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Asset price dynamics with heterogeneous beliefs and local network interactions

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ABSTRACT

In this paper we investigate the effects of network topologies on asset price dynamics. We introduce network communications into a simple asset pricing model with heterogeneous beliefs. The agents may switch between several belief types according to their performance. The performance information is available to the agents only locally through their own experience and the experience of other agents directly connected to them. We model the communications with four commonly considered network topologies: a fully connected network, a regular lattice, a small world, and a random graph. The results show that the network topologies influence asset price dynamics in terms of the regions of stability, amplitudes of fluctuations and statistical properties.

1. Introduction

Interpersonal communication plays an important role in the diffusion of information across social and business communities (Shiller, 1995). In a survey of institutional investors in USA, Shiller and Pound (1989) found that money managers who invested in stocks with extremely high growth of the price/earnings ratio were often discussing their trades with colleagues. Arnswald (2001) found that among fund managers in Germany information exchange with other financial and industry experts was the second most important factor influencing their investment decisions, complemented by conversations with their colleagues and reports from media. Similarly, a study of fund managers by Hong et al. (2005) provided strong support for the importance of informal communication. Cohen et al. (2008) provide empirical evidence that

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connections between mutual fund managers and corporate board members via shared education networks have a significant effect on the mutual fund portfolio performance.

Household investment decisions are also affected by interpersonal communication. Duflo and Saez (2002) showed that employees are more likely to join an investment retirement scheme if their colleagues have done so. Hong et al. (2004) suggested, by reviewing data from the University of Michigan health and retirement study, that interaction with neighbors and church attendance increased the likelihood of a household investing in stocks.

Given this evidence, we study the impact that local interactions between investors have on the asset price dynamics in a theoretical model of asset pricing. We bring together ideas from various streams of literature: the rapidly developing literature on networks, the literature on heterogeneous agent models and agent-based models. We explore a range of local interaction patterns by introducing different types of communication network topologies, a fully connected network, a regular lattice, a small world, and a random graph, into the stylized heterogeneous agent model of Brock and Hommes (1998).

We show that our model with the fully connected network converges to the original Brock and Hommes (1998) model when the number of agents is large, excluding the degenerate case when all agents are of the same type. For the two-type model with the random graph we derive a low-dimensional representation for which we perform stability and bifurcation analysis. The other network topologies are analyzed by simulations. We find that qualitatively the asset price dynamics of the two-type Brock and Hommes (1998) model are preserved under local interactions, but communication network topologies influence the regions of stability, amplitudes of fluctuations and statistical properties. In particular, in the two-type model the latency in the information transmission caused by a specific communication network translates into earlier bifurcation, smaller regions of stability and higher price fluctuations. However, the impact of a network topology may depend on a particular agent ecology, that is, the specifications of belief types and their numbers. In particular, for the four-type model we find qualitatively different dynamics for some of the network topologies compared to the four-type Brock and Hommes (1998) model.

In the next section we survey models with boundedly rational and heterogeneous agents. Section 3 examines different network topologies and their properties. Section 4 introduces our model with network interactions, derives a reduced form approximation and offers stability and bifurcation analysis for the two-type model with the random graph. Section 5 uses simulations to analyze the two-type model under various network topologies, extends the analysis to the four-type model, presents and discusses the results of the simulations. Section 6 concludes and discusses further extensions.

2. Bounded rationality and heterogeneity in asset pricing

The rational expectations theory in finance (Friedman, 1953) asserts that rational investors would drive irrational traders out of financial markets. Numerous empirical studies, however, have shown that successful traders follow a variety of investment strategies (e.g. Frankel and Froot, 1987; Ito, 1990). DeLong et al. (1990) was among the first studies to analytically demonstrate that irrational noise traders may survive in a market with fully rational traders. This survival is possible because these noise traders bear a higher risk which leads to higher returns in the long run. Other researchers used heterogeneity of expectations to explain asset price dynamics. Day and Huang (1990), Chiarella (1992), Kirman (1993) and Lux (1995) showed that transactions between different agents that follow simple behavioral rules and interact with each other lead to endogenous price fluctuations. Alfarano and Milakovic (2009) enriched the Kirman-Lux model with explicit network structures. Anufriev and Bottazzi (2012) analyzed the implication of heterogeneity in investment horizons.

Brock and Hommes (1998) introduced a structural asset pricing model with heterogeneous agents switching between several belief types according to their performance (denoted the BH model henceforth). The belief types differed in their expectation about the future price of the risky asset. The performance measure of each type was freely available to all agents. The BH model showed that the rational expectation type do not necessarily drive out boundedly rational types. In fact, these types could co-exist in a market. BH conducted stability analysis of the steady states and derived the conditions for the occurrence of certain bifurcations. The BH model was able to produce excess volatility and positive volatility/volume correlations, the stylized facts which were not reproduced by the rational expectations models.

Various extensions of the BH model have been considered in the literature. Hommes (2002) adapted the model to better reproduce volatility clustering. Brock et al. (2005) extended the BH model to many trader types and Brock et al. (2009) studied how the presence of risk hedging instruments in the form of Arrow securities affects the market dynamics. Hommes et al. (2005) included a market maker into the market pricing mechanism, while Anufriev and Panchenko (2006, 2009) investigated the changes in the model outcomes due to various market architectures. Boswijk et al. (2007) estimated the parameters of the BH model using annual US stock price data. Chang (2007) enriched the model with exogenous social interactions of the Brock and Durlauf (2001) type. Diks and van der Weide (2005) studied continuous distribution of the agent beliefs. Gerasymchuk (2008) introduced prospect theory-like preferences of the agents into a modified BH model. Diks and Dindo (2008) and Goldbaum and Panchenko (2010) considered BH-type switching and learning in the context of financial models with informational differences. Anufriev et al. (forthcoming) applied the BH model to form interest rate expectations and study the resulting dynamics in a macroeconomic setting. Anufriev and Hommes (2012a, 2012b) introduced a BH-style heuristics switching model and estimated it using data from a learning-to-forecast experiment. For a detailed survey of the heterogeneous agent literature see Hommes (2006).

Another stream of asset pricing literature focuses on large-scale models of evolving, interacting artificial agents. Examples of this approach include the Santa Fe artificial stock market (Arthur et al., 1997; LeBaron et al., 1999; Ehrentreich, 2006) and the models of Chen and Yeh (2001) and Chen et al. (2001). A major advantage of these models over the smaller scale heterogeneous agent models previously discussed is that they allow for higher flexibility, richer behavioral assumptions, and more realistic market architectures. This, however, comes at the price of increased complexity. Analytical solutions are not typically attainable for these models, and therefore computer simulations are often used to study their properties. The literature on agent-based finance has also been influenced by the contributions of interdisciplinary statistical physicists. In particular, Iori (2002) and Cont and Bouchaud (2000) explicitly considered network structures in their models of financial markets. For further details and references on agent-based finance we refer the interested reader to the review by LeBaron (2006).

In this paper we combine the work on networks with heterogeneous agent models by introducing local interactions into the stylized BH model. Our aim is to study the effects of different networks of local interactions on the asset price dynamics. In our setting information about the performance of their investments is only available to the agents locally through their own experience and the experience of other agents directly connected to them. We derive transition equations reflecting these local interactions and offer analytical approximations for some network topologies. When analytical approximations are not available, we investigate the model using computer agent-based simulations.

3. Social networks

Social networks are important in our lives. Decision making, trade activity, job searching and disease transmission are all heavily influenced by the social and economic networks. Network modeling is a rapidly growing part of the economic literature (see Jackson, 2008 for a detailed treatment). Watts (1999) indicates that a typical social network has the following properties: (1) there are many participants in the network; (2) each participant is connected to a small fraction of the entire network, i.e., the network is sparse; (3) even the most connected node is still connected only to a small fraction of the entire network, i.e., the network is decentralized; (4) neighborhoods overlap, i.e., the network is clustered; and yet (5) the characteristic path length or diameter of the network, i.e., the shortest path between the furthest pair of nodes, is small.

To capture these properties Watts and Strogatz (1998) introduced a network model called a *small world*. It is an intermediate network between a *regular lattice* network, in which the agents (nodes) are linked in a geometrically regular way, and a *random graph*, in which the links are random. The small world network model approximates social interactions in real life. Networks with the small world properties include social networks of the US corporate elite (Davis et al., 2003), partnerships of investment banks in Canada (Baum et al., 2003), and many more. Small world networks emerge when participating agents form networks through a mix of random and strategic interactions (Baum et al., 2003; Morone and Taylor, 2004).

Fig. 1 shows four examples of network topologies. The *degree* of a node is the number of links the node has to other nodes. In the fully connected network, all nodes are linked to all other nodes. Denote the total number of nodes in the network by N. In the regular lattice, each node is linked to a *fixed* number of neighboring nodes, and hence, all nodes have the same degree, which we denote by K; in our example in Fig. 1 $K=4$. In order to form a small world network, a link is rewired to a different randomly chosen node on the lattice (avoiding self- and double-connections) with a given rewiring probability π, $0<\pi<1$. Such rewiring of the nodes continues until all the links are processed. In the limit when $\pi=1$ the network becomes the random graph of Erdős and Rényi (1959), with N nodes and the probability of the link between any two nodes equals to $K/(N-1)$.

The structural properties of a network can be quantified in terms of three additional characteristics (Newman, 2003): a degree distribution, a clustering coefficient, C, and a characteristic path length, L. The *degree distribution* is the distribution of the degrees of all nodes in the network. Denote the average degree of the network by k. In the Watts and Strogatz (1998) model k equals to K for any π and the degree distribution is closely concentrated around its average K (Barrat and Weigt, 2000). The *clustering coefficient* of a node is calculated by dividing the number of links between the direct neighbors of this

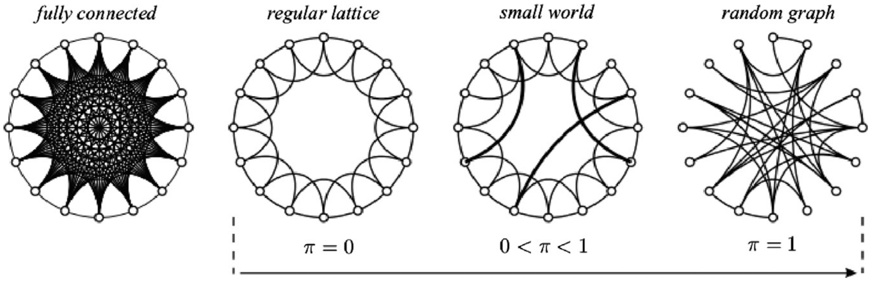

Fig. 1. Network topologies (adapted from Watts and Strogatz, 1998). π indicates a link rewiring probability.

