independent random variables with B_{lk} following a Bernoulli distribution with parameter $\delta^{k-1} \cdot e^*(\mathbf{d}_M)$.

Now, let us characterize the distribution of R. Again, we can denote the cdf and pdf of R by $H(r)$ and $h(r)$ respectively. Observe that:

$$\begin{aligned}
H(r) &= \Pr(R \leq r) \\
&= \Pr(\theta + vK + \alpha_M \leq r) \\
&= \Pr(\theta \leq r - \alpha_M - vK) \\
&= \sum_k F(r - \alpha_M - v(s_M^* + k)) \Pr(K = s_M^* + k).
\end{aligned}$$

Differentiating, we find that:

$$h(r) = \sum_k f(r - \alpha_M - v(s_M^* + k)) \Pr(K = s_M^* + k).$$