Housing Consumption CAPM and the Term Structure of Interest Rates

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Abstract

This paper develops an affine term structure model of nominal interest rates based on a housing consumption CAPM framework proposed in Piazzesi et al. (2007). Different from standard CCAPM, the agent is not only concerned with consumption risk but also the composition of that consumption. This gives rise to a new pricing kernel that depends not only on the standard non-housing consumption but also on the relative share of housing and non-housing expenditures. We model the dynamics of this pricing kernel in a long-run risk specification where each component of the pricing kernel contains a small long-run predictable component. We calibrate the model to U.S. data on non-housing and housing consumptions. We find that the model can generate positive term spreads and bond yields that are very close to the data, which therefore resolves the term premium puzzle faced by standard CCAPM.

Keywords: Housing, Consumption CAPM, Term Structure Models

JEL Classification: E43, E44, G12

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1 Introduction

Consumption-based capital asset pricing model (CCAPM) is an important branch of asset pricing models. It derives pricing equation from the first-order condition of an optimizing investor who decides how much to save and how much to consume. The basic intuition is that the marginal utility loss of consuming a little less today and buying a little more of the asset should equal the marginal utility gain of consuming a little more of the asset’s payoff in the future. This follows that the asset’s price should equal the expected discounted value of the asset’s payoff, with the investor’s marginal utility as the discount factor.\(^1\) Despite this intuitively appealing characterization for the economy and hence the asset price associated with it, standard CCAPM has not found much empirical support, producing a bunch of asset pricing puzzles, for example, the equity premium puzzle (Mehra and Prescott 1985) and the term premium puzzle (Backus et al. 1989).\(^2\)

Different methods have been developed to resolve these empirical difficulties, for example, the Campbell-Cochrane model of habit persistence that challenges the standard model by modifying preferences; the Constantinides-Duffie model of heterogeneous agents that brings uninsured idiosyncratic risks into the economy. Recent focus in the literature has switched to the composition risk of consumption, that is, agents are not only concerned with the overall consumption risk but also the composition risk of that consumption. For example, Piazzesi et al. (2007) puts forward a consumption-based asset pricing model that separates housing services from other forms of consumption in the utility function and find that the presence of composition risk does help at generating a sizeable and volatile equity premium observed in the data.\(^3\)

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\(^1\)An investor’s first-order condition gives the basic consumption-based pricing formula, in which \(P_t = E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}]\). The marginal rate of substitution \(\beta \frac{u'(c_{t+1})}{u'(c_t)}\) is therefore the discount factor or pricing kernel in the economy.

\(^2\)The equity premium puzzle - excess stock returns predicted by the standard consumption model are too low to match the US data, first documented in Mehra and Prescott, 1985. The term premium puzzle - model generated long rate in excess of the short rate is negative instead of positive, first documented in Backus et al., 1989.

\(^3\)Other studies that consider the explicit role of housing consumption for asset pricing also include
While the existing application is the resolution of equity premium puzzle, this paper uses the insight of composition risk to study another risk premium puzzle, specifically, the behavior of the term premium. The term premium puzzle was first documented in Backus et al. (1989), in which the authors find that standard consumption model predicts an average return of long bonds in excess of the short rate negative and small, whereas the observed term premium in the data is positive. As pointed out in Salyer (1995), the term premium puzzle is in some sense a stronger challenge of the underlying asset pricing model since instead of a problem with the degree of risk associated with an asset as in the equity premium puzzle, the failure of the term structure model implies it is the nature (sign) of risk associated with bonds that is the issue.

As in Piazzesi et al. (2007), we model aggregate consumption as a quantity index that aggregates housing services and numeraire (nonhousing) consumption through a CES contemporaneous utility function. This non-separable preference between housing and nonhousing consumption gives rise to a new pricing kernel that depends not only on consumption but also on the component of consumption, measured by the relative share of housing and nonhousing expenditures. As a result, the representative agent in the economy is worried not only when numeraire consumption tomorrow is low, but also when the relative consumption of housing services tomorrow is low. This is exactly the extra composition risk brought into the model.

With respect to the term premium, the composition risk alone is not enough, as in a CCAPM framework, an asset commands a premium when its conditional covariance with the pricing kernel is negative, for a fixed-income asset like the bond, which has no random cash flows, this covariance is reduced to the covariance between the pricing kernel with its expected future values. To capture the positive sign of term premium, we therefore also need the second ingredient: a model describing the dynamic processes for the pricing kernel that can produce negative autocorrelation in the pricing kernel. We consider a long-run risk specification like those used in Bansal and Yaron (2004) for the three factors in the pricing kernel. Specifically, each pricing factor (i.e., numeraire

Lustig and Van Nieuwerburgh (2005), Davis and Martin (2005) and so on.
consumption growth, consumption expenditure share growth, inflation) contains (1) a long-run predictable component that determines its conditional expectation and (2) a random disturbance that is correlated across the factors. These dynamics allow for both persistent and correlated changes in the endowment processes, which are important in generating negative correlations among the pricing factors.

Under normality assumptions, our model of bond yields falls within the affine bond pricing with three factors. We then estimate this affine term structure model using quarterly data on consumption and bond yields in the US. We get our measures of housing and nonhousing consumption from the National Income and Product Account (NIPA). We also construct the corresponding price index for nonhousing consumption in a similar way to that in Piazzesi et al. (2007). The model is then estimated through maximum likelihood estimation based on a state-space representation with Kalman Filter. Our estimation results show that (1) the long-run risk specification is consistent with many salient features of observed data on housing and nonhousing consumption in the U.S. (2) Evaluated at model estimates, the model predicted bond yields move very closely to the data, especially that the average spread between long-term and short-term bonds is positive. Moreover, our empirical results are based on a risk aversion of 3.47 and an elasticity of substitution between housing and nonhousing consumption of 1.6, which are reasonable values supported by the literature. These all suggest that a housing CCAPM with long-run risks for the endowment processes can resolve the term premium puzzle and capture the behavior of bond yields very well.

The rest of the paper is organized as follows. In section 2, we develop an affine term structure model of nominal interest rates based on the housing CCAPM framework proposed in Piazzesi et al. (2007): we first describe the insight of a housing CCAPM and the form of the pricing kernel; we then put forward the dynamic processes for the pricing factors; we finally derive bond yields with different maturities and get the affine term structure representation. In section 3, we go to the empirical part: we first describe the data and then estimate the model using maximum likelihood estimation. In section 4, we examine the model’s quantitative performance in explaining bond yields. Section 5
concludes.

2 Model

In this section we develop our model of nominal interest rates. We first introduce the setup of housing CCAPM and derive the pricing kernel. We then model the dynamics of the pricing kernel and derive the term structure of interest rates.