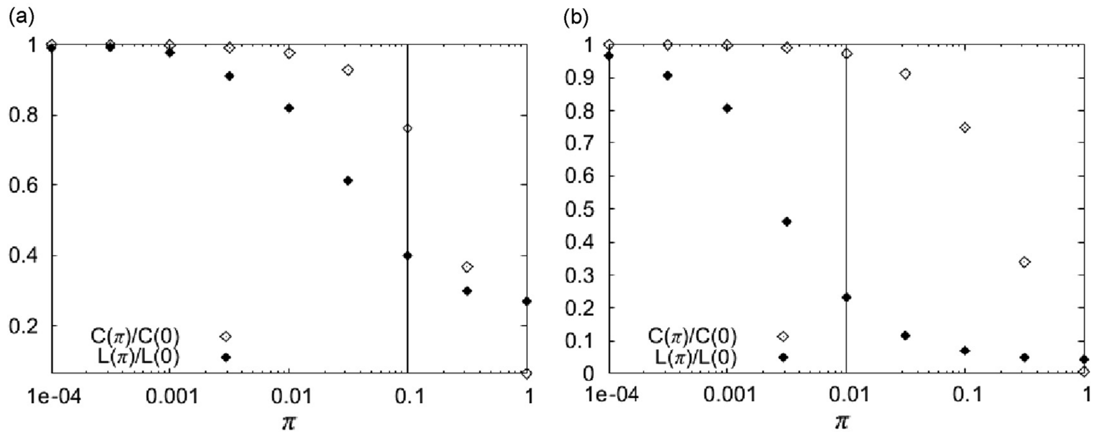

Fig. 2. Clustering coefficient and characteristic path length for networks of different sizes. *Note*: logarithmic scale is used for abscissa. Vertical line indicates small world value of π. (a) $N=100$. (b) $N=1000$.

node by the maximum possible number of links between them. It indicates how well the neighborhood of the node is connected or, in other words, the 'cliquishness' of the neighborhood. By averaging over the clustering coefficients of all the nodes in a network we obtain the clustering coefficient of the network, C. The *characteristic path length or diameter* of the network, L, measures the shortest path (minimum number of links) between any two nodes averaged over all the nodes in the network.

Latora and Marchiori (2001) relate the clustering coefficient and the characteristic path length to the local and global efficiency of the network, respectively. Local efficiency measures fault tolerance, that is, how efficient is the communication between the immediate neighbors of node i, when i is removed. Global efficiency is related to the signal transmission through the whole network.

For each value of the rewiring probability, π, we obtain a network with new structural properties. These properties also depend on N and K. A *small world* network can be formally defined as a decentralized, sparsely connected network with a high clustering coefficient, C, and a small characteristic path length, L. Fig. 2 shows the values of C and L, normalized by the corresponding characteristics of the regular lattice, for different rewiring probabilities π for two network sizes $N=100$ and $N=1000$ and $K=4$ in both cases. The small world network properties emerge in a setting around $\pi=0.1$ for $N=100$ and $\pi=0.01$ for $N=1000$ (see, e.g., Albert and Barabási, 2002 for a detailed discussion about the rewiring probability and small world properties).

The Watts and Strogatz (1998) model is a popular choice in the social networks literature, but as with any model it has some limitations. One of its main drawbacks is that the model is unable to produce the degree distribution observed in typical real social networks. Scale free networks suggested by de Solla Price (1965) and advanced by Barabási and Albert, 1999 address this problem, but are often unable to generate realistic clustering as observed in social networks. Hence, in this paper, we limit ourselves to the Watts and Strogatz (1998) networks.

4. Heterogeneous belief model with local interactions

4.1. Brock–Hommes model

In this section we first describe the BH model, and then extend it by allowing for local interactions. There are two assets that are traded in discrete time: a risk-free asset paying a constant gross return, $R=1+r$, and a risky asset paying a stochastic dividend, y_t, at the beginning of each trading period t. The dividend is assumed to be independently and identically normally distributed (i.i.d.) with mean $\overline{y}$ and variance Var[y]. The price, p_t, per-share (ex-dividend) of the risky asset in period t is obtained from the Walrasian market clearing condition. The wealth dynamics is specified by

$$W_{t+1}=R(W_t-p_tz_t)+(p_{t+1}+y_{t+1})z_t=RW_t+(p_{t+1}+y_{t+1}-Rp_t)z_t, \tag{1}$$

where W_t and W_{t+1} are the wealth levels in period t and $t+1$ correspondingly, and z_t is the number of shares of the risky asset purchased at date t.

The agents are myopic maximizers of the mean–variance expected wealth

$$\max_{z_t}\left\{\mathrm{E}_{t-1}[W_{t+1}]-\frac{a}{2}\mathrm{V}_{t-1}[W_{t+1}]\right\}, \tag{2}$$

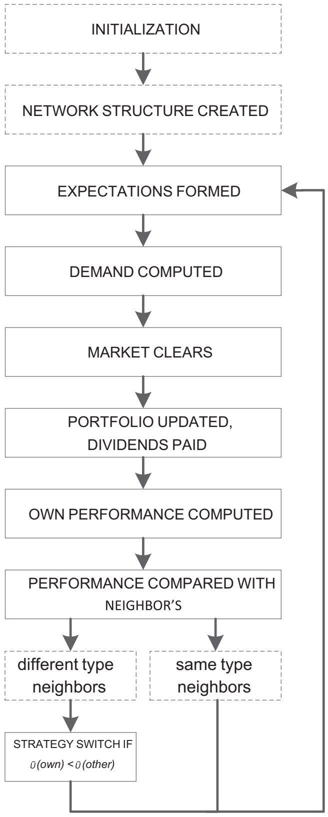

Fig. 3. Temporal flow.

where a is the absolute risk aversion coefficient, and E_t and V_t denote conditional expectation and conditional variance that are based on the publicly available information set $I_t = \{p_t, p_{t-1}, p_{t-2}, \ldots; y_t, y_{t-1}, y_{t-2}, \ldots\}$. There are H belief types which differ in their expectations about the future price. The demand for the risky asset of type h is then given by

$$z_t^h(p_t) = \frac{\mathrm{E}_{t-1}^h[p_{t+1} + y_{t+1}] - Rp_t}{a\mathrm{V}_{t-1}^h[p_{t+1} + y_{t+1}]} = \frac{\mathrm{E}_{t-1}^h[p_{t+1} + y_{t+1}] - Rp_t}{a\sigma^2}. \tag{3}$$

Operators $\mathrm{E}_{t-1}^h[\cdot]$ and $\mathrm{V}_{t-1}^h[\cdot]$ are the expectations, or the predictors, of type h about mean and variance, respectively. The predictors for period $(t+1)$ depend on $(t-1)$ information because the price at period t is not realized at the moment when the predictors are produced (see Fig. 3 for timing in the model). It is assumed that all the types expect the same variance, $\mathrm{V}_t^h = \sigma^2$ and have the same value for the risk-aversion coefficient, a.

Set the supply of outside shares of the risky asset to zero.[1] Let n_t^h be the fraction of type h agents determined in the end of period t. The equilibrium of supply and demand then determines the price, p_t, in the market-clearing equation:

$$\sum_{h=1}^{H} n_{t-1}^h \frac{\mathrm{E}_{t-1}^h[p_{t+1} + y_{t+1}] - Rp_t}{a\sigma^2} = 0. \tag{4}$$

Under the assumption of homogeneous beliefs ($H=1$), the fundamental price, p^*, is the unique constant solution to the market-clearing equation (4). It is equal to the discounted infinite sum of the expected future dividends, i.e., $p^* = \overline{y}/r$.

All beliefs are of the form

$$\mathrm{E}_{t-1}^h[p_{t+1} + y_{t+1}] = b^h + p^* + \overline{y} + g^h(p_{t-1} - p^*), \tag{5}$$

where b^h is a constant bias and g^h is an extrapolation parameter.

The main focus of this paper is on the ecology with two types ($H=2$), *fundamentalists* and *chartists*. Both of these types have zero bias, $b=0$. Fundamentalists believe that price will be at the fundamental level, p^*, and set the extrapolation parameter, g, to 0, while chartists expect persistent deviations from the fundamental value and use a positive extrapolation

[1] This is a standard assumption of the baseline BH model. Hommes et al. (2005) consider a positive supply of the risky asset.

parameter, $g>0$ (see Brock and Hommes, 1998, Section 4.1.2). In Section 5.3 this ecology is extended to four types by introducing positively and negatively biased types (see Brock and Hommes, 1998, Section 4.3).

Define the performance measure, U_t^h, as a net profit of type h, that is

$$U_t^h = (p_t + y_t - Rp_{t-1})z_{t-1}^h - c^h, \tag{6}$$

where $p_t + y_t - Rp_{t-1}$ is the excess return earned per unit of the risky asset, z_{t-1^h}, held in the agents' portfolio at the end of period $t-1$, and c^h is the costs of type h. In the ecology with two types these costs are set to zero for chartists and are strictly positive for fundamentalists.[2] The costs are set to 0 for all types in the ecology with four types.

The belief types are updated over time depending on the relative utility from following a rule of a specific type compared to other types. The utility is based on the observed performance measure, U_t^h, and an unobserved idiosyncratic random component, ε_t^h, that is

$$\tilde{U}_t^h = U_t^h + \frac{1}{\beta}\varepsilon_t^h, \tag{7}$$

where β is the *intensity of choice* parameter, which controls the level of the random component. The sources of randomness in the satisfaction are unobserved variations in preferences of agents and in the attributes of alternatives, and agents' errors of perception and behavioral biases (Hirshleifer, 2001). In the case when the (noisy) performances of all types are observed by all agents, the probability that an agent selects type h at period t is given by $P_t^h = P(\tilde{U}_t^h > \tilde{U}_t^\ell,\ \text{for all } \ell \neq h)$.

For a sufficiently large number of the agents, the fraction of agents of type h, n_t^h, converges to probability P_t^h. Moreover, if we assume that the idiosyncratic random component, ε_t^h, in (7) follows the standard Gumbel (extreme value) distribution, n_t^h can be described by a discrete choice logit model (Manski and McFadden, 1990)[3]:

$$n_t^h = \frac{\exp(\beta U_t^h)}{\sum_{\xi=1}^H \exp(\beta U_t^\xi)}. \tag{8}$$

The dynamics of the model is described through the co-evolution of the fractions of the types, and the market equilibrium price.

4.2. Local interactions

In our setup, the agents are located on the nodes of a network and can observe the performance measure of the predictor types employed only by those agents who reside on the nodes directly connected with them. Hence, they cannot observe the performance of the types adopted by agents located two or more links away. Therefore, contrary to the BH model, we do not assume that the performance of every type is available to all the agents. Instead, we allow only for local information exchange in the market.

In particular, if an agent is directly connected only to the agents of the same type, they are not able to switch as there is no information about the performance of the alternative type(s). If an agent has at least one neighbor of a different type,[4] they are able to compare the utility from their own type with the utility from the alternative observed type(s) and make a choice. Note that under local information exchange, the fractions of the belief types, n_t^h, do not follow the discrete choice fractions specified in Eq. (8) because some agents are not able to switch.

As a motivation for our model, imagine a world populated by many individuals who invest their money following the recommendations of financial advisors. The advisors use the mean–variance framework to recommend an optimal portfolio allocation between the risk-free and the risky assets (see Eq. (3)). The advice is given at regular time periods. The advisors are classified into a small number of types based on the predictors they use to derive the optimal portfolio allocation. In the two-type model, there are two types of the financial advisors: fundamentalists and chartists. Moreover, in the two-type model, the fundamentalist advisors charge the higher fees. The individuals are not professional investors and do not know or understand the methods used by their advisors. The individuals are free to choose and change their advisors. Every period, say, every quarter, the individuals receive reports on the performance of their financial portfolios. The individuals interact with their friends and if their friends follow financial advisors of a different type, they are able to compare the relative performance of their investments and choose their advisors accordingly.

The model progresses in the following way (see Fig. 3). After the expectations (predictors) of the different types of advisors are formed and the demands are ascertained, trades occur and the market clears. Next, the performances are released. Then, when possible, agents (individual investors) compare their utility with their neighbors utility and switch to another type of advisors or remain with their current advisor. Finally, the expectations are formed again and the cycle repeats.

[2] This is a standard assumption of the two-type BH model. Brock and Hommes (1998) suggest attributing these costs to 'training' costs required to understand the fundamental theory.

[3] If we assume normal distribution for the idiosyncratic random component, a probit model will arise instead. The dynamics implied by both models are similar, but the logit model is more analytically tractable.

[4] If there is more than one neighbor of the different types, only one of them is consulted to compare utilities. A more general setting is discussed in model extensions.

For simplicity, next we consider the model with two types (of advisors), fundamentalists and chartists as introduced before. Denote the probability that agent i chooses the chartist type at period t by $P_{i,t}$. Then, the probability of choosing the fundamentalist type is simply $(1-P_{i,t})$. The evolution of $P_{i,t}$ can be described by

$$P_{i,t} = I_{i,t-1}\prod_{j\in\mathbb{G}_i} I_{j,t-1} + \left[I_{i,t-1}\left(1-\prod_{j\in\mathbb{G}_i} I_{j,t-1}\right) + (1-I_{i,t-1})\left(1-\prod_{j\in\mathbb{G}_i}(1-I_{j,t-1})\right)\right]\Delta_t, \tag{9}$$

where $I_{i,t}$ is an indicator variable taking value of 1 if agent i chose the chartist type in period t and 0 otherwise, $P(I_{i,t}=1)=P_{i,t}$; $\mathbb{G}_i$ denotes the neighborhood of agent i, i.e., the set of agents directly connected to i excluding i, and $\Delta_t = (1+\exp[\beta(U_t^{\text{fund}}-U_t^{\text{chart}})])^{-1}$ is the discrete choice logistic probability of choosing the chartist type over the fundamentalist type when both type performances are observed. $P_{i,t}$ may take values of 0, 1 or Δ_t. It equals to 1 when the first component of Eq. (9) is 1, that is, the first component indicates that agent i chose the chartist type in period $(t-1)$ and is unable to switch because she is surrounded by the neighbors who chose the same type. The second component consists of two parts (multiplied by Δ_t). The first part indicates whether agent i chose the chartist type, is neighbored by at least one agent who chose the fundamentalist type and, hence, is able to compare these two types and switch if necessary. The second part indicates whether agent i chose the fundamentalist type, is neighbored by at least one agent who chose the chartist type and, hence, also is able to compare these two types and switch if necessary.

In general, we have to keep track of every agent in the system. There are two cases, however, where it is possible to reduce the dimensionality of the system. The first case is the fully connected network, in which all agents are in one large neighborhood. Ignoring the degenerate situations when all agents are of the same type, in this network topology all agents have access to the performances of all types and hence, $P_{i,t}=\Delta_t$. Furthermore, when $(N\to\infty)$ we can apply the law of large numbers and recover the original BH model with the fraction of chartists $n_t=\Delta_t$. The second case is the random graph. In this network topology the links between the agents are random and the neighborhoods are not clustered which makes the network-induced dependence between any two nodes fairly small. Under these conditions, by the argument of symmetry, for $N\to\infty$ we can drop individual agent indices, replace the realizations by probabilities or fractions in Eq. (9) and approximate the evolution of the fractions of the chartists[5] by

$$\begin{aligned} n_t &= n_{t-1}n_{t-1}^k + [n_{t-1}(1-n_{t-1}^k) + (1-n_{t-1})(1-(1-n_{t-1})^k)]\Delta_t = \\ &= n_{t-1}^{k+1} + [1-n_{t-1}^{k+1}-(1-n_{t-1})^{k+1}]\Delta_t, \end{aligned} \tag{10}$$

where k is the average degree of the random graph; $k=K$ in the Watts and Strogatz (1998) model. In the limit, $k\to\infty$, we recover the original BH model with $n_t=\Delta_t$.

Note that Eq. (10) resembles the BH model with *asynchronous updating* (Hommes et al., 2005; Anufriev and Hommes, 2012a, 2012b), in which a fraction of one type is determined by $n_t=\alpha_1 n_{t-1}+\alpha_2\Delta_t$, in which $\alpha_1,\alpha_2\geq 0, \alpha_1+\alpha_2=1$. Weights α_1 and α_2 determine the fractions of agents who retain their previous type ignoring any performance information and choose between the two types according to Δ_t, respectively. The difference is that in our case the weights, α_1 and α_2, are state-dependent endogenously determined by the model and do not typically add up to 1. In the context of the BH model with two types, values of β for which bifurcations are observed do not typically change when asynchronous updating is introduced (see, e.g., Anufriev and Hommes, 2012a). As we show below in our model local information exchange leads to significant quantitative and qualitative implications including changing critical values of β for which bifurcations are observed.

4.3. Steady states, stability and bifurcations for two-type model with random graph

Eqs. (3)–(6) and (10) jointly determine a system of difference equations governing the dynamics of the system. The analysis of the system is easier when the price is written in terms of deviations from the fundamental price, $x_t=p_t-p^*$. Specifically, for the two-type ecology with fundamentalists and chartists and n_t denoting the fraction of chartists the system becomes

$$x_t = \frac{g}{R} n_{t-1} x_{t-1} \tag{11}$$

$$n_t = n_{t-1}^{k+1} + [1-n_{t-1}^{k+1}-(1-n_{t-1})^{k+1}]/[1+\exp(\beta(-Dgx_{t-2}(x_t-Rx_{t-1})-c))], \tag{12}$$

where $D=1/a\sigma^2$, c is the extra costs of fundamentalists (the costs of chartists are normalized to 0), g is the extrapolation parameter of chartists. Eqs. (11) and (12) define a three-dimensional dynamical system of difference equations for which we produce some analytical results summarized in Proposition 1. We call the steady states, in which the price is at the fundamental level, that is $x_t=0$, *fundamental steady states* and all other steady states *non-fundamental steady states*.

[5] For brevity, hereafter, we will simply use *fundamentalists* or *chartists* to refer to the investors who chose the fundamentalist or the chartist type at period t, respectively.

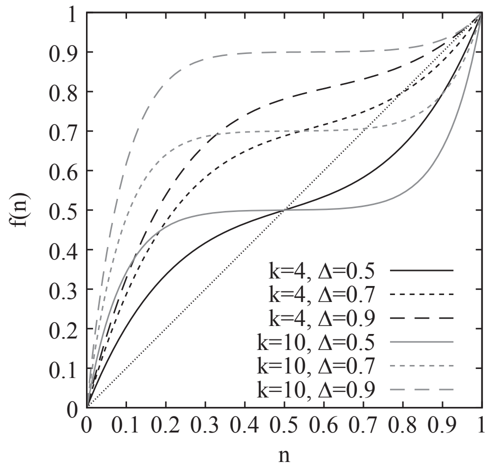

Fig. 4. Dependence of map $f(n)=n^{k+1}+(1-n^{k+1}-(1-n)^{k+1})\Delta$, on k and Δ.

Proposition 1 (*Existence and stability*[6] *of steady states for the dynamical system in Eqs. (11) and (12)*). *Let $n^\diamond$, $n^\diamond \in [1/2,1)$, be the interior solution of $n=n^{k+1}+(1-n^{k+1}-(1-n)^{k+1})/(1+\exp(-\beta c))$, $n^*=R/g$, $c \geq 0$, $\beta \geq 0$ and*

$$x^*=\sqrt{(\ln[(1/n^*-1)(1-(1-n^*)^k)/(1-(n^*)^k)]/\beta+c)/(Dg(R-1))},$$
$$\beta_1=\ln(k)/c,$$
$$\beta^*=-\ln[(1/n^*-1)(1-(1-n^*)^k)/(1-(n^*)^k)]/c,$$
$$\beta^{**}=\beta^*+\frac{1-2R-A(R-1)+\sqrt{(1+A)^2-2(4+A+A^2)R+(8+A^2)R^2}}{2BRc},$$

where

$$A=\frac{(k+1)((1-n^*)^k n^*+(1-n^*)(n^*)^k-((1-n^*(n^*)^k)}{1-(1-n^*)^k+n^*((1-n^*)^k-(n^*)^k)},\quad B=\frac{(1-n^*)(1-(1-n^*)^k)(1-(n^*)^k)}{1-(1-n^*)^k+n^*((1-n^*)^k-(n^*)^k)}.$$

Denote all possible steady states $(\overline{x},\overline{n})$ of the system as follows: fundamental steady states $E_\diamond=(0,n^\diamond)$, $E_0=(0,0)$, $E_1=(0,1)$ and non-fundamental steady states $E_+=(x^*,n^*)$, $E_-=(-x^*,n^*)$. Depending on the parameters of the model we may observe the following outcomes:

1. *$0<g<R$. For $\beta<\beta_1$ there exist three fundamental steady states, $E_\diamond$ is stable and E_0 and E_1 are unstable. At $\beta=\beta_1$ a transcritical bifurcation occurs when $E_\diamond$ collides into E_1. For $\beta>\beta_1$, $E_\diamond$ ceases to exist and E_1 becomes stable, E_0 remains unstable.*
2. *$R \leq g<2R$. For $\beta<\beta^*$ there exist three fundamental steady states, $E_\diamond$ is stable and E_0 and E_1 are unstable. At $\beta=\beta^*$ a (primary) pitchfork bifurcation occurs and $E_\diamond$ looses stability and two non-fundamental steady states E_+ and E_- emerge for $\beta>\beta^*$. For $\beta^*<\beta<\beta^{**}$ steady states E_+ and E_- are stable. At $\beta=\beta^{**}$ a (secondary) Neimark–Sacker bifurcation takes place and E_+ and E_- loose stability for $\beta>\beta^{**}$. E_0 and E_1 remain unstable for any β.*
3. *$g \geq 2R$. There exist three fundamental steady states $E_\diamond$, E_0 and E_1 and two non-fundamental steady states E_+ and E_-. All three fundamental steady states are unstable. The two non-fundamental steady states are stable for $\beta<\beta^{**}$.*

Proof. See Appendix A.