2.1 Housing CCAPM and the Pricing Kernel

The housing CCAPM proposed in Piazzesi et al. (2007) considers whether separating housing services from other forms of consumption may improve the performance of standard consumption models. The model decomposes aggregate consumption $C_t$ into housing service consumption $h_t$ and nonhousing consumption $c_t$, which is the consumption of all nondurables and services except housing services:

$$C_t = g(c_t, h_t) = (c_t^{(\varepsilon - 1)/\varepsilon} + \omega h_t^{(\varepsilon - 1)/\varepsilon})^{\varepsilon/(\varepsilon - 1)}$$

Preferences over aggregate consumption are time separable

$$E_t \left[ \sum_{j=0}^{\infty} \beta^j u(C_{t+j}) \right]$$

with the contemporaneous utility function taking the standard form

$$u(C_t) = \frac{C_t^{1-1/\theta}}{1 - 1/\theta}$$

The parameter $\theta$ stands for the inter-temporal elasticity of substitution, and $\varepsilon$ represents the intra-temporal elasticity of substitution between housing consumption and nonhousing consumption. For high values of $\varepsilon$, the agent is willing to substitute housing and nonhousing consumptions within each period. The agent in a housing CCAPM setup not only wants to smooth the consumption across time, but also try to smooth consumption across two goods. $\beta$ is his standard subjective discount factor. $\omega$ is the weight given to housing consumption in the utility function.
The central asset pricing formula comes from the Euler equation that describes the agent’s optimal choice of consumption

\[ 1 = E[M_{t+1}R_{t+1}] \]

where \( M_{t+1} \) is the stochastic discount factor or the pricing kernel, \( R_{t+1} \) is the return on real assets. It is worth to note that for keeping a more convenient track for asset prices, the housing CCAPM literature has used the nonhousing consumption \( c_t \) as the numeraire consumption (Piazzesi et al., 2007). Therefore, the real returns here are measured in units of nohousing consumption.

The pricing kernel \( M_{t+1} \) is the present value of an extra unit of numeraire consumption tomorrow, that is,

\[
M_{t+1} = \beta \frac{u'(C_{t+1})g_c(c_{t+1}, h_{t+1})}{u'(C_t)g_c(c_t, h_t)} = \beta \left( \frac{C_{t+1}}{c_t} \right)^{-\frac{\beta}{\gamma}} \left( 1 + \omega \left( \frac{h_{t+1}}{c_{t+1}} \right)^{(\varepsilon-1)/\varepsilon} \right)^{\frac{\phi-\varepsilon}{\phi}}
\]

(1)

It represents the date \( t \) prices of contingent claims that pay off in \( t+1 \). \( M_{t+1} \) is large over states that the agent feels bad. In the housing CCAPM, it consists of two terms. The first term is familiar from the standard CCAPM with power utility: the agent feels bad when future consumption growth \( \left( \frac{c_{t+1}}{c_t} \right) \) is low. We call this a concern with consumption risk. The lower the inter-temporal elasticity of substitution \( \theta \), the larger is the effect of consumption risk. The second term is new. If preference is separable across two goods, i.e. when \( \theta = \varepsilon \), this term collapses to one and consumption risk alone matters for asset pricing. If preference is nonseparable, the second term reflects the agent’s concern with composition risk. Suppose that the intra-temporal elasticity of substitution is larger than one and the inter-temporal elasticity \( (\varepsilon > 1, \varepsilon > \theta) \), the agent also feels bad when the ratio of housing consumption \( \left( \frac{h_{t+1}}{c_{t+1}} \right) \) in the future is relatively lower than today.

The pricing kernel (1) involves the real relative quantities \( \frac{h_t}{c_t} \). However, the price \( p_t^h \) and quantity \( h_t \) of housing services are difficult to measure.\(^4\) Piazzesi et al. (2007) therefore gets around this problem by showing that the pricing kernel can be equivalently written

\(^4\)The major concern is that readily available measures of real housing consumption such as square
in terms of expenditure shares. Now define
\[ \alpha_t \equiv 1 + \omega \left( \frac{h_t}{c_t} \right)^{(\varepsilon -1)/\varepsilon} \]

It can be verified from the first-order condition \( \frac{p_t^c}{p_t^h} = \frac{g_t(c_t, h_t)}{g_t(c_t, h_t)} = \omega^{-1} \left( \frac{c_t}{h_t} \right)^{-1/\varepsilon} \) that \( \alpha_t \) equals exactly the share of expenditures of nonhousing consumption to total consumption, that is, \( \alpha_t = \frac{p_t^c c_t}{p_t^h c_t + p_t^h h_t} \). The pricing kernel (1) can therefore be rewritten as the following equivalent form, in terms of nonhousing consumption and its expenditure share:
\[ M_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\theta}} \left( \frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{\varepsilon - \theta}{\varepsilon (\varepsilon - 1)}} \] (2)

The pricing kernel (2) is exactly the two-factor kernel used to price real assets under the housing CCAPM framework. To price nominal assets that pay off in dollars, such as nominal bonds, whose data are much more widely available than real bonds, we need to model the nominal pricing kernel as well. Denote the nominal price index for numeraire consumption at time \( t \) as \( \Pi_t \), the Euler equation must hold for real returns (in units of numeraire consumption) on nominal bonds
\[ \frac{1^S}{\Pi_t} = E_t \left[ M_{t+1} \frac{R_{t+1}^S}{\Pi_{t+1}} \right] \]
\[ 1^S = E_t \left[ M_{t+1} \frac{\Pi_{t+1}}{\Pi_t} R_{t+1}^S \right] \]

This gives rise our nominal pricing kernel as
\[ M_{t+1}^S \equiv M_{t+1} \left( \frac{\Pi_{t+1}}{\Pi_t} \right)^{-1} \]
\[ = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\frac{1}{\theta}} \left( \frac{\alpha_{t+1}}{\alpha_t} \right)^{\frac{\varepsilon - \theta}{\varepsilon (\varepsilon - 1)}} \left( \frac{\Pi_{t+1}}{\Pi_t} \right)^{-1} \] (3)

In log terms, we have
\[ m_{t+1}^S = \ln \beta - \frac{1}{\theta} \Delta \ln c_{t+1} + k \Delta \ln \alpha_{t+1} - \Delta \ln \Pi_{t+1} \] (4)
where $\Delta \ln c_{t+1}$ is the difference in log numeraire consumption, which is essentially numeraire consumption growth rate. Similarly, $\Delta \ln \alpha_{t+1}$ is nonhousing consumption expenditure share growth rate, and $\Delta \ln \Pi_{t+1}$ is numeraire inflation rate, $k = \frac{\varepsilon - \theta}{\theta(\varepsilon - 1)}$.