For a relatively small parameter of extrapolation, $g<R$, we observe only the fundamental steady states in the system. An important difference from the BH model is that in our model in addition to the “interior” fundamental steady state, $E_\diamond$, we observe two “corner” fundamental states which are E_0 with all agents being fundamentalists and E_1 with all agents being chartists. The steady states, in terms of $\overline{n}$, are shown in Fig. 4. The figure plots the map, $f(n)$, derived from Eq. (12) for various values of k and Δ, where Δ denotes the steady state fraction of chartists in the original BH model; in the fundamental steady states, $\Delta=1/(1+\exp(-\beta c))$. The steady states are the fixed points of the map, that is, the points where the map crosses the diagonal line. The “corner” steady states arise due to the possibility of a lock-in effect, that is, the situation when all agents adopt one specific type in one of the periods and, because the performance of the other type is not observed, no agent is able to switch to the other type in any subsequent period. Note that E_0 is always

[6] By stability we mean asymptotic local stability.

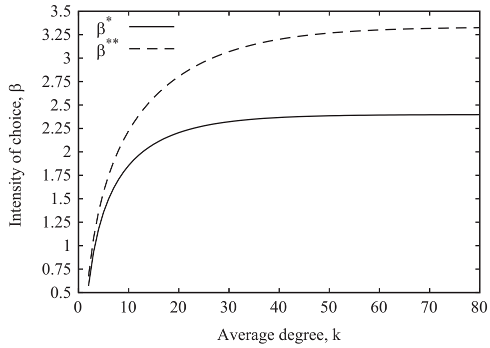

Fig. 5. Bifurcation values of β for given k, $g=1.2,\ R=1.1,\ c=1$.

unstable and E_1 is unstable for small β when $E_\diamond$ exists. When the costs, c, or the intensity of choice, β, are equal to zero, the stable steady state fractions of chartists, $n^\diamond$, and fundamentalists $(1-n^\diamond)$ are both equal to $1/2$. Moreover, $n^\diamond$ is increasing in c and β. This is consistent with economic intuition, i.e., in the fundamental steady state, when the prices fully reflect the fundamental value, the performance of the fundamental type is inferior due to the positive costs. The fraction of fundamentalists is non-zero, only due to the idiosyncratic component in the utility (in combination with the network effect). As β reaches β_1 the fractions of chartists reach one and a transcritical bifurcation takes place, that is, $E_\diamond$ collides into E_1 and ceases to exist. Fig. 4 shows that only two fundamental steady states exist when $\Delta=0.9$, which in terms of β for given k is equivalent to $\beta>\beta_1$.

For intermediate values of the extrapolation parameter, $R\leq g<2R$, as in the BH model, when $\beta<\beta^*$ only the fundamental steady states exist and when $\beta>\beta^*$ both the fundamental steady states and the non-fundamental steady states exist. At $\beta=\beta^*$ the (interior) fundamental steady state fraction of chartists, $n^\diamond$, reaches critical level $n^*=R/g$, and a (primary) pitchfork bifurcation occurs. The fundamental steady state $E_\diamond$ looses stability and stable non-fundamental steady states E_+ and E_- arise. The bifurcation value, β^*, is lower in our model with the random graph than in the original BH model that follows from $n^\diamond\geq\Delta$ (for the latter relation see the proof of Proposition 1). At $\beta=\beta^{**}$ the (secondary) Neimark–Sacker bifurcation takes place and the non-fundamental steady states loose stability. The bifurcation values of β are increasing in k as shown in Fig. 5. In the limit $k\to\infty$, the bifurcation values converge to the BH bifurcation values.[7]

For strong values of the extrapolation parameter, $g\geq 2R$, the fundamental steady state is always unstable and the non-fundamental steady state is stable in a relatively small region $0<\beta<\beta^{**}$.

We focus on the case with the intermediate values of the extrapolation parameter, $R\leq g<2R$. To investigate the behavior of the system after the secondary bifurcation, we carry out the numerical analysis in the E&F Chaos[8] for the typical BH parameter values, $g=1.2,\ R=1.1,\ c=1,\ D=1$. After the Neimark–Sacker bifurcation the system exhibits two coexisting quasi-periodic attractors (above and below the fundamental). The largest Lyapunov exponent computed numerically becomes positive for β just above the Neimark–Sacker bifurcation critical value, β^{**}, which suggests weakly chaotic dynamics.

5. Agent-based simulations

The low-dimensional approximation analyzed in the previous section cannot be easily derived for networks with clustered neighborhoods or regularly structured links such as the regular lattice or the small world networks. Intuitively, having directly connected neighbors, whose types are correlated, decreases the *informational content* of the neighborhood. In the context of the random graph network, this may be viewed as a reduction in the "effective" neighborhood size or the average degree of the network, k. Hence, by fixing the "nominal" value of k, we may expect earlier (for smaller β) bifurcations for more clustered networks than for the random networks. We proceed with agent-based simulations to investigate this hypothesis and, more generally, to compare the effects of various network topologies.

We conduct the agent-based simulations[9] for four different network topologies of local interactions, i.e., for a fully connected graph, a regular lattice, a small world graph and a random graph (see Fig. 1). In the baseline regular lattice network each node has $K=4$ links. We further consider extensions to $K=6$ and $K=8$. All the graphs are connected, that is, there are no nodes that do not have any links. The fully connected graph is used as a benchmark corresponding to the finite number of agents implementation of the original BH model. Note that in the fully connected graph each node has $N-1$ links which is a much larger number relatively to other considered network topologies. As a baseline model we consider the model with two types of agents, fundamentalists and chartists. The simulations are further extended to the model with four types (Section 5.3). We analyze the asset price dynamics for $N=1000$ agents and, hence, the rewiring probability to obtain

[7] The bifurcation values for the two-type BH model are $\beta^*=-\ln(g/R-1)/c$ and $\beta^{**}=\beta^*+(1-2R+\sqrt{1-8R+8R^2})/2Rc(1-R/g)$.

[8] The E&F chaos is a software package for nonlinear economic dynamics (Diks et al., 2008). Code for this model and the generated plots are available on request.

[9] The C++ code for our simulations is partially adapted from the code of Bottazzi et al. (2005) and is available on request.

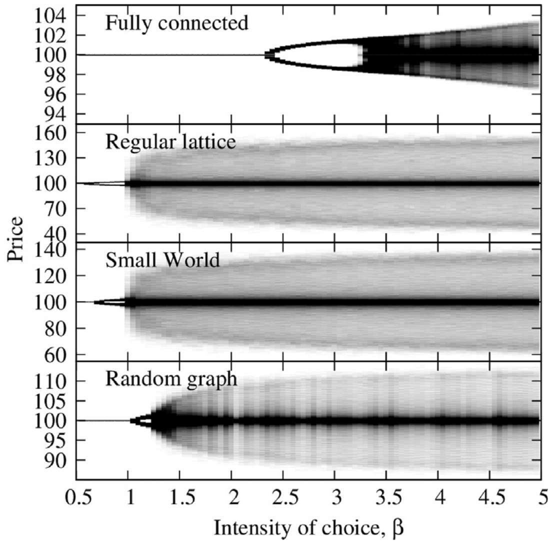

Fig. 6. Bifurcation diagrams for various network topologies.

the small world network is set to $\pi = 0.01$. We found that $N = 1000$ is sufficient for convergence of the discrete choice probabilities (in Eq. (8)) to the observed fractions in the case of the fully connected graph.[10] Given that the number of agents in our simulations is finite the system may get locked in the state with only one type. To avoid this, we introduce two "die-hard" agents who never change their type. They are located on the opposite sides of the network. For comparison we choose the basic parameter values of the model similar to those used for the two-type BH model, that is, $r = 0.1, \overline{y} = 10, D = 1, c = 1, g = 1.2$.

5.1. Evolution of prices and beliefs

The asset price dynamics for a range of values of β are shown by means of bifurcation diagrams in Fig. 6. These bifurcation diagrams depict the dependence of the price distribution on the intensity of choice parameter, β. Each bifurcation diagram combines two parts: one is initialized at positive deviations from the fundamental price, and the other for the negative deviations of the same magnitude. The price distribution for each level of β is represented by a gray-shade histogram. Darker shades correspond to areas of higher density. The histograms are computed using price levels from 10000 periods after 2000 transient periods with β ranging from 0.5 to 5 and a linear step of 0.05.

The primary and secondary bifurcations occurring in the fully connected network are similar to the pitchfork and Neimark–Sacker bifurcations occurring in the original BH model for $\beta^* = 2.40$ and $\beta^{**} = 3.33$ respectively. During the pitchfork bifurcation, the steady state loses its stability and two additional stable steady states are created. The Neimark–Sacker bifurcation leads to the emergence of periodic or quasi-periodic cycles. The economic intuition behind these bifurcations is as follows. Fundamentalists bring the price to the fundamental level, while chartists destabilize the fundamental price by extrapolating the trend. The difference in the fractions of these two types determines the price behavior. When the price is close to the fundamental level the excess returns of fundamentalists and chartists are equal, but the former incur the costs. When $\beta < \beta^*$ this relative difference in the past performance is not important for the choice of the type. Thus, the difference in the fractions is not large enough and the price remains at the fundamental level. However, when $\beta^* < \beta < \beta^{**}$, the relative past performance becomes more important and a larger fraction of agents chooses the less costly chartist type. This results in the deviation of the equilibrium price from the fundamental level. When $\beta > \beta^{**}$, that is, when the agents become highly reactive to the difference in excess returns, we observe cyclical behavior. When the price is near the fundamental level, the fraction of chartists rapidly increases, amplifying any small deviation from the fundamental level and creating a bubble. The bubble ends since the extrapolative behavior of chartists is not strong enough to sustain the trend and at some point fundamentalists start dominating the market bringing the price back to the fundamental level and this sequence recurs.

For the random graph model the bifurcation values of β are close to the values in the low-dimensional approximation derived in the previous section, namely, $\beta^* = 1.07$ and $\beta^{**} = 1.35$. The other networks show dynamics consistent with our previous predictions. In particular, in terms of the occurrence of the primary bifurcation with respect to the critical value of β the networks can be arranged as follows (in decreasing order): the fully connected network, the random graph, the small

[10] We also analyzed networks with $N = 100$. Qualitatively the results were similar. However, the level of noise due to the finite sample implementation was much higher.

world network, and the regular lattice. These results along with orderings in terms of other characteristics are collected in Table 1 in the end of the paper. The reverse order holds with respect to the price amplitude for fixed $\beta = 4$. These results can be explained by typical characteristics of the network, namely, the average size of the neighborhood, or the average degree, k, adjusted by its informational content, that is, by the level of independence (disconnectedness) of the neighbors between themselves. The latter is inversely related to the clustering coefficient, C. The information about the performance of the alternative type reaches all the nodes in the fully connected network within one time period. As we remove some links, the average neighborhood size, k decreases, the characteristic path length, L, increases and the information transmission between the agents who are not directly connected slows down. In addition to the decreased k and increased L, in the regular lattice we observe high clustering, C, which means that, as we discussed earlier, the informational content of the neighborhood is impaired by the direct connections between the neighbors. In particular, in the neighborhood of size four there are two directly connected neighbors. This makes the overall speed of the information transmission the slowest for the regular lattice. Slower information transmission results in higher persistence of a prevalent type over time, or, in other words, it delays the switching. Thus, in the regular lattice the fraction of chartists becomes relatively large for smaller values of β in comparison to the fully connected network. This translates into earlier bifurcations. The post-bifurcation region of price instability becomes larger and the amplitude of price fluctuations becomes higher. As we start rewiring some of the nodes with probability π, both the clustering, C, and the characteristic path length, L, of the network reduce. A reduction in C increases informative content of the neighborhood and a reduction in L decreases the minimum distance between any two nodes. Both C and L are decreasing in π (recall Fig. 2) and, therefore, the information transmission in the network is increasing in π.

Importantly, k, L and C can be measured from empirical data for any network, while π and K are the parameters of the Watts and Strogatz (1998) model. Therefore, a natural question is whether a single measurable characteristic k, L or C on its own is able to characterize the information transmission and predict price bifurcation values and amplitudes in our model. Note that both C and L are functions of π and K; and $k = K$ in the Watts and Strogatz (1998) model (see Newman, 2003, for more details on these relations). Therefore, fixing any two of the three measurable characteristics is not possible without predefining the third. Fig. 6 already presented the case in which we fixed k, i.e. $k = K$, except for the fully connected network, and gradually reduced π. Because bifurcation diagrams keep changing for fixed k, we may conclude that k on its own is not able to characterize the information transmission. Fig. 7 shows the bifurcation diagrams for the networks in which we vary $k = K$ as $K \in 4, 6, 8$ and (a) fix L (C becomes predetermined) and (b) fix C (L becomes predetermined). All these networks exhibit small world properties (relatively high C and small L). The bifurcation diagrams keep changing for fixed L and C. Hence, we conclude that neither L nor C by itself is a sufficient measure to represent the speed of information transmission in our model. As we previously discussed a "clustering-adjusted" degree could be a promising measure for our set-up. However, the development of this measure is left for future work.

Fig. 8 depicts the time series of the price for two values of the intensity of the switching parameter, $\beta = 1$ (left panel) and $\beta = 3.5$ (right panel), and the four networks: the fully connected graph (FC), the regular lattice (RL), the small world network (SW) and the random graph (RG). We use this abbreviation in subsequent figures as well. For $\beta = 1$, the price dynamics corresponding to the fully connected graph and the random network converges to a steady state; the regular lattice and the small world network lead to irregular asset price fluctuations. For $\beta = 3.5$, somewhat regular fluctuations emerge for all the network topologies, however, the regularity and the amplitudes of fluctuations vary considerably among them. The price

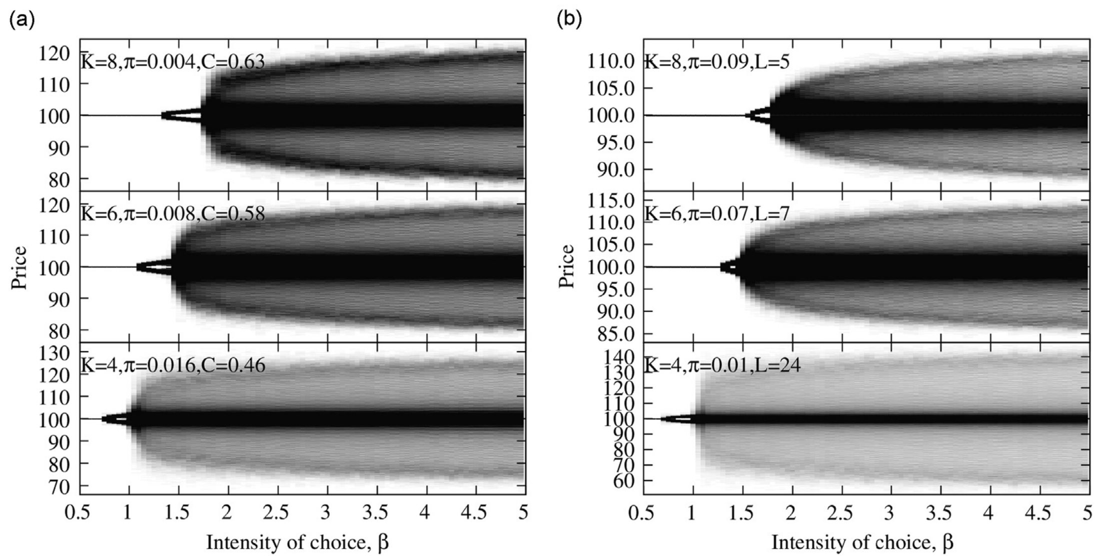

Fig. 7. Bifurcation diagrams for varying K, π, L and C. (a) Fixed $L = 17$. (b) Fixed $C = 0.48$.

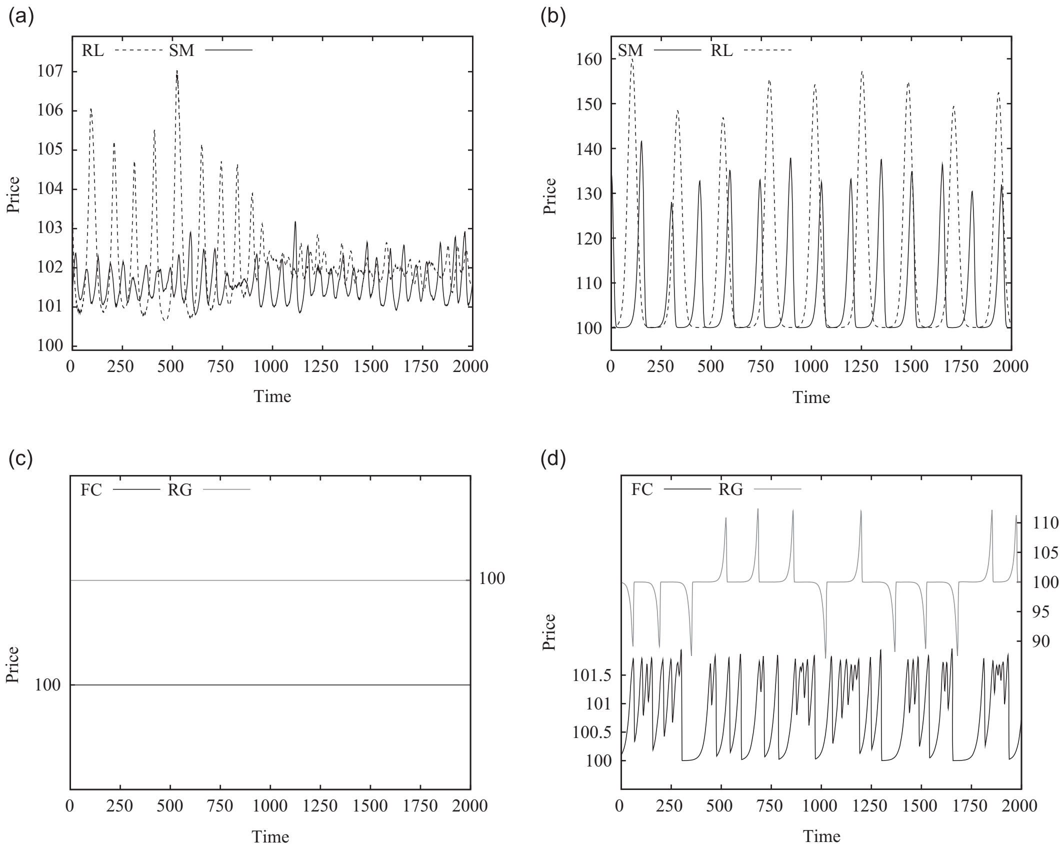

Fig. 8. Time series of price. (a) $\beta = 1$. (b) $\beta = 3.5$. (c) $\beta = 1$. (d) $\beta = 3.5$.

dynamics of the random network are close to those of the fully connected network. The price dynamics produced by the regular lattice are the most distinct from the fully connected network. The small world network produces price dynamics similar to the regular lattice with some shift towards the random graph. The observed price behavior is consistent with the previously inferred bifurcation values of β for different network topologies.

To provide insights into the effects of different network topologies on market behavior, we track how individual agents change their forecasting beliefs over time. Fig. 9 shows a typical set of patterns that emerge during simulations. This set is for $\beta = 3.5$. The figure shows the evolution of the forecasting type for all 1000 agents at every time step from 0 to 1000. Each point on a vertical line represents an agents' type: a black point indicates the fundamentalist type, while a blank point indicates the chartist type. Agent's ID indicates a spacial location of the agent on the initial regular lattice (Fig. 1). The agents are numbered sequentially clockwise. The circular lattice is broken between agent 0 and agent 1000 to be represented as a line. The inner-circle connections are not explicitly shown on the line, but the network configuration can be deduced from the time-evolution of agents' types. Agent 0 is a “die-hard” chartist and agent 500 is a “die-hard” fundamentalist. These two agents never change their types.

The periods of the highest concentration of fundamentalists correspond to the time when the price falls to the fundamental level, while the lowest concentration of fundamentalists corresponds to the highest deviation from the fundamental value of the price. Since the bifurcation values of β depend on the network topology, the four models may be at different stages of development for fixed $\beta = 3.5$. Hence, the direct comparison of the models is not formally possible. For each network topology, however, the observed patterns are somewhat representative of the behavior subsequent to the secondary bifurcation. They can be used to better understand price dynamics on the right side in Fig. 8. Overall, the fraction of fundamentalists is relatively high in the fully connected network. This is consistent with smaller deviations from the fundamental price and frequent price oscillations. Fundamentalists are relatively uniformly distributed across the network. Large spikes in the fractions of fundamentalists correspond to price falls. In the case of the regular lattice we observe high clustering of fundamentalists around the fundamental “core”. This is consistent with the high clustering coefficient of this network. In the small world network we also observe clusters, but they are smaller and more disperse in space. Again this is consistent with sparsity and a high clustering coefficient typical for this network. In the case of the random graph we do not observe any clusters of fundamentalists. This is due to a very small clustering coefficient for this network and a relatively small number of fundamentalists in the market in most of the periods.