The nominal pricing kernel (3) enables us to price any nominal assets in the economy, including nominal bonds. Denote the price of a bond that pays one dollar $n$ periods later as $P_{nt}^\delta$. The price is therefore determined as the expected value of its payoff tomorrow weighted by the pricing kernel. Solving it forward suggests that it is determined by the expected values of future pricing kernel:

$$P_{nt}^\delta = E_t(P_{n-1,t+1}^\delta M_{t+1}^\delta) = E_t(\prod_{i=1}^{n} M_{t+i}^\delta)$$  \hspace{1cm} (5)

Assuming that the log pricing kernel is normally distributed, then by the property of log-normal distribution, we get log price

$$p_{nt}^\delta = E_t(\sum_{i=1}^{n} m_{t+i}^\delta) + \frac{1}{2} \text{var}_t(\sum_{i=1}^{n} m_{t+i}^\delta)$$  \hspace{1cm} (6)

The yield for a continuously compounded $n$-period nominal bond is then defined from the relation

$$y_{nt}^\delta = -\frac{1}{n} \ln P_{nt}^\delta = -\frac{1}{n} E_t(\sum_{i=1}^{n} m_{t+i}^\delta) - \frac{1}{2n} \text{var}_t(\sum_{i=1}^{n} m_{t+i}^\delta)$$  \hspace{1cm} (7)

For a fixed date $t$, the yield curve maps the maturity $n$ of a bond to its yield $y_{nt}^\delta$.

Equations (6) and (7) show that log prices and yields of bonds are determined by expected values of future numeraire consumption growth rate, expenditure share growth rate and inflation rate. Take the short rate for example

$$y_{1t}^\delta = -\ln \beta + E_t(\frac{1}{\theta} \Delta \ln c_{t+1} - k \Delta \ln \alpha_{t+1} + \Delta \ln \Pi_{t+1})$$

$$-\frac{1}{2} \text{var}_t(\frac{1}{\theta} \Delta \ln c_{t+1} - k \Delta \ln \alpha_{t+1} + \Delta \ln \Pi_{t+1})$$

The effects of expected numeraire consumption growth and inflation are familiar from the standard CCAPM: a lower consumption in the future than today makes the agent save more today, thus decreases interest rate - consumption risk; a higher inflation in the future
makes the agent prefer to consume today rather than tomorrow hence raises interest rate - inflation risk. Now within the housing CCAPM, the third factor of expenditure share matters: a lower expenditure share on housing consumption in the future also makes the agent save more today, thus raises bond prices and decreases interest rate - composition risk.

Since nominal bond yields and hence the nominal term structure of interest rates are essentially determined by the conditional expectations and volatilities of future housing and nonhousing consumption in this endowment economy. Therefore, it’s necessary to model the dynamic processes of the endowment. That’s what the next subsection is about.

2.2 Dynamic Processes of the Economy

We model the dynamics for numeraire consumption growth, nonhousing expenditure share growth and numeraire inflation in a long-run risk specification like those used in Bansal and Yaron (2004), that is, each of them contains a long-run predictable component that determines its conditional expectation and a random disturbance that is correlated across each other:

\[
\Delta \ln c_{t+1} = \mu_c + x_t + e_{t+1} \\
x_t = \rho_x x_{t-1} + \sigma_x e_t \\
\Delta \ln \alpha_{t+1} = \mu_\alpha + y_t + \eta_{t+1} \\
y_t = \rho_y y_{t-1} + \sigma_y \eta_t \\
\Delta \ln \Pi_{t+1} = \mu_\Pi + z_t + \psi_{t+1} \\
z_t = \rho_z z_{t-1} + \sigma_z \psi_t
\]

Predictable components \(x_t, y_t, z_t\) have AR(1) form, with parameters \(\rho_x, \rho_y, \rho_z\) governing the persistence of expected processes and \(\sigma_x, \sigma_y, \sigma_z\) governing the disturbances to the expected processes. Shocks to the economy are allowed to be correlated and governed by
a variance-covariance matrix $\Omega$:

$$e_{t+1}, \eta_{t+1}, \psi_{t+1} \sim N(0, \Omega), \text{ where } \Omega = \begin{pmatrix}
\sigma_e^2 & \sigma_{e,\eta} & \sigma_{e,\psi} \\
\sigma_{\eta,e} & \sigma_{\eta}^2 & \sigma_{\eta,\psi} \\
\sigma_{\psi,e} & \sigma_{\psi,\eta} & \sigma_{\psi}^2
\end{pmatrix}$$

This specification allows for both persistent and correlated changes in the endowment dynamics. In the following sections, we will show that this specification not only captures the dynamics of housing and nonhousing consumption observed in the data but also plays a critical role in our bonds pricing and producing yields properties that are consistent with the data.

2.3 The Term Structure of Interest Rates

Under normality and linearity of the model specified in (8), we show that log bond prices defined in equation (6) have the following affine form

$$p_{nt}^3 = A_n + B_n x_t + C_n y_t + D_n z_t$$

where $A_n, B_n, C_n, D_n$ can be calculated as follows (see the Appendix for detailed derivations):

$$A_n = A_{n-1} + \ln \beta - \frac{1}{\theta} \mu_c + k \mu_c + \frac{1}{2} \lambda_n \Omega \lambda_n$$

$$B_n = -\frac{1}{\theta} \cdot \frac{1 - \rho_x^n}{1 - \rho_x}$$

$$C_n = k \cdot \frac{1 - \rho_y^n}{1 - \rho_y}$$

$$D_n = -\frac{1 - \rho_z^n}{1 - \rho_z}$$

with $\lambda_n = (-\frac{1}{\theta} + B_{n-1} \sigma_x k + C_{n-1} \sigma_y - 1 + D_{n-1} \sigma_z)'$ and $A_0 = B_0 = C_0 = D_0 = 0$.

The nominal term structure is therefore given by:
\[ y_{n,t}^s = \frac{1}{n}p_{n,t}^s \]
\[ = \frac{1}{n}(A_n + B_n x_t + C_n y_t + D_n z_t) \]
\[ = \tilde{A}_n + \tilde{B}_n x_t + \tilde{C}_n y_t + \tilde{D}_n z_t \]

This falls within the affine term structure model with three factors, where yields are affine functions of the economy's state variables. Here the state variables are the persistent components of consumption growth, consumption expenditure share growth and inflation.

### 3 Data and Estimation

The term structure model described in Section 2 generates nominal bond yields which are linear functions of state variables. In this section, we estimate the model using U.S. data on consumption and yields. Before laying out the estimation method we use, we first give an introduction of the data and the method we use to construct each index.