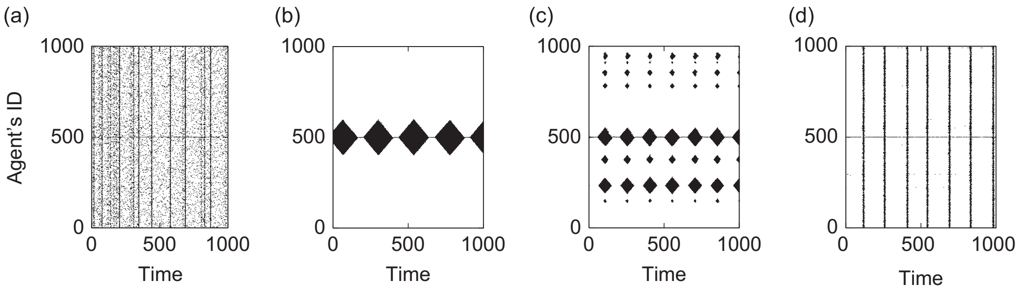

Fig. 9. Evolution of agent population. (a) Fully connected. (b) Regular lattice. (c) Small world. (d) Random graph.

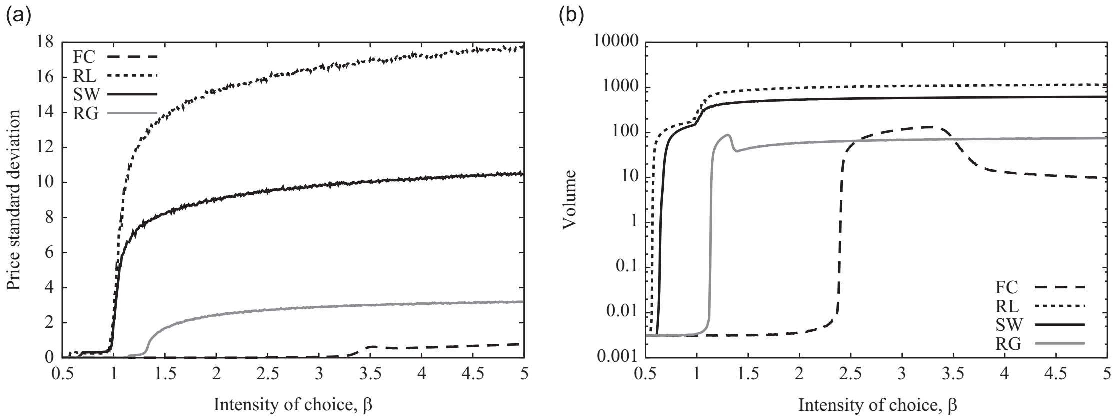

Fig. 10. Measures of market information inefficiency. (a) Standard deviation of price. (b) Average traded volume.

The informational efficiency is closely related to the speed of information transmission and can be measured by comparing the volatility of the observed price with the volatility of the fundamental dividend process as suggested by Shiller (1981). In order to abstract from the effect of the time-varying dividend in our model, we keep the dividend process constant. Under this assumption, the Efficient Market Hypothesis would predict constant price over time and zero trading volume. In Fig. 10 we analyze the standard deviation of the price (panel a) and the average traded volume (panel b) for values of β ranging from 0.5 to 5 for the four topologies. We ignore the first 2000 transitory iterations and compute the standard deviation of the price and the average traded volume for the following 2000 periods. To eliminate the dependence of our results on a particular realization of the random seed, we report averages for 100 simulation runs, each run having its own random seed. The same simulation setup is used for all the other statistics reported below. We observe that the random graph and the fully connected network exhibit the most informational efficient outcomes for any values of β which is consistent with the highest speed of information transmission in these two networks. The regular lattice exhibits the least informationally efficient outcome.

5.2. Statistical properties of the returns

Below we analyze the properties of the returns generated by the four considered networks and relate them to the stylized facts of financial time series.

Fig. 11 depicts the skewness of the returns and the kurtosis of the returns. The former statistic (Fig. 11a) measures the asymmetry of the distribution. It is close to zero for all the networks for all post-bifurcation values of β. The returns generated by the model with the small world network are slightly negatively skewed.

The kurtosis plot (Fig. 11b) reveals that all the four networks generate return distributions with different kurtosis values. The small world network return distribution exhibits the kurtosis value around 8, which is relatively close to the one observed for the returns on the financial markets.

By computing the autocorrelation of the returns for the four network structures, we analyze their linear (un)predictability. Fig. 12a depicts the autocorrelation of returns for the first five lags as a function of the intensity of choice. Usually empirical stock return series exhibit small or no autocorrelation. The regular lattice and the small world network produce high autocorrelations at all lags. This, again, can be attributed to a less efficient information transmission in these networks. Although the random graph and the fully connected network display large autocorrelations at the first two lags, they converge to zero autocorrelation values at lags three to five. The significant positive autocorrelations are resulting predominantly from the persistence of the chartist type. It is

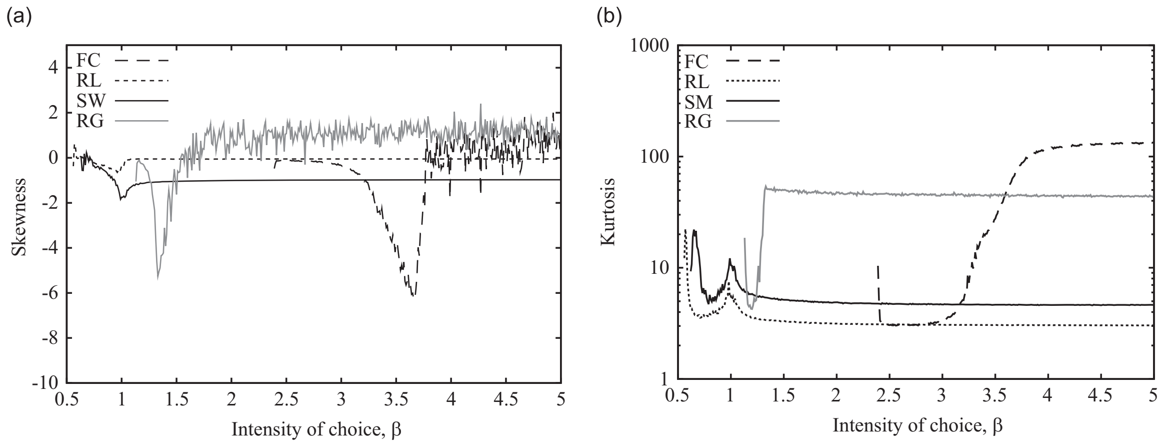

Fig. 11. (a) Skewness and (b) Kurtosis of returns.

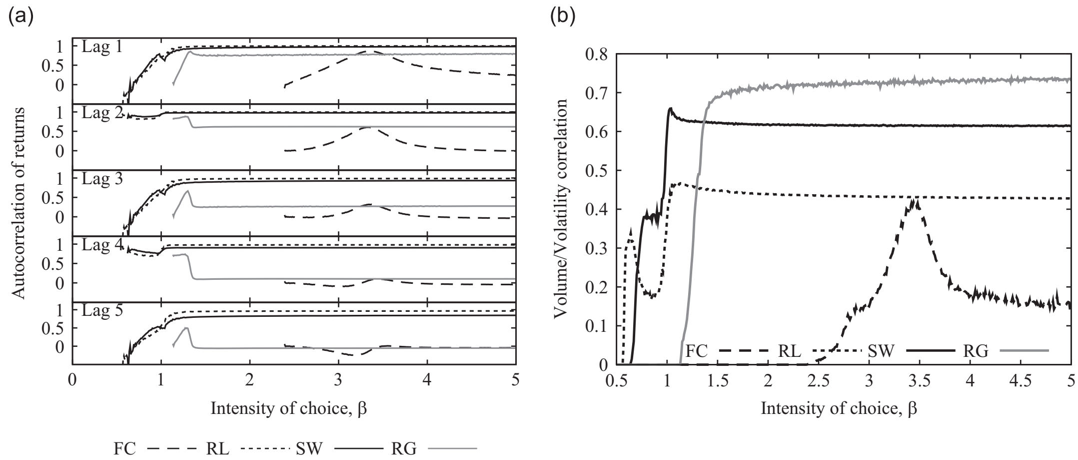

Fig. 12. Properties of returns (a) Autocorrelations of returns and (b) Volume/volatility correlations.

possible to reduce the autocorrelations by adding a sufficient amount of dynamic noise into the price as in Hommes (2002). However, we do not aim to reproduce stylized facts in this paper and therefore do not pursue this route.

Fig. 12b shows the correlations between the squared returns and the volume of trades. In real financial markets, high trade volumes are associated with high volatility. Many standard asset pricing models, however, fail to reproduce this relation. Our model produces positive volume-volatility correlations for all networks. The highest values in the post-bifurcation region are observed under the random graph network, followed by the small world network.

A universal property of the empirical return series is the volatility clustering, i.e., the presence of slow decaying autocorrelations in the squared returns. Fig. 13a shows the autocorrelations of the squared returns at the first five lags as a function of β, while Fig. 13b shows the autocorrelation function of the squared returns for 20 lags with $\beta = 3.5$. The autocorrelations of the squared returns under the fully connected network and random graph vanish after the first few lags, which is not consistent with stylized facts. In turn the autocorrelations under the regular lattice and the small world network remain positive and large at many lags for the regular lattice and the small world network, indicating the volatility clustering of the returns.

The above analysis reveals that different local interaction arrangements in the market affect the dynamics and the time series properties. The effect of the change in the behavior parameter β also depends on a particular network configuration.

5.3. Extension to four belief types

In this section we extend the baseline BH model with fundamentalists and chartists to the model with four belief types by introducing two additional types, i.e., positively biased extrapolators and negatively biased extrapolators (see Brock and Hommes, 1998, Section 4.3). This model is chosen because for some parametrization it exhibits almost no autocorrelations in the chaotic return series (Hommes, 2006). The general form of all belief types is given in Eq. (5). The parameters used for fundamentalists and chartists are set to $b^f = 0,\ g^f = 0$ and $b^c = 0,\ g^c = 1.21$ and the parameters for the two new types, the

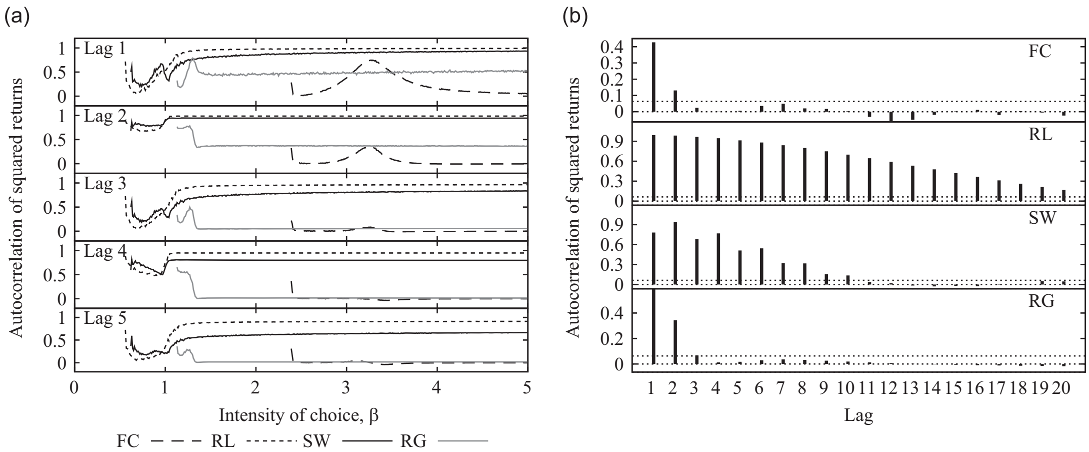

Fig. 13. Autocorrelation of squared returns. (a) Autocorrelations. (b) ACF.