#### 3.1 Data

The consumption data we use are from the National Income and Product Accounts (NIPA). First, we define nonhousing consumption as nondurables (excluding clothing and footwear) and services subtracting housing and utilities. The expenditure share data are easy to construct. NIPA Table 2.3.5 reports personal consumption expenditures over products. The housing expenditure data correspond to line 15. The nonhousing consumption expenditures correspond to summing up lines 8 and 13, subtracting lines 10 and 15. The expenditure share of nonhousing consumption \((\alpha_t)\) is therefore constructed as the expenditure on nonhousing consumption divided by the total expenditure on both nonhousing and housing consumption. The data are at a quarterly frequency from 1959Q1

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5We follow the convention of excluding clothing and footwear, because they may be viewed as durable (see, for example, Lettau and Ludvigson, 2001).
to 2005Q3.\textsuperscript{6}

As for real nonhousing consumption itself ($c_t$), which serves as the numeraire consumption, we get its measure by first constructing a nonhousing consumption price index, then deflating the nonhousing consumption expenditure by the constructed price index. The nonhousing consumption price index is constructed as a weighted average of the price indexes reported in NIPA Table 2.3.4 for major consumption products that correspond to our definition of $c_t$. The weight we are using is the 2005 expenditure share of each item in the chosen category.\textsuperscript{7}

Figure 1 plots the constructed data on the expenditure share $\alpha_t$ as well as its growth rate during our sample period. It can be seen that $\alpha_t$ varies little over time, centering in the range between 0.77 and 0.79. This implies that agents spend around the same fraction of their expenditures on housing service, although their total consumption keeps increasing over time.

Figure 1: Non-housing Expenditure Share and its Growth Rate

Figures 2 and 4 plot the constructed numeraire consumption growth rate and inflation rate in the data, together with our model estimated counterparts. We will discuss these

\textsuperscript{6}NIPA has consumption data available much longer than this. However, since the yields data we have start from 1959, we build our sample from 1959 as well.

\textsuperscript{7}Year 2005 is the base year used in NIPA.
two figures in more details in the next section. But one direct feature about the data on nonhousing consumption and inflation is that they behave much like the aggregate consumption growth and CPI inflation.

The nominal term structure data are from CRSP. Bond yields with maturities one year and longer are from CRSP Fama-Bliss discount data file. The one-quarter short rate is from CRSP Fama riskfree rate file.

3.2 Estimation

We estimate the model by first rewriting the long-run risk specification (8) in a state-space representation. The persistent components \( x_t, y_t, z_t \) are the state variables that determine both endowment dynamics and bond yields in the economy. Their AR(1) dynamics can therefore be modeled exactly as the state equation. The measurement equation consists of not only nonhousing consumption growth, nonhousing consumption expenditure share growth and numeraire inflation but also two yields. Bond yields predicted by the model can be written as affine functions of the state variables, as we’ve shown in the previous section. Therefore, including yields in the measurement equation can help recover the underlying state dynamics.

State equation:

\[
\begin{pmatrix}
  x_{t+1} \\
  y_{t+1} \\
  z_{t+1}
\end{pmatrix} =
\begin{pmatrix}
  \rho_x & 0 & 0 \\
  0 & \rho_y & 0 \\
  0 & 0 & \rho_z
\end{pmatrix}
\begin{pmatrix}
  x_t \\
  y_t \\
  z_t
\end{pmatrix} +
\begin{pmatrix}
  \sigma_x & 0 & 0 \\
  0 & \sigma_y & 0 \\
  0 & 0 & \sigma_z
\end{pmatrix}
\begin{pmatrix}
  e_{t+1} \\
  \eta_{t+1} \\
  \psi_{t+1}
\end{pmatrix}
\]

Measurement equation:

\[
\begin{pmatrix}
  \Delta \ln c_{t+1} \\
  \Delta \ln \alpha_{t+1} \\
  \Delta \ln \Pi_{t+1} \\
  y_{1,t} \\
  y_{4,t}
\end{pmatrix} =
\begin{pmatrix}
  \mu_c \\
  \mu_\alpha \\
  \mu_\pi \\
  -A_1 \\
  -A_4/4
\end{pmatrix}
+ \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -B_1 & -C_1 & -D_1 \\
  -B_4/4 & -C_4/4 & -D_4/4
\end{pmatrix}
\begin{pmatrix}
  x_t \\
  y_t \\
  z_t \\
  u_t \\
  \omega_t
\end{pmatrix} +
\begin{pmatrix}
  e_{t+1} \\
  \eta_{t+1} \\
  \psi_{t+1}
\end{pmatrix}
\]
Here $y_{1,t}^s$ and $y_{4,t}^s$ represent three-month and one-year nominal interest rates. $A_n, B_n, C_n, D_n$ are as defined in (??), which are functions of deep time series parameters such as $\rho$’s and $\sigma$’s as well as preference parameters such as $\theta$ and $\varepsilon$. We also allow for measurement errors in bonds pricing so that there would be more degrees of freedom in estimation to match both endowment dynamics and yields.

The system is then estimated through maximum likelihood estimation with Kalman Filter applied to sequentially update the estimates for state variables. Appendix B presents the details of derivation. The estimation results are reported in Table 1.

<table>
<thead>
<tr>
<th>$\rho_x$</th>
<th>$\rho_y$</th>
<th>$\rho_z$</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>$\sigma_z$</th>
<th>$\mu_c$</th>
<th>$\mu_\alpha$</th>
<th>$\mu_\pi$</th>
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<td>0.200</td>
<td>0.897</td>
<td>0.510</td>
<td>0.250</td>
<td>0.800</td>
<td>0.0126</td>
<td>-0.0002</td>
<td>0.0153</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>$\sigma_\eta$</td>
<td>$\sigma_\psi$</td>
<td>$\rho_{e\eta}$</td>
<td>$\rho_{e\psi}$</td>
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<td>$\varepsilon$</td>
<td>$\theta$</td>
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<tr>
<td>0.027</td>
<td>0.020</td>
<td>0.020</td>
<td>0.800</td>
<td>-0.490</td>
<td>0.290</td>
<td>1.604</td>
<td>0.288</td>
<td></td>
</tr>
</tbody>
</table>

The unconditional means for nonhousing consumption growth, expenditure share growth and inflation are obtained from the sample means directly, that’s why there are no standard errors reported. It is worth to note that the coefficient of relative risk aversion ($1/\theta \approx 3.47$) and the intratemporal elasticity of substitution ($\varepsilon = 1.6$) between housing and numeraire consumption are reasonable numbers supported by existing evidences. For example, Mehra and Prescott (1985) argue that a risk aversion below 10 seems reasonable. Estimates of the intratemporal elasticity are more difficult to obtain, but existing studies generally support $\varepsilon$ to be above one. For example, Ogaki and Reinhart (1998) give $[1.04, 1.43]$ as a 95 percent confidence interval; Benhabib et al. (1991) obtain $\varepsilon = 2.5$. Piazzesi et al. (2007) use values around 1.05 and 1.25. The subjective discount factor $\beta$ is set at 0.986. It is chosen to match the average three-month short rate (the low-end of our average yield curve) in the data.