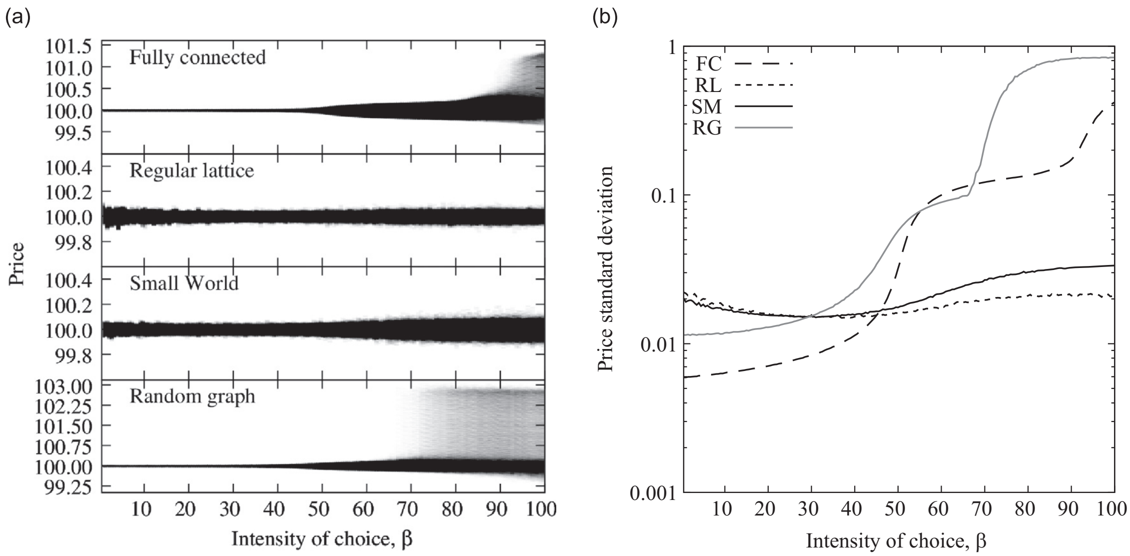

Fig. 14. Bifurcations in four-type models. (a) Bifurcation diagrams. (b) Standard deviation of the price.

positively and negatively biased extrapolators, are set to $b^p = 0.2,\ g^p = 1.1$ and $b^n = -0.2,\ g^n = 0.9$; cost c were set to 0 for all types, $r = 0.1, \bar{y} = 10$.[11] The number of agents is set to $N = 1000$ and similarly to the two-type case, the network is populated by four equidistant "die-hard" agents who never change their types.[12]

Conceptually it is possible to extend a low-dimensional approximation of the two-type model with the random network (Eq. (10) in Section 4.2) to the four-type model. However, the extension is tedious in a sense that we would have to keep track of all possible situations when an agent has information about the performance of various subsets of all possible types and consider multiple Δ−s depending on these subsets. Hence, we resort directly to simulations.

Fig. 14 shows (a) the bifurcation diagrams and (b) the standard deviation of the price as a function of β for the model with four belief types. For the fully connected graph, the bifurcation appears to occur around the value of $\beta^* = 50$. This is the theoretically derived critical value for which the Neimark–Sacker bifurcation occurs in the BH model with four belief types. For large values of $\beta > 85$ chaotic behavior can be inferred in the BH model. This coincides with the region for β, in which we observe the highest price fluctuations in the case of the fully connected network. In the case of the random network the bifurcation seems to be occurring for the values of β just before 50 and the observed price fluctuations are higher relative to the fully connected network. From the standard deviations of the price we deduce that for small world and regular lattice

[11] This is a standard specification used in Brock and Hommes (1998). We also have tried an alternative specification in Hommes (2006), but the results did not change qualitatively.

[12] Given that we have four types now, there are several possibilities on how to locate "die-hard" agents relative to each other. We have tried various permutations, but they did not influence the results.

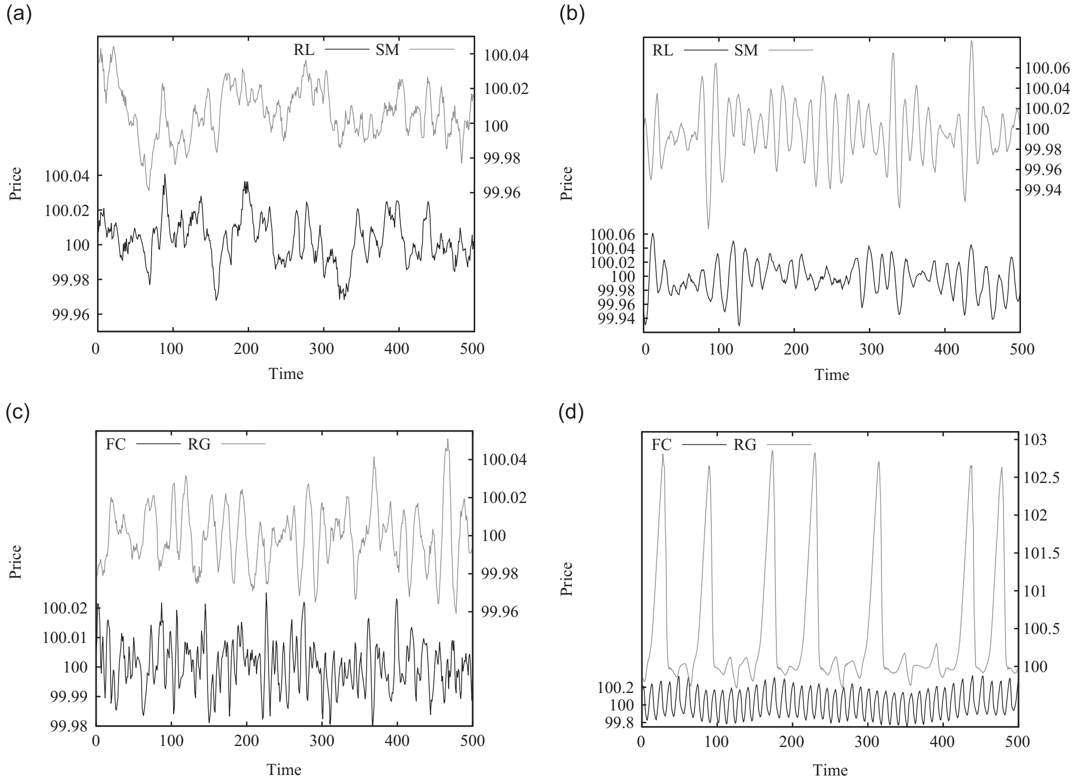

Fig. 15. Time series of price for four-type models. (a) $\beta=30$. (b) $\beta=85$. (c) $\beta=30$. (d) $\beta=85$.

bifurcations may be occurring for β around 35. This ordering in terms of the bifurcation values is somewhat similar to what we have observed in the two-type model. Contrary to the case of the two-type model, we do not observe large price fluctuations for the regular lattice and the small world networks. Fig. 15 shows typical time series for the four-type models for different values of β. For small $\beta=30$ we observe small amplitude noise fluctuations, closely resembling random walk series for the regular lattice and the small world networks. For larger $\beta=85$, the price deviations for the regular lattice and small world exhibit more regularity and some volatility clustering, but their amplitude remains small. The fully connected graph shows symmetric fluctuations of some regularity, while the random graph shows asymmetric fluctuation with large positive price spikes. Given that bifurcations occur at different values for different networks, care needs to be taken in comparing these results. To understand differences in the amplitude of the price, we look at the distribution of the fractions of the four types. For small $\beta=30$ all fractions are approximately equal to 0.25 for all network topologies. For large $\beta=85$, the distribution of the fraction differs substantially. In the fully connected network and the random graph, the fraction of chartists is close to zero, while the fractions of all the other types vary between zero and one, which causes the observed price fluctuations. Moreover the fractions of the active types change rapidly, a fraction may grow from 0 to 1 within 5 time periods. In the regular lattice and the small world, all the fractions fluctuate in a relatively small range, fundamentalists seem to dominate most of the time and their average fraction is close to 0.35, chartists are sometimes overtaking fundamentalists and their average fraction isaround 0.25, the fractions of the positively and negatively biased trades are both around 0.20. As β increases, in the regular lattice and the small world network the range of price and fraction fluctuations does not change much, but the fractions of fundamentalists and chartists increase, while the fractions of the positively and negatively biased agents decrease. One possible explanation of these differences in the distributions of the fractions is that in the networks with a relatively slow information transmission some types (positively and negatively biased chartists) become unsustainable because their fractions cannot grow fast enough.

We considered other network topologies with $K=6$ and $K=8$ but the results were qualitatively similar. In particular, as the average degree, $k=K$, increases holding the clustering coefficient, C, fixed, the dynamics for the small world case becomes closer to the dynamics observed for the random graph. We further qualitatively compare various characteristics of the two-type and four-type models for the considered networks with $K=4$ Table 1. Depending on the computed values of the characteristics the network topologies are reported in increasing order from left to right. First, we note that the ordering is rather different for the two-type and four-type models. We also note that in the two-type model the ordering in terms of the timing of the primary bifurcation, amplitude, length of the instability interval, statistical properties of prices and returns,

Table 1
Characteristics depending on the network in increasing order left to right. The characteristics depending on values of β are compared at fixed β: $\beta=4$ for the two-type model and $\beta=85$ for the four-type model. *Note*: $A \approx B$ indicates that there is no clear ranking between A and B, *neg* stands for negative values.

Characteristic	Two-type model	Four-type model
Latency in information transmission	FC RG SW RL	FC RG SW RL
$1/\beta$ of the primary bifurcation	FC RG SW RL	FC RG SW $\approx$ RL
Length of instability interval	FC RG SW RL	FC RG SW $\approx$ RL
Amplitude of price fluctuation	FC RG SW RL	RL SW FC RG
Std. deviation of price	FC RG SW RL	RL SW FC RG
Average trading volume	FC RG SW RL	RL SW FC RG
Skewness of returns	SW RL FC RG	RL SW FC RG
Kurtosis of returns	RL SW RG FC	SW RL FC RG
Autocorrelations of returns	FC RG SW RL	inconclusive
Volume/volatility correlations	FC RL SW RG	FC (*neg*) SM RL RG
Autocorrelations of squared returns	FC RG SW RL	FC RG RL SW

except for skewness and volume/volatility correlations, is consistent with the ordering of the latency in the information transmission. This ordering is different for the four-type model which is more complicated. For this model we observe large qualitative difference in the dynamics between the networks with low latency (FC and RG) and high latency (RL and SW) which may be the cause for the non-linear ordering.