How well do our model estimates capture the dynamics of housing and nonhousing consumption? We present some direct evidences on this by comparing the model predicted series with the actual data. The model predicted series are achieved by first recovering the estimated state variables through Kalman Filter and then constructing each series ac-
cordingly, for example, $\Delta \ln c_{t+1|t} = \mu_c + \hat{x}_{t|t}$. Figures 2 to 4 plot the estimated numeraire consumption growth, expenditure share growth and inflation predicted by the model and their comparisons with the data. It can be seen that the model predicted series move closely with the data and capture many fluctuations in the data, especially for non-housing consumption and inflation. The estimated series for expenditure share growth is much less volatile than that in the data and the model only produces the lower-frequency fluctuations. This is mainly related to the fact that expenditure share growth in the data is by itself very small and close to white noise. All the growth rates reported are annualized percentages.

Figure 2: Numeraire Consumption Growth (Data and Model)

![Figure 2: Numeraire Consumption Growth (Data and Model)](image)

Figure 3: Non-housing Consumption Expenditure Share (Data and Model)

![Figure 3: Non-housing Consumption Expenditure Share (Data and Model)](image)
We go further to make a comparison between some key statistics for the estimated series in the model and their counterparts in the data. We focus on the mean, standard deviation, first-order autocorrelation and cross-correlation. These statistics represent the level, volatility, persistence and comovement of each series respectively. Table 2 reports the results. Generally, all estimated series have their means close to the data. The standard deviations in the estimated series are all smaller than those in the data. This is because what we compute are generally the persistent movements in the data, which determine the conditional expectations of the agent. The expenditure share growth by itself is very small and close to white noise. That’s why the estimated persistent component is small and unable to dominate the random disturbances as well. The degree of persistence for each series can also be seen from the data and model estimates clearly, with inflation being most persistent and expenditure share growth being most close to white noise. The cross-correlations show that (1) numeraire consumption growth and inflation are negatively correlated: high inflation is correlated with low consumption growth (2) high inflation is correlated with high expenditure share growth on nonhousing consumption (low expenditure share on housing consumption) (3) high numeraire consumption growth is correlated with a low expenditure share growth on housing consumption. These cross-correlations are important in bonds pricing, especially in explaining the risk premium associated with long-term bonds, as we will show in the next section.
Table 2: Moments of the Time Series Processes of the Economy

<table>
<thead>
<tr>
<th>Variables</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean(Δ ln c_t)</td>
<td>3.20%</td>
<td>3.80%</td>
</tr>
<tr>
<td>Mean(Δ ln α_t)</td>
<td>−0.0079%</td>
<td>−0.0061%</td>
</tr>
<tr>
<td>Mean(Δ ln Π_t)</td>
<td>4.16%</td>
<td>4.40%</td>
</tr>
<tr>
<td>Std(Δ ln c_t)</td>
<td>1.94%</td>
<td>1.13%</td>
</tr>
<tr>
<td>Std(Δ ln α_t)</td>
<td>0.69%</td>
<td>0.17%</td>
</tr>
<tr>
<td>Std(Δ ln Π_t)</td>
<td>2.63%</td>
<td>2.29%</td>
</tr>
<tr>
<td>Corr(Δ ln c_t, Δ ln α_t)</td>
<td>0.464</td>
<td>0.447</td>
</tr>
<tr>
<td>Corr(Δ ln c_t, Δ ln Π_t)</td>
<td>−0.273</td>
<td>−0.286</td>
</tr>
<tr>
<td>Corr(Δ ln α_t, Δ ln Π_t)</td>
<td>0.177</td>
<td>0.153</td>
</tr>
<tr>
<td>AC(Δ ln c_t)</td>
<td>0.354</td>
<td>0.571</td>
</tr>
<tr>
<td>AC(Δ ln α_t)</td>
<td>0.088</td>
<td>0.035</td>
</tr>
<tr>
<td>AC(Δ ln Π_t)</td>
<td>0.798</td>
<td>0.833</td>
</tr>
</tbody>
</table>

4 Model Implications for Bond Yields

We go further to examine the model’s performance in explaining the term structure of interest rates. Based on the estimated parameters and state variables \( \hat{x}_t, \hat{y}_t, \hat{z}_t \), we simulate model predicted series of yields through affine pricing formula (9) and then compare them with the quarterly data we have for nominal yields with maturities one-quarter, one-year and so on between 1959Q1 to 2005Q3. We’re particularly interested in the model’s performance in explaining the term premium.

4.1 Average Nominal Yields

Table 3 reports the average nominal bond yields predicted by the model and their counterparts in the data. The numbers are all annualized percentages. We choose the subjective discount factor \( \beta = 0.986 \) to approximately match the short end of the yield curve. This can also be seen from the average yield curve plotted in figure 5. The model successfully produces a positive-sloping average yield curve, with the term premia asso-
ciated with long-term nominal bonds significantly positive. This is not plausible in a standard CCAPM with power utility. By introducing housing consumption and thus a third factor into the standard model, the term premium puzzle is solved. Figure 6 also plots the time series dynamics of the yield spread between five-year bonds and three-month bonds. On average, the yield spread predicted by the model is above zero and matches the fluctuations in the data, though there are still some periods, especially the early 1980s, during which the model misses the quick decreasing in spreads.

Table 3: Average Bond Yields (Data and Model)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Data (%)</th>
<th>Model (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-month</td>
<td>5.61</td>
<td>5.62</td>
</tr>
<tr>
<td>1-year</td>
<td>6.05</td>
<td>6.16</td>
</tr>
<tr>
<td>2-year</td>
<td>6.26</td>
<td>6.61</td>
</tr>
<tr>
<td>3-year</td>
<td>6.43</td>
<td>6.88</td>
</tr>
<tr>
<td>4-year</td>
<td>6.57</td>
<td>7.05</td>
</tr>
<tr>
<td>5-year</td>
<td>6.64</td>
<td>7.18</td>
</tr>
</tbody>
</table>

Figure 5: Average Yield Curve

Figure 6: Dynamics of Yield Spreads between 5-year and the Short Rate
4.2 The Term Premium Decomposition

The intuitive explanation behind the positive term premium captured by the housing CCAPM can be understood in a decomposition of the risk premium associated with long-term nominal bonds. Define \( r_{x,n,t+1} = p_{n-1,t+1}^s - p_{n,t}^s - y_{1,t}^s \) as the holding return on buying a \( n \)-period nominal bond at time \( t \) for \( p_{n,t}^s \) and selling it at time \( t + 1 \) for \( p_{n-1,t+1}^s \) in excess of the one-period short rate. Based on the pricing equation, we can derive that

\[
E_t(r_{x,n,t+1}) = -\text{cov}_t(p_{n-1,t+1}^s, m_t^s) - \frac{1}{2} \text{var}_t(p_{n-1,t+1}^s)
\]

or

\[
E_t(\sum_{i=1}^{n-1} m_{t+1+i}^s, m_t^s) - \frac{1}{2} \text{var}_t(p_{n-1,t+1}^s)
\]

The covariance term on the right-hand side is the risk premium, while the variance term is due to Jensen’s inequality. The risk premium on long-term nominal bonds relative to short rate is positive when the pricing kernel and long bond prices are negatively correlated, or when the autocorrelation of the pricing kernel is negative. In this case, long-term bonds are less attractive than short-term bonds, because their payoffs tend to be low when the pricing kernel is high (marginal utility is high). Over long samples, the average excess return on a \( n \)-period bond is approximately equal to the average spread between the \( n\)-
period yield and the short rate\textsuperscript{8}. This means that the yield curve is on average upward sloping if the risk premium is positive on average.

In this model, the pricing kernel is determined by numeraire consumption growth, nonhousing expenditure share growth and inflation. Plugging equation (4) into the covariance term, we can decompose the risk premium into individual conditional covariances among the three factors. More specifically,

\[
\text{Risk Premium} = -\text{cov}_t\left(m_{t+1}^n, E_{t+1} \sum_{i=1}^{n-1} m_{t+1+i}^n\right)
\]

\[
= -\text{cov}_t\left(-\frac{1}{\theta} \Delta \ln c_{t+1} + k \Delta \ln \alpha_{t+1} - \Delta \ln \Pi_{t+1},
\sum_{i=1}^{n-1} \left(-\frac{1}{\theta} \Delta \ln \alpha_{t+1+i} + k \Delta \ln \alpha_{t+1+i} - \Delta \ln \Pi_{t+1+i}\right)\right)
\]

Figure 7 plots the individual terms that determine the risk premium as a function of maturity. Several terms contribute to a positive term premium. The most dominant one is \(k \cdot \text{cov}_t(\Delta \ln \alpha_{t+1}, E_{t+1} \sum_{i=1}^{n-1} \Delta \ln c_{t+1+i})\), which states a positive correlation between non-housing expenditure share growth and expected future numeraire consumption growth. A low expected numeraire consumption growth in the future would increase the agent’s desire to save more, thus raising the future bond price as well as the return of holding bond. However, this happens in a good state when the housing expenditure share growth is high (nonhousing expenditure share growth is low). This means that holding bonds with longer terms does not give protection against the composition risk Therefore the agent would require a positive premium of holding them. Other terms are useful in producing the positive term premium include \(kcov(\Delta \ln \alpha_{t+1}, E_{t+1} \sum_{i=1}^{n-1} \Delta \ln \Pi_{t+1+i})\), 
\(-\frac{1}{\theta} \text{cov}(\Delta \ln c_{t+1}, E_{t+1} \sum_{i=1}^{n-1} \Delta \ln \Pi_{t+1+i})\) and so on. In fact, all the cross-covariance terms contribute to the positive term premium, whereas the three auto-covariance terms are all

\textsuperscript{8}To see this, we can write the excess return as

\[
 p_{n-1,t+1}^s - p_{n,t}^s - y_{1,t}^s = ny_{nt}^s - (n-1)y_{n-1,t+1}^s - y_{1t}^s
 = y_{nt}^s - y_{1t}^s - (n-1)(y_{n-1,t+1}^s - y_{nt}^s)
\]

For large \(n\) and a long sample, the difference between the average \((n-1)\)-period yield and the average \(n\)-period yield is approximately zero.
negative. This justifies the importance of modeling the dynamics of the pricing factors in the economy jointly rather than separately.

![Figure 7: Risk Premium Decomposition](attachment:image.png)

5 Conclusion

In this paper, we build a term structure model of nominal interest rates based on the housing consumption CAPM framework proposed in Piazzesi et al. (2007). In this framework, consumption is a CES aggregate of housing service consumption and numeraire nonhousing consumption. So the agent is concerned with not only consumption risk, but also the composition risk of consumption, which leads to a pricing kernel that depends on both numeraire consumption and the expenditure share of nonhousing-housing consumption. We combine this pricing kernel with a dynamic long-run risk specification that allows for persistent and correlated changes in the endowment economy and obtain an affine term structure model for nominal interest rates. We then estimate the model using data on housing and nonhousing consumption and simulate model predicted bond yields. We find that the model can explain many salient features of bond yields very well, especially that it produces a positive term premium associated with long-term bonds, that standard CCAPM fail to explain.
The housing CCAPM framework adopted here does not differentiate between obtaining house consumption through home-ownership or obtaining house consumption as a tenant. So future research along the line to improve the housing consumption based model can be introducing more realistic assumptions about the housing market. Agents could either own a house or rent a house, and house plays a dual role: it is a consumption good which gives agents utility, and also a financial asset which gives agents capital return. Furthermore, people can also investigate the nature of house as a social status good: agents care about the interpersonal comparison of the housing consumption, other than the physical size and quality of their house. To sum up, house property takes up a significant fraction of people’s wealth and it is quite different from the usual consumption goods discussed in standard asset pricing theory. Therefore, people might get more interesting results once incorporating different roles of housing goods into the model.

References


A Derivation of the nominal term structure

We conjecture that the nominal log bond price has the following affine structure:

\[ p^n_{n,t} = A_n + B_n x_t + C_n y_t + D_n z_t \]

Substituting this into the L.H.S and R.H.S of \( p^n_{nt} = E_t(p^n_{n-1,t+1} + m^t_{t+1}) + \frac{1}{2} \text{Var}_t(p^n_{n-1,t+1} + m_{t+1}) \), we have

\[
E_t(p^n_{n-1,t+1} + m^t_{t+1}) = A_{n-1} + B_{n-1} \rho_x x_t + C_{n-1} \rho_y y_t + D_{n-1} \rho_z z_t + \\
\ln \beta - \frac{1}{\theta} (\mu_c + x_t) + k(\mu_{n} + y_t) - (\mu_{n} + z_t)
\]

\[
\text{Var}_t(p^n_{n-1,t+1} + m^t_{t+1}) = \text{Var}_t\{(-\frac{1}{\theta} + B_{n-1} \sigma_x) \epsilon_{t+1} + (k + C_{n-1} \sigma_y) \eta_{t+1} \\
+ (-1 + D_{n-1} \sigma_z) \psi_{t+1}\}
\]

\[
= \lambda_n' \Omega \lambda_n, \text{ where } \lambda_n = (-\frac{1}{\theta} + B_{n-1} \sigma_x + k + C_{n-1} \sigma_y - 1 + D_{n-1} \sigma_z)'
\]