6. Conclusions and extensions

In this paper we expanded the model of Brock and Hommes (1998) by introducing local information exchange via communication networks. We studied how different network structures affect asset price dynamics. We derived a low-dimensional system to represent dynamics in the two-type model with a random graph and proved some stability results for this case. Other network structures were investigated by simulations. We observed that the stability regions with respect to the intensity of choice parameter β depend on the parameters of the communication network. In the two-type model, the latency in the information transmission, which is the highest for the regular lattice and the small world networks, creates greater information inefficiencies and induces greater instabilities and higher deviations in the price dynamics. In the four-type network model, the latency in information transmission causes qualitatively different results.

The work in this paper may be extended in a number of directions. (1) The basic principles used to derive the low-dimensional analytically tractable model for the random network may be used to derive similar models for small world networks. Moreover, it would be interesting to extend this work to the scale free networks and other topologies popular in the literature. Other agent ecologies may also be extended to incorporate various network structures. (2) Another interesting direction is to make the strength of the noise, $1/\beta$, dependent on the number of neighbors of own and alternative types. This to some degree would endogenize parameter β. (3) In addition to this, it would be important to consider agents with longer memory who would consider the investment rules they used more than one period ago or perform some counterfactual analysis if the performance of an alternative type is not observed. (4) In many real-life networks there is a feedback between network performance and network formation. The performance of agents may gradually influence the network topology they are active in. Extending the model to include endogenous network formation would also be of great interest.

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Appendix A

Proof of Proposition 1. By imposing steady state $(\overline{x},\overline{n})$ on Eqs. (11) and (12) we obtain

$$\overline{x}=\frac{g}{R}\overline{n}\,\overline{x} \tag{13}$$

$$\overline{n}=\overline{n}^{k+1}+[1-\overline{n}^{k+1}-(1-\overline{n})^{k+1}]/[1+\exp(\beta(Dg\overline{x}^2(R-1)-c))], \tag{14}$$

Existence. From Eq. (13) we find that a necessary condition for a steady state is either $\overline{x}=0$ or $\overline{n}=n^*$, where $n^*=R/g$. The former condition leads to fundamental steady states, while the latter to non-fundamental steady states. Additionally a steady state needs to satisfy the conditions implied by Eq. (14). For the *fundamental steady states*, $\overline{x}=0$, and $\overline{n}$ is a fixed point of map *f*, defined as

$$f(n)=n^{k+1}+(1-n^{k+1}-(1-n)^{k+1})\Delta, \tag{15}$$

where $\Delta=1/(1+\exp(-\beta c))$ and $k>1$. See Fig. 4 for plots of the map.

The map, $f:[0,1]\to[0,1]$, is continuously differentiable and increasing, $f'(n)=(k+1)(n^k(1-\Delta)+(1-n)^k\Delta)>0$. There are two corner fixed points $\overline{n}=f(0)=0$ and $\overline{n}=f(1)=1$ which correspond to fundamental steady states $E_0(0,0)$ and $E_1(0,1)$. We now prove that for $\beta<\beta_1=\ln(k)/c$ there exists the unique interior fixed point and for $\beta>\beta_1=\ln(k)/c$ no fixed point exists. Let $n^\diamond$ be an interior fixed point such that $n^\diamond\in(0,1)$. By setting $f''(n)=0$, we find the unique interior inflection point $n^\sim=1/[(1/\Delta-1)^{1/(k-1)}+1]$ such that $f''(n)<0$ for $n<n^\sim$ and $f''(n)>0$ for $n>n^\sim$. Hence, we conclude that there may exist at most one interior fixedpoint.[13] Next we derive the condition for which the interior fixed point exists. We find $f'(0)=(k+1)\Delta>1$ and $f'(1)=(k+1)(1-\Delta)$. There may be two cases: (1) $f'(1)>1$ which is equivalent to $\beta<\beta_1$ and for which the (unique) interior point exists[14] and $E_\diamond(0,n^\diamond)$ is the fundamental steady of the system; and (2) $f'(1)\le 1$ which is equivalent to $\beta>\beta_1$ and for which no interior fixed point exists.[15] At $\beta=\beta_1$, $E_\diamond$ collides into E_1. Note that $1/2\le\Delta<1$ because $\beta c\ge 0$. Using this we find that $n^\diamond\ge\Delta$ and $1/2\le n^\diamond<1$.[16]

Next, we derive the *non-fundamental steady states* for which $\overline{n}=n^*=R/g$. Because $0\le n\le 1$, the non-fundamental steady states may exist only for $g\ge R$. By substituting $\overline{n}=n^*$ in Eq. (14), we find $\overline{x}=\pm x^*$, where

$$x^*=\sqrt{\left(\ln\left[\left(\frac{g}{R}-1\right)\left(1-\left(1-\frac{R}{g}\right)^k\right)\Big/\left(1-\left(\frac{R}{g}\right)^k\right)\right]\Big/\beta+c\right)\Big/(Dg(R-1))}.$$

It is easy to see that a real value of x^* exists if and only if $\beta\ge\beta^*$, where

$$\beta^*=-\ln\left[\left(\frac{g}{R}-1\right)\left(1-\left(1-\frac{R}{g}\right)^k\right)\Big/\left(1-\left(\frac{R}{g}\right)^k\right)\right]\Big/c.$$

Importantly, $\beta^*=0$ when $g/R=2$ and β^* becomes negative and hence nonbinding for existence of x^* when $g>2R$. Note that x^* reaches its minimum value $x^*=0$ at $\beta=\beta^*$. Therefore, $n^\diamond=n^*$ at $\beta=\beta^*$. This way, we establish the existence of the non-fundamental steady states, $E_+=(x^*,n^*)$, $E_-=(-x^*,n^*)$, for $\beta>\beta^*$ when $R\le g\le 2R$ and for any values of β when $g>2R$.

Stability. To verify the stability of the steady states we compute the eigenvalues of the corresponding Jacobian matrix at steady states. A steady state is stable when all the eigenvalues lie inside of the unit circle. For the *fundamental steady states* at $x=0$ the eigenvalues are

$$\left(0,\frac{g}{R}n,\frac{(k+1)(\exp(\beta c)(1-n)^k+n^k)}{1+\exp(\beta c)}\right).$$

Steady state $E_0=(0,0)$ is always unstable. Steady state $E_1=(0,1)$ is unstable for $g\ge R$, while for $g<R$ it is stable only when $\beta>\beta_1$. At $\beta=\beta_1$, the transcritical bifurcation takes place at which $E_\diamond$ ceases to exist by colliding into E_1 and E_1 gains stability.

To derive the stability conditions for steady state $E_\diamond=(0,n^\diamond)$, we consider the third eigenvalue equal to $f'(n^\diamond)$. We have previously established that $f'(n^\diamond)>0$, $f'(0)>1$ and that $f'(1)>1$ whenever the unique interior fixed point, $n^\diamond$, exists. Because, the map is continuously differentiable and there is a unique interior inflection point, it holds that $0<f'(n^\diamond)<1$.[17] From the second eigenvalue we establish that $E_\diamond$ is unstable for $g\ge 2R$, it is always stable for $g<R$ and it is stable for $g<2R$

[13] Consider continuously differentiable map $g(n)=f(n)-n$ defined on interval [0,1]. By the mean value theorem between any two real roots of $g(n)$ there should be at least one turning point $g'=0$ or $f'=1$. By the same argument between any two turning points there should be at least one inflection point of *f*.

[14] Apply the intermediate value theorem to map $g(n)=f(n)-n$: for infinitesimally small $\varepsilon>0$, $g(0+\varepsilon)>0$, $g(1-\varepsilon)<0$ and hence $\exists n_0\in[\varepsilon,1-\varepsilon]$ such that $g(n_0)=0$.

[15] Any such fixed point would require the existence of at least two inflection points.

[16] To show this, impose $f(n)=n$ to obtain $n/\Delta=(1-n^{k+1}-(1-n)^{k+1})/(1-n^k)$. Note that $(n/\Delta)\ge 1$ when $n\ge 1/2$ and also that $n=1/2$ when $\Delta=1/2$. To complete the argument, use the implicit function theorem and show that n is increasing in Δ on interval $(0,1)$.

[17] In other words under these conditions, the only way $g(n)=f(n)-n$ may cross $n=0$ line is from the above which can be proved using the mean value and the intermediate value theorems.

when $n^\diamond < R/g$. Because $n^\diamond$ is an increasing function of β and $n^\diamond = n^*$ at $\beta = \beta^*$, we can express this stability condition for $E_\diamond$ in terms of β, that is, $\beta < \beta^*$. Also, note that $\beta^* < \beta_1$. At $\beta = \beta^*$ one of the three real eigenvalues becomes equal to one and the pitchfork bifurcation takes place. For $\beta > \beta^*$ the fundamental steady state, $E_\diamond$, looses stability and the two non-fundamental steady states, $E_+ = (x^*, n^*)$ and $E_- = (-x^*, n^*)$, emerge and gain stability as we show below.

The characteristic polynomial for the stability of the *non-fundamental steady states* $E_+ = (x^*, n^*)$ and $E_- = (-x^*, n^*)$ is

$$\mathcal{P}(\lambda) = \lambda(\lambda-1)(\lambda-A) + (\beta-\beta^*)B(\lambda^2 - R(1+\lambda)+1)/(c(R-1)),$$

where

$$A = \frac{(k+1)((1-n^*)^k n^* + (1-n^*)(n^*)^k - ((1-n^*(n^*)^k)}{1-(1-n^*)^k + n^*((1-n^*)^k - (n^*)^k)},$$

$$B = \frac{(1-n^*)(1-(1-n^*)^k)(1-(n^*)^k)}{1-(1-n^*)^k + n^*((1-n^*)^k - (n^*)^k)}$$

and $n^* = R/g$. Note that $0 < A \le 1$, $0 \le B \le 1$.

When $\beta = \beta^*$, $x^* = 0$ and all the three eigenvalues are real and equal to 0, 1 and A. In order to find regions of β for which all the eigenvalues lie inside the unit circle we apply the Schur–Cohn criterion.[18] We find critical value $\beta^{**} = \beta^* + (1-2R-A(R-1) + \sqrt{(1+A)^2 - 2(4+A+A^2)R + (8+A^2)R^2})/(2BRc)$, such that for $\beta^* < \beta < \beta^{**}$ the non-fundamental steady states are stable.

The discriminant of the characteristic polynomial at $\beta = \beta^{**}$ is negative which signals one real and two complex (conjugate) eigenvalues. This, in turn, indicates the occurrence of Neimark–Sacker bifurcation. □

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[18] The Schur–Cohn criterion states that all the roots of the polynomial $x^3 + bx^2 + cx + d$ lie inside the unit circle if and only if $|bd-c| < 1-d^2$ and $|b+d| < |1+c|$. Some calculations to use the criterion were performed in Wolfram Mathematica 8.0 and are available upon request.

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