Therefore

\[
A_n + B_n x_t + C_n y_t + D_n z_t = A_{n-1} + B_{n-1} \rho_x x_t + C_{n-1} \rho_y y_t + D_{n-1} \rho_z z_t + \ln \beta \\
- \frac{1}{\theta} (\mu_c + x_t) + k(\mu_{n} + y_t) - (\mu_{n} + z_t) + \frac{1}{2} \lambda_n' \Omega \lambda_n
\]

which implies

\[
A_n = A_{n-1} + \ln \beta - \frac{1}{\theta} \mu_c + k \mu_{n} - \mu_{n} + \frac{1}{2} \lambda_n' \Omega \lambda_n
\]

\[
B_n = \rho_x B_{n-1} - \frac{1}{\theta}
\]

\[
C_n = \rho_y C_{n-1} + k
\]

\[
D_n = \rho_z D_{n-1} - 1
\]

Specifically, the one period real bond price is given by

\[
p_{1,t}^r = E_t(m_{t+1}) + \frac{1}{2} \text{Var}_t(m_{t+1})
\]

where we have

\[
E_t(m^t_{t+1}) = \ln \beta - \frac{1}{\theta} (\mu_c + x_t) + k(\mu_{n} + y_t) - (\mu_{n} + z_t)
\]

\[
\text{Var}_t(m^t_{t+1}) = \lambda_1' \Omega \lambda_1
\]
So the value of $A_n, B_n, C_n$ can be given by

$$
A_n = A_{n-1} + \ln \beta \frac{1}{\theta} \mu_c + k \mu_\alpha - \mu_\pi + \frac{1}{2} \lambda_n \Omega \lambda_n
$$

$$
B_n = -\frac{1}{\theta} \cdot \frac{1 - \rho_x^n}{1 - \rho_x}
$$

$$
C_n = k \cdot \frac{1 - \rho_y^n}{1 - \rho_y}
$$

$$
D_n = -\frac{1 - \rho_z^n}{1 - \rho_z}
$$

with $A_0 = B_0 = C_0 = D_0 = 0$

The complete term structure is therefore given by

$$
y_{n,t} = -\frac{p_{n,t}^4}{n} = -\frac{1}{n} (A_n + B_n x_t + C_n y_t + D_n z_t)
$$

## B Kalman Filter Derivation of the Model

The dynamic of our economy can be rewritten in the following state space representation:

$$
\begin{pmatrix}
    x_{t+1} \\
    y_{t+1} \\
    z_{t+1}
\end{pmatrix} =
\begin{pmatrix}
    \rho_x & 0 & 0 \\
    0 & \rho_y & 0 \\
    0 & 0 & \rho_z
\end{pmatrix}
\begin{pmatrix}
    x_t \\
    y_t \\
    z_t
\end{pmatrix} +
\begin{pmatrix}
    \sigma_x & 0 & 0 \\
    0 & \sigma_y & 0 \\
    0 & 0 & \sigma_z
\end{pmatrix}
\begin{pmatrix}
    \epsilon_{t+1} \\
    \eta_{t+1} \\
    \psi_{t+1}
\end{pmatrix}
$$

\begin{pmatrix}
    \Delta c_{t+1} \\
    \Delta \ln \alpha_{t+1} \\
    \Delta \pi_{t+1} \\
    y_{1,t}^s \\
    y_{4,t}^s
\end{pmatrix} =
\begin{pmatrix}
    \mu_c \\
    \mu_\alpha \\
    \mu_\pi \\
    -A_1 \\
    -A_4/4
\end{pmatrix}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -B_1 & -C_1 & -D_1 \\
    -B_4/4 & -C_4/4 & -D_4/4
\end{pmatrix}
\begin{pmatrix}
    x_t \\
    y_t \\
    z_t
\end{pmatrix} +
\begin{pmatrix}
    \epsilon_{t+1} \\
    \eta_{t+1} \\
    \psi_{t+1}
\end{pmatrix}
$$

Here we use vector notations:

$$
S_{t+1} = \begin{pmatrix}
    x_{t+1} \\
    y_{t+1} \\
    z_{t+1}
\end{pmatrix}^T, \quad
Y_{t+1} = \begin{pmatrix}
    \Delta c_{t+1} & \Delta \ln \alpha_{t+1} & \Delta \pi_{t+1} & y_{1,t}^s & y_{4,t}^s
\end{pmatrix}^T, \quad
\epsilon_{t+1} =
\begin{pmatrix}
    \epsilon_{t+1} \\
    \eta_{t+1} \\
    \psi_{t+1}
\end{pmatrix}
$$

and

$$
\mu = \begin{pmatrix}
    \mu_c & \mu_\alpha & \mu_\pi & -A_1 & -A_4/4
\end{pmatrix}^T, \quad
\Omega = E (\epsilon_t \epsilon_t^T)
$$
Kalman recursion gives the updating of state variable based on new observations:

\[ S_{t|t} = F S_{t-1|t-1} + K_{t|t-1}(Y_t - \mu - H S_{t|t-1}) \]

where \( S_{t|t} \) is the conditional estimate of state variables based on information up till date \( t \), and \( K_{t|t-1} \) is the Kalman gain:

\[ K_{t|t-1} = (FP_{t-1|t-1}H^T + GΩ)(HP_{t-1|t-1}H^T + Ω)^{-1} \]

where \( P_{t|t} \) is the corresponding MLE of the estimates:

\[
\begin{align*}
P_{t|t} &= FP_{t-1|t-1}F^T + GΩG^T - \\
&\quad (FP_{t-1|t-1}H^T + GΩ)(HP_{t-1|t-1}H^T + Ω)^{-1}(FP_{t-1|t-1}H^T + GΩ)^T
\end{align*}
\]

We use the unconditional mean and variance as the initial condition:

\[
\begin{align*}
E(S_t) &= 0 \\
Var(S_t) &= FVar(S_{t-1})F^T + GΩG^T
\end{align*}
\]

Denote \( Var(S_t) = Σ \), then \( Σ = FΣF^T + GΩG^T \) and

\[ vec(Σ) = [I - (F \otimes F)]^{-1} vec(GΩG^T) \]

Thus we get \( S_{0|0} = 0, P_{0|0} = Σ \). With the state variables \( \{x_t\}, \{y_t\}, \{z_t\} \), we can recover the model predicted consumption growth, nonhousing expenditure share growth, and inflation by \( \hat{Y}_t = \mu + HS_{t-1|t-1} \).
C Decomposition of term premium

\[
E_t(r_{n,t+1}) - y_{1,t} + \frac{1}{2} \text{Var}_t(r_{n,t+1}) = -\text{Cov}_t(r_{n,t+1}, m_{t+1})
\]

\[-\text{Cov}_t(r_{n,t+1}, m_{t+1}) = -\text{Cov}_t(p_{n-1,t+1} - p_{n,t}, m_{t+1})
\]

\[= -\text{Cov}_t(p_{n-1,t+1}, m_{t+1})
\]

\[= -\text{Cov}_t(E_{t+1}(m_{t+2} + \cdots + m_{t+n}), m_{t+1})
\]

Let \(n = 2\) at first,

\[
m_{t+1} = \ln \beta - \frac{1}{\theta} \Delta \ln C_{t+1} + k\Delta \alpha_{t+1} - \Delta \pi_{t+1}
\]

\[m_{t+2} = \ln \beta - \frac{1}{\theta} \Delta \ln C_{t+2} + k\Delta \alpha_{t+2} - \Delta \pi_{t+2}
\]

\[E_{t+1}(m_{t+2}) = \ln \beta - \frac{1}{\theta} (\mu_c + x_{t+1}) + k(\mu_a + y_{t+1}) - (\mu_{\pi} + z_{t+1})
\]

Hence,

\[-\text{Cov}_t(E_{t+1}(m_{t+2}), m_{t+1}) = -\text{Cov}_t(-\frac{1}{\theta} x_{t+1} + ky_{t+1} - z_{t+1}, m_{t+1})
\]

\[= -\text{Cov}_t(-\frac{1}{\theta} \sum_{e=1}^n \sigma_e e_{t+1} + k\sum_{e=1}^n \eta_{e,t+1} - \sum_{e=1}^n \psi_{e,t+1}, -\frac{1}{\theta} e_{t+1} + k\eta_{e,t+1} - \psi_{e,t+1})
\]

\[= \frac{1}{\theta^2} \sum_{e=1}^n \sigma_e^2 + k^2 \sum_{e=1}^n \eta_{e,t+1}^2 - \sum_{e=1}^n \psi_{e,t+1}^2 + \sum_{e=1}^n \sigma_e \eta_{e,t+1} + \frac{1}{\theta} \sum_{e=1}^n \sigma_e \psi_{e,t+1} - \frac{1}{\theta} k \sum_{e=1}^n \sigma_e \eta_{e,t+1} - \frac{1}{\theta} \sum_{e=1}^n \sigma_e \psi_{e,t+1} - \frac{1}{\theta} k \sum_{e=1}^n \eta_{e,t+1} \psi_{e,t+1} + \frac{1}{\theta} \sum_{e=1}^n \sigma_e \eta_{e,t+1} - k \sum_{e=1}^n \sigma_e \psi_{e,t+1}
\]

Next we turn to arbitrary \(n\), firstly we notice that

\[E_{t+1}(\Delta c_{t+n}) = \rho_z^{n-2} x_{t+1}, E_{t+1}(\Delta \alpha_{t+n}) = \rho_y^{n-2} y_{t+1}, E_{t+1}(\Delta \pi_{t+n}) = \rho_z^{n-2} z_{t+1}
\]

Then we can plug (??) into the equation below

\[E_{t+1}(m_{t+n}) = \ln \beta - \frac{1}{\theta} E_{t+1}(\Delta c_{t+n}) + kE_{t+1}(\Delta \alpha_{t+n}) - E_{t+1}(\Delta \pi_{t+n})
\]

which gives,

\[E_{t+1}(m_{t+n}) = \ln \beta - \frac{1}{\theta} \rho_z^{n-2} x_{t+1} + k\rho_y^{n-2} y_{t+1} - \rho_z^{n-2} z_{t+1}
\]

\[E_{t+1}\left(\sum_{i=2}^n m_{t+i}\right) = (n - 1) \ln \beta - \frac{1}{\theta} (1 + \cdots + \rho_z^{n-2}) x_{t+1} + k(1 + \cdots + \rho_y^{n-2}) y_{t+1} - (1 + \cdots + \rho_z^{n-2}) z_{t+1}
\]
Hence,

\[-Cov_t(r_{n,t+1}, m_{t+1}) = -Cov_t(E_{t+1}(m_{t+2} + \cdots + m_{t+n}), m_{t+1})\]

\[= -Cov_t(-\frac{1}{\theta} E_{t+1}(\sum_{i=2}^{n} \Delta \ln c_{t+i}) + k E_{t+1}(\sum_{i=2}^{n} \Delta \ln \alpha_{t+i}) - E_{t+1}(\sum_{i=2}^{n} \Delta \ln \pi_{t+i}),\]

\[-\frac{1}{\theta} \Delta \ln c_{t+1} + k \Delta \ln \alpha_{t+1} - \Delta \ln \pi_{t+1})\]

\[= \frac{1}{\theta} B_{n-1} \sum_x \sigma_e^2 - k C_{n-1} \sum_y \sigma_{\eta}^2 + D_{n-1} \sum_z \sigma_{\psi}^2\]

\[-(k B_{n-1} \sum_x - \frac{1}{\theta} C_{n-1} \sum_y) \sigma_{e,\eta}\]

\[+(B_{n-1} \sum_x + \frac{1}{\theta} D_{n-1} \sum_z) \sigma_{e,\psi}\]

\[+(C_{n-1} \sum_y - k D_{n-1} \sum_z) \sigma_{\eta,\psi}\]

So the term premium can be decomposed into nine terms, which are the follows

\[-\frac{1}{\theta^2} Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln c_{t+i}), \Delta \ln c_{t+1}) = \frac{1}{\theta} B_{n-1} \sum_x \sigma_e^2\]

\[-k^2 Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \alpha_{t+i}), \Delta \ln \alpha_{t+1}) = -k C_{n-1} \sum_y \sigma_{\eta}^2\]

\[-Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \pi_{t+i}), \Delta \ln \pi_{t+1}) = D_{n-1} \sum_z \sigma_{\psi}^2\]

\[k \theta Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln c_{t+i}), \Delta \ln c_{t+1}) = -kB_{n-1} \sum_x \sigma_{e,\eta}\]

\[k \theta Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \alpha_{t+i}), \Delta \ln c_{t+1}) = \frac{1}{\theta} C_{n-1} \sum_y \sigma_{e,\eta}\]

\[-1 \theta Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln c_{t+i}), \Delta \ln \pi_{t+1}) = B_{n-1} \sum_x \sigma_{e,\psi}\]

\[-\frac{1}{\theta} Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \pi_{t+i}), \Delta \ln c_{t+1}) = \frac{1}{\theta} D_{n-1} \sum_z \sigma_{e,\psi}\]

\[k Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \alpha_{t+i}), \Delta \ln \pi_{t+1}) = C_{n-1} \sum_y \sigma_{\eta,\psi}\]

\[k Cov_t(E_{t+1}(\sum_{i=2}^{n} \Delta \ln \pi_{t+i}), \Delta \ln \alpha_{t+1}) = -k D_{n-1} \sum_z \sigma_{\eta,\psi}\